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ON THE HAUSDORFF MEASURE OF A CLASS OF SELF-SIMILAR SETS

Abstract

We develop a new combinatorial method to estimate Hausdorff measures of various self-similar sets. This method can be applied to the evaluation of Hausdorff measures which are induced by various Hausdorff functions including power functions. Moreover, a few examples for evaluations of the lower and upper bounds of Hausdorff measures of uniform Cantor sets are introduced.

1 Introduction

Throughout the paper, we use $\mathbb{N}_0$, $\mathbb{N}$ and $\mathbb{R}$ to denote the set of all non-negative integers, of all positive integers and of all real numbers, respectively. $\mathbb{R}^n$ denotes n-dimensional Euclidean space, and D will be some fixed closed subset of $\mathbb{R}^n$. By $d(A)$ we denote the diameter of any subset A of $\mathbb{R}^n$, and the cardinal number of a set C will be denoted by $\#C$.

A monotonically increasing function $h : [0, \infty) \to [0, \infty)$ is called a Hausdorff function if and only if $h(t) > 0$ for $t > 0$, $h(0) = 0$ and h is continuous from the right. It is well-known that every Hausdorff function h induces a corresponding Hausdorff measure μ^h as follows

$$\mu^h(C) = \lim_{\delta \to 0} \inf \sum_i h(d(U_i)), \tag{1}$$

where we take the infimum over all δ-coverings $\{U_i\}_i$ of C (see [10]).

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A mapping $S : D \to D$ is called a similarity on D if there is a constant c $(0 < c < 1)$ such that $|S(x) - S(y)| = c|x - y|$ for all $x, y \in D$. The constant c is called the (similarity) ratio of S.

Throughout the paper, suppose $m\,(> 1)$ is a fixed integer. Let $\{S_1, \ldots, S_m\}$ be a family of similarities on D. We say that a subset F of D is self-similar under $S_1, \ldots, S_m$ if

$$F = \bigcup_{i=1}^{m} S_i(F).$$

Definition 1. A family $\{S_1, \ldots, S_m\}$ of similarities on D is said to be disjoint if $S_i(D) \cap S_j(D) = \emptyset$ for all $i \neq j$.

Hutchinson [6] proved that for every disjoint family $\{S_1, \ldots, S_m\}$ of similarities on D there exists a unique non-empty compact set which is self-similar under the S_i's. Many self-similar sets are well-known, e.g., Cantor sets, Cantor dusts, the Sierpiński gasket, the von Koch curve, etc.

For the time being, let $\{S_1, \ldots, S_m\}$ be any disjoint family of similarities on D with the ratios $c_1, \ldots, c_m$, respectively. Let $\ell \in \mathbb{N}$ be fixed. By $\mathcal{I}_\ell$ we denote the family of all finite sequences $(a_i)_{i=1,\ldots,\ell}$ satisfying $a_i \in \{1, \ldots, m\}$, i.e.,

$$\mathcal{I}_\ell = \{(a_i)_{i=1,\ldots,\ell} : a_i \in \{1, \ldots, m\}\}.$$

For every finite sequence $a = (a_i) \in \mathcal{I}_\ell$, let $S_a = S_{a_1} \circ S_{a_2} \circ \cdots \circ S_{a_\ell}$ with the convention

$$S_a(x) = S_{a_1}(S_{a_2}(\cdots(S_{a_\ell}(x))\cdots)).$$

We denote by $\mathcal{H}$ the class of all Hausdorff functions and by $\mathcal{H}_m^c$ the class of all Hausdorff functions satisfying

$$h(ct) = \frac{1}{m} h(t) \tag{2}$$

for all sufficiently small $t > 0$.

In this paper, a new combinatorial method to estimate Hausdorff measures (which are induced not only by power functions but also by other Hausdorff functions different from power functions) of self-similar sets are investigated (see Theorem 8), and the method is applied to the estimation of Hausdorff measures of uniform Cantor sets.

2 Preliminaries

Definition 2. Suppose $\{S_1, \ldots, S_m\}$ is a disjoint family of similarities on D. Let C be an arbitrary subset of $\mathbb{R}^n$. For any non-negative integer i let

$$\alpha_i(C) = \lim_{\ell\to\infty} \frac{\#\{a \in \mathcal{I}_{i+\ell} : S_a(D) \subset C\}}{m^\ell}.$$

Lemma 1. *Let $\{S_1, \ldots, S_m\}$ be a disjoint family of similarities on D, and let C be an arbitrary subset of D. For each positive integer i*

(a) $\alpha_i(C) = m\alpha_{i-1}(C)$,

(b) $\alpha_i(C) = \alpha_{i+1}(S_j(C)) \quad (j = 1, \ldots, m)$,

(c) $\alpha_i(C) = \alpha_i\Big(\bigcup_{j=1}^m S_j(C)\Big)$.

Proof. (a) It follows from Definition 2 that

$$\alpha_i(C) = \lim_{\ell\to\infty} \frac{m\#\{a \in \mathcal{I}_{i-1+\ell+1} : S_a(D) \subset C\}}{m^{\ell+1}} = m\alpha_{i-1}(C).$$

(b) Let A, B be arbitrary subsets of D and $j \in \{1, \ldots, m\}$ fixed. If $x \in A\backslash B$ and $S_j(A) \subset S_j(B)$, then there exists some $y \in B$ such that $S_j(y) = S_j(x)$ contrary to the injectivity of S_j. So $S_j(A) \not\subset S_j(B)$ if $A \not\subset B$. On the other hand, it is obvious that $S_j(A) \subset S_j(B)$ if $A \subset B$. Hence, we obtain that $S_j(A) \subset S_j(B)$ if and only if $A \subset B$. So, for all $j \in \{1, \ldots, m\}$, we have

$$\alpha_i(C) = \lim_{\ell\to\infty} \frac{\#\{a \in \mathcal{I}_{i+\ell} : S_j \circ S_a(D) \subset S_j(C)\}}{m^\ell}.$$

Consequently if $S_a(D) \subset C$, then $S_j \circ S_a(D) \subset S_j(C) \subset S_j(D)$ and $S_k \circ S_a(D) \subset S_k(C) \subset S_k(D)$ for all $j, k \in \{1, \ldots, m\}$. If $j \neq k$, then $S_k \circ S_a(D) \not\subset S_j(C)$, since $\{S_1, \ldots, S_m\}$ is disjoint. Thus, combining this fact with the above equality, we get

$$\alpha_i(C) = \lim_{\ell\to\infty} \frac{\#\{a \in \mathcal{I}_{i+1+\ell} : S_a(D) \subset S_j(C)\}}{m^\ell} = \alpha_{i+1}(S_j(C)).$$

(c) Since $\{S_1, \ldots, S_m\}$ is disjoint, by (a) and (b), we obtain

$$\alpha_i\Big(\bigcup_{j=1}^m S_j(C)\Big) = \sum_{j=1}^m \alpha_i(S_j(C)) = \sum_{j=1}^m \alpha_{i-1}(C) = m\alpha_{i-1}(C) = \alpha_i(C). \qquad \square$$

3 Method of Substitution

Let $D \neq \emptyset$ be a closed subset of $\mathbb{R}^n$ and let $\{S_1, \ldots, S_m\}$ be a disjoint family of similarities on D with common ratio c $(0 < c < 1)$. Suppose D_o is a subset of D with non-empty interior such that $c^{i_*+1}d(D) < d(D_o) \leq c^{i_*}d(D)$ for a fixed non-negative integer i_*. Let D_v be a subset of D for which $c^{i_*+1}d(D) < d(D_v) \leq c^{i_*}d(D)$, and suppose that there exist an integer $i_v (\geq i_*)$ and a finite sequence $a \in \mathcal{I}_{i_v}$ satisfying $S_a(D) \subset D_v$. (The last hypothesis guarantees that $\alpha_0(D_v) > 0$.)

We now introduce a new method to estimate Hausdorff measures of self-similar sets.

(a) Let $\varepsilon > 0$ be given such that $\varepsilon/(1-\alpha_0(D_v))$ is sufficiently small (cf. the proof of Lemma 2 below). According to Definition 2, it is possible to choose a positive integer i_0 such that

$$m^{i_0}\left(\alpha_{i_*}(D_o) - \varepsilon\right) < n_0 = \#\{a \in \mathcal{I}_{i_*+i_0} : S_a(D) \subset D_o\} < m^{i_0}\left(\alpha_{i_*}(D_o) + \varepsilon\right).$$

Let $I_0^1, \ldots, I_0^{n_0}$ be an enumeration of the set $\{S_a(D) : S_a(D) \subset D_o;\, a \in \mathcal{I}_{i_*+i_0}\}$. For every $j = 1, \ldots, n_0$ there is exactly one $a \in \mathcal{I}_{i_*+i_0}$ with $I_0^j = S_a(D)$. Divide each I_0^j $(j = 1, \ldots, n_0)$ into $V_0^j = S_a(D_v)$ (where $a \in \mathcal{I}_{i_*+i_0}$ with $I_0^j = S_a(D)$) and $R_0^j = I_0^j \setminus V_0^j$.

We further describe the process of our method by induction on $\ell = 0, 1, 2, \ldots$.

(b) Assume that we have already chosen a sufficiently large integer i_ℓ $(\ell \geq 1)$ such that

$$m^{i_\ell}\sum_{j=1}^{n_{\ell-1}}\left(\alpha_{i_*+i_0+\cdots+i_{\ell-1}}(R_{\ell-1}^j) - \varepsilon\right) < n_\ell < m^{i_\ell}\sum_{j=1}^{n_{\ell-1}}\left(\alpha_{i_*+i_0+\cdots+i_{\ell-1}}(R_{\ell-1}^j) + \varepsilon\right),$$

where

$$n_\ell = \#\{a \in \mathcal{I}_{i_*+i_0+\cdots+i_\ell} : \text{there exists some } j \in \{1, \ldots, n_{\ell-1}\} \text{ with } S_a(D) \subset R_{\ell-1}^j\}.$$

Then, let $I_\ell^1, \ldots, I_\ell^{n_\ell}$ be an enumeration of the set

$$\{S_a(D) : S_a(D) \subset \bigcup_{j=1}^{n_{\ell-1}} R_{\ell-1}^j;\, a \in \mathcal{I}_{i_*+i_0+\cdots+i_\ell}\}.$$

For every $j = 1, \ldots, n_\ell$ there exists a unique $a \in \mathcal{I}_{i_*+i_0+\cdots+i_\ell}$ such that $I_\ell^j = S_a(D)$. Divide every I_ℓ^j $(j = 1, \ldots, n_\ell)$ into $V_\ell^j = S_a(D_v)$ (where $a \in \mathcal{I}_{i_*+i_0+\cdots+i_\ell}$ with $I_\ell^j = S_a(D)$) and $R_\ell^j = I_\ell^j \setminus V_\ell^j$. By Definition 2, we can again choose a large integer $i_{\ell+1}$ such that

$$m^{i_{\ell+1}} \sum_{j=1}^{n_\ell} \Big(\alpha_{i_*+i_0+\cdots+i_\ell}(R_\ell^j) - \varepsilon\Big) < n_{\ell+1} < m^{i_{\ell+1}} \sum_{j=1}^{n_\ell} \Big(\alpha_{i_*+i_0+\cdots+i_\ell}(R_\ell^j) + \varepsilon\Big),$$

where

$$\begin{aligned} n_{\ell+1} = \#\{a \in \mathcal{I}_{i_*+i_0+\cdots+i_{\ell+1}} &: \text{there exists some } j \in \{1, \ldots, n_\ell\} \\ &\text{with } S_a(D) \subset R_\ell^j\}. \end{aligned}$$

(c) Repeat the process (b) for $\ell + 1$.

Definition 3. Let $h \in \mathcal{H}$. Suppose $\{S_1, \ldots, S_m\}$ is a disjoint family of similarities on D with common ratio c. The above process is called D_v-substitution of D_o with respect to the sequence (i_ℓ). Every V_ℓ^j $(\ell = 0, 1, 2, \ldots; j = 1, \ldots, n_\ell)$ is called an element of D_v-substitution of D_o. The D_v-substitution of D_o is said to be efficient if

$$\sigma(D_o; D_v) = \lim_{\varepsilon \to 0} \sum_{\substack{V \text{ is an element of} \\ D_v\text{-substitution of } D_o}} h(d(V)) < h(d(D_o)).$$

Remark 1. Obviously, $\{V_\ell^j\}$ which was obtained from the above process covers almost all of the self-similar set F under $S_1, \ldots, S_m$. Indeed, for every $h \in \mathcal{H}$ we can select a covering of F consisting of $\{V_\ell^j\}$ and $\{E_j\}$ such that the values of $\sum h(d(E_j))$ and $\sum \alpha_{i_*}(E_j)$ are as small as desired by taking the values of i_ℓ's sufficiently large in the above process.

Lemma 2. *For every positive integer* ℓ

$$\begin{aligned} m^{i_0+\cdots+i_\ell}(\alpha_{i_*}(D_o) - \varepsilon)(1 - \alpha_0(D_v) - \varepsilon)^\ell &< n_\ell \\ &< m^{i_0+\cdots+i_\ell}(\alpha_{i_*}(D_o) + \varepsilon)(1 - \alpha_0(D_v) + \varepsilon)^\ell. \end{aligned}$$

Proof. By Lemma 1 (b) we have, for every positive integer ℓ,

$$\begin{aligned} \alpha_{i_*+i_0+\cdots+i_{\ell-1}}\Big(R_{\ell-1}^j\Big) &= \alpha_{i_*+i_0+\cdots+i_{\ell-1}}\Big(I_{\ell-1}^j \setminus V_{\ell-1}^j\Big) \\ &= \alpha_{i_*+i_0+\cdots+i_{\ell-1}}\left(S_a(D) \setminus S_a(D_v)\right) \\ &= \alpha_{i_*+i_0+\cdots+i_{\ell-1}}\left(S_a(D)\right) - \alpha_{i_*+i_0+\cdots+i_{\ell-1}}\left(S_a(D_v)\right) \\ &= 1 - \alpha_0(D_v) \end{aligned}$$

where $a \in \mathcal{I}_{i_*+i_0+\cdots+i_{\ell-1}}$ with $S_a(D) = I_{\ell-1}^j$. Combining this result with the inequalities for n_ℓ's in the above process (b) and by induction on ℓ, we complete the proof. $\square$

Theorem 3. *Let $\{S_1, \ldots, S_m\}$ be a disjoint family of similarities on D with common ratio c. Suppose $h \in \mathcal{H}_m^c$. If i_* is so large that the relation* (2) *holds for all $0 < t \leq c^{i_*} d(D)$, then we have $\sigma(D_o; D_v) = \alpha_{i_*}(D_o) \frac{h(d(D_v))}{\alpha_{i_*}(D_v)}$.*

PROOF. It follows from Definition 3, Lemma 2 and Lemma 1 (a) that

$$\begin{aligned}
\sigma(D_o; D_v) &= \lim_{\varepsilon \to 0} \sum_{\ell=0}^{\infty} \sum_{j=1}^{n_\ell} h(d(V_\ell^j)) \\
&= \lim_{\varepsilon \to 0} \sum_{\ell=0}^{\infty} n_\ell h(c^{i_*+i_0+\cdots+i_\ell} d(D_v)) \\
&= \lim_{\varepsilon \to 0} \sum_{\ell=0}^{\infty} \frac{n_\ell}{m^{i_*+i_0+\cdots+i_\ell}} h(d(D_v)) \\
&= \frac{\alpha_{i_*}(D_o)}{m^{i_*}} \frac{h(d(D_v))}{\alpha_0(D_v)} \\
&= \alpha_{i_*}(D_o) \frac{h(d(D_v))}{\alpha_{i_*}(D_v)}.
\end{aligned}$$
$\square$

Definition 4. Suppose $\{S_1, \ldots, S_m\}$ is a disjoint family of similarities on D with common ratio c. Let i be a non-negative integer. Define

$$\Phi_i(D_1) = \inf \left\{ \frac{h(d(I))}{\alpha_i(I)} : I \subset D_1;\ c^{i+1} d(D) < d(I) \leq c^i d(D) \right\}$$

for all subsets D_1 of D with $d(D_1) \geq c^i d(D)$, where we follow the convention that if $a > 0$, then $a/0 := \infty$.

In the definition of Φ_i, the infimum has to be taken over all 'test' sets I whose diameters lie between $c^{i+1} d(D)$ and $c^i d(D)$. Therefore, the Φ_i can be defined on the only sets D_1 with $d(D_1) \geq c^i d(D)$.

Definition 5. For any compact subsets A, B of D the distance $\rho(A, B)$ between A and B is defined by $\rho(A, B) = \min\{|x - y| : x \in A;\ y \in B\}$. Suppose $\{S_1, \ldots, S_m\}$ is a disjoint family of similarities on D with common ratio c. Let $\delta = \min\{\rho(S_i(D), S_j(D)) : i \neq j\}$ and $\tau = \min\{i \in \mathbb{N}_0 : c^{i+1} d(D) \leq \delta\}$. The constant τ is called the index of the self-similar set under $S_1, \ldots, S_m$.

Lemma 4. *Let τ be the index of the self-similar set under $S_1, \dots, S_m$ with common ratio c. Assume that $i \geq \tau$ and $a \in \mathcal{I}_{i-\tau}$ are fixed. Let $h \in \mathcal{H}$. Then $\Phi_i(S_a(D)) = \Phi_i(D)$, where we set $S_a(D) = D$ for $a \in \mathcal{I}_0$.*

PROOF. If $i = \tau$, the statement of the lemma is obvious. Now, let $i > \tau$, then we have by the fact that $S_1, \dots, S_m$ are similarities with common ratio c and by Definition 5

$$\begin{aligned}\min\{\rho(S_b(D), S_{b'}(D)) : b, b' \in \mathcal{I}_{i-\tau};\ b \neq b'\}& \\ &= c^{i-\tau-1}\min\{\rho(S_j(D), S_k(D)) : j \neq k\} \\ &\geq c^{i-\tau-1}c^{\tau+1}d(D) \\ &= c^i d(D).\end{aligned} \tag{3}$$

For any subset A of D let

$$\begin{aligned}\mathcal{C}_1(A) &= \{I \subset D : I \subset A;\ c^{i+1}d(D) < d(I) \leq c^i d(D)\}, \\ \mathcal{C}_2(A) &= \{I \subset D : I \cap A \neq \emptyset;\ I \not\subset A;\ c^{i+1}d(D) < d(I) \leq c^i d(D)\}\end{aligned}$$

and

$$\mathcal{C}_3(A) = \{I \subset D : I \cap A = \emptyset;\ c^{i+1}d(D) < d(I) \leq c^i d(D)\}.$$

Then we have

$$\mathcal{C}_1(D) = \bigcup_{b \in \mathcal{I}_{i-\tau}} \mathcal{C}_1(S_b(D)) \ \cup \bigcup_{b \in \mathcal{I}_{i-\tau}} \mathcal{C}_2(S_b(D)) \ \cup\ \mathcal{C}_3\left(\bigcup_{b \in \mathcal{I}_{i-\tau}} S_b(D)\right).$$

Since the structure in $S_b(D)$ is congruent to that in $S_a(D)$ and (3) implies that if $I \in \mathcal{C}_2(S_b(D))$, then $\#(I \cap S_{b'}(D)) \leq 1$ for any $b' \in \mathcal{I}_{i-\tau}$ with $b' \neq b$, the above equality implies that

$$\sup\{\alpha_i(I) : I \in \mathcal{C}_1(D)\} = \sup\{\alpha_i(I) : I \in \mathcal{C}_1(S_a(D))\}.$$

Further, since the structure in $S_b(D)$ is congruent to that in $S_a(D)$, we see that

$$\inf\left\{\frac{h(d(I))}{\alpha_i(I)} : I \in \mathcal{C}_1(S_b(D))\right\} = \inf\left\{\frac{h(d(I))}{\alpha_i(I)} : I \in \mathcal{C}_1(S_a(D))\right\}$$

for any $b \in \mathcal{I}_{i-\tau}$. As it was already stated, (3) implies that if $I \in \mathcal{C}_2(S_b(D))$, then $\#(I \cap S_{b'}(D)) \leq 1$ and hence $\alpha_i(I \cap S_{b'}(D)) = 0$ for each $b' \in \mathcal{I}_{i-\tau}$ with $b' \neq b$. Hence, we get

$$\inf\left\{\frac{h(d(I))}{\alpha_i(I)} : I \in \mathcal{C}_2(S_b(D))\right\} \geq \inf\left\{\frac{h(d(I))}{\alpha_i(I)} : I \in \mathcal{C}_1(S_a(D))\right\}.$$

Trivially, we have

$$\inf\left\{\frac{h(d(I))}{\alpha_i(I)} : I \in \mathcal{C}_3\Big(\bigcup_{b\in\mathcal{I}_{i-\tau}} S_b(D)\Big)\right\} = \infty.$$

Therefore, we may conclude that

$$\begin{aligned}\Phi_i(D) &= \inf\left\{\frac{h(d(I))}{\alpha_i(I)} : I \in \mathcal{C}_1(D)\right\} \\ &= \inf\left\{\frac{h(d(I))}{\alpha_i(I)} : I \in \mathcal{C}_1(S_a(D))\right\} \\ &= \Phi_i(S_a(D)).\end{aligned}$$

□

Lemma 5. *Suppose $\{S_1, \ldots, S_m\}$ is a disjoint family of similarities on D with common ratio c. Let $h \in \mathcal{H}_m^c$. Suppose i_* is given such that (2) holds for all $0 < t \le c^{i_*}d(D)$. Assume that D_1, D_2 are subsets of D such that*

$$c^{i_*+1}d(D) < d(D_1) \le c^{i_*}d(D);\ c^{i_*+1}d(D) < d(D_2) \le c^{i_*}d(D);\ \ \frac{h(d(D_1))}{\alpha_{i_*}(D_1)} < \frac{h(d(D_2))}{\alpha_{i_*}(D_2)}.$$

(a) There exists an efficient D_1-substitution of D_2.

(b) There exists no efficient D_2-substitution of D_1.

PROOF. (a) Using Theorem 3 we obtain

$$\sigma(D_2; D_1) = \alpha_{i_*}(D_2)\frac{h(d(D_1))}{\alpha_{i_*}(D_1)} < \alpha_{i_*}(D_2)\frac{h(d(D_2))}{\alpha_{i_*}(D_2)} = h(d(D_2)).$$

(b) As in the proof of (a), it is easy to see

$$\sigma(D_1; D_2) = \alpha_{i_*}(D_1)\frac{h(d(D_2))}{\alpha_{i_*}(D_2)} > \alpha_{i_*}(D_1)\frac{h(d(D_1))}{\alpha_{i_*}(D_1)} = h(d(D_1))$$

by using Theorem 3 again. □

Lemma 6. *Let τ be the index of the self-similar set under $S_1, \ldots, S_m$ with common ratio c. Let $h \in \mathcal{H}_m^c$. Suppose $i\,(\ge \tau)$ is a given positive integer such that (2) holds for all $0 < t \le c^{i-\tau}d(D)$. Then we have $m^i\Phi_i(D) = m^{i+1}\Phi_{i+1}(D)$.*

Proof. By Lemma 1 (a) and (b) we obtain, for any $j \in \{1, \ldots, m\}$,

$$\begin{aligned}\Phi_i(D) &= \inf\left\{\frac{h(d(I))}{\alpha_i(I)} : I \subset D; c^{i+1}d(D) < d(I) \le c^i d(D)\right\}\\ &= m \cdot \inf\left\{\frac{h(d(S_j(I)))}{\alpha_{i+1}(S_j(I))} : I \subset D;\ c^{i+1}d(D) < d(I) \le c^i d(D)\right\}\\ &\ge m \cdot \inf\left\{\frac{h(d(J))}{\alpha_{i+1}(J)} : J \subset D;\ c^{i+2}d(D) < d(J) \le c^{i+1}d(D)\right\}\\ &= m\Phi_{i+1}(D),\end{aligned} \tag{4}$$

where the second equality holds because, by (2) and our hypothesis for i, we have

$$h(d(S_j(I))) = (1/m)h(d(I)) \quad \text{and} \quad \alpha_{i+1}(S_j(I)) = \alpha_i(I).$$

On the other hand, let I be a subset of D with $c^{i+2}d(D) < d(I) \le c^{i+1}d(D)$.

Case I. Assume that there exists a finite sequence $a = (a_1, \ldots, a_{i+1-\tau}) \in \mathcal{I}_{i+1-\tau}$ such that $I \subset S_a(D)$. Then we may choose a subset I' of $S_{a_2} \circ \cdots \circ S_{a_{i+1-\tau}}(D)$ such that $I = S_{a_1}(I')$. Hence, using the properties of S_j and Lemma 1 (b) we have $d(I) = cd(I')$ and $\alpha_{i+1}(I) = \alpha_i(I')$ and so, using the properties of h

$$\frac{h(d(I))}{\alpha_{i+1}(I)} = \frac{1}{m}\frac{h(d(I'))}{\alpha_i(I')}. \tag{5}$$

Case II. Now assume that every $S_a(D)$, $a \in \mathcal{I}_{i+1-\tau}$, does not include I. In view of (3) we have

$$\min\{\rho(S_b(D), S_{b'}(D)) : b, b' \in \mathcal{I}_{i+1-\tau}; b \ne b'\} \ge c^{i+1}d(D) \ge d(I).$$

Therefore, we may merely consider the case where there exists a unique $a \in \mathcal{I}_{i+1-\tau}$ such that the interior of the intersection of I and $S_a(D)$ is non-empty. Due to the properties of the self-similar sets there exists a set $I_1 \subset \mathbb{R}^n$ similar to I with $\alpha_i(I_1) = \alpha_i(I_1 \cap D) = \alpha_{i+1}(I)$ and $cd(I_1) = d(I)$. Thus, we may choose a subset I' of D including $I_1 \cap D$ and satisfying $\alpha_i(I') \ge \alpha_{i+1}(I)$ and $cd(I') = d(I)$. Since

$$d(I_1) = c^{-1}d(I) \le c^i d(D) < d(D),$$

we may select an appropriate subset I_2 of D such that $d((I_1 \cap D) \cup I_2) = d(I_1)$. If we put $I' = (I_1 \cap D) \cup I_2$, then I' satisfies the desired properties. Hence, using the properties of h

$$\frac{h(d(I))}{\alpha_{i+1}(I)} \ge \frac{1}{m}\frac{h(d(I'))}{\alpha_i(I')}. \tag{6}$$

Finally, by (5) and (6)

$$\begin{aligned}\Phi_{i+1}(D) &= \inf\left\{\frac{h(d(I))}{\alpha_{i+1}(I)} : I \subset D;\ c^{i+2}d(D) < d(I) \le c^{i+1}d(D)\right\}\\ &\ge \frac{1}{m}\cdot\inf\left\{\frac{h(d(I'))}{\alpha_i(I')} : I' \subset D;\ c^{i+1}d(D) < d(I') \le c^{i}d(D)\right\}\\ &= \frac{1}{m}\cdot\Phi_i(D).\end{aligned}$$

The assertion of the lemma follows from (4) and the last inequality. □

Lemma 7. *Let F be the self-similar set under $S_1, \ldots, S_m$ with common ratio c and let τ be the index of F. Let $h \in \mathcal{H}_m^c$. Suppose $i\,(> 2\tau)$ is a given integer such that* (2) *holds for all $0 < t \le c^{i-2\tau}d(D)$. Then for any $a \in \mathcal{I}_{i-\tau}$*

$$\Phi_{i-\tau}(D) = \inf\left\{\sum_j h(d(U_j)) : \{U_j\} \text{ is a } c^i d(D)\text{-covering of } F \cap S_a(D)\right\}.$$

PROOF. Let $\varepsilon > 0$ be arbitrarily small, and let $\mathcal{U}_i$ be the set of all $c^i d(D)$-coverings of $F \cap S_a(D)$. By Definition 4, we can choose a subset D' of D such that

$$c^{i-\tau+1}d(D) < d(D') \le c^{i-\tau}d(D)$$

and

$$\Phi_{i-\tau}(D) \le \frac{h(d(D'))}{\alpha_{i-\tau}(D')} \le \Phi_{i-\tau}(D) + \varepsilon. \tag{7}$$

Let $\mathcal{U}_i^\varepsilon$ be the set of all $c^i d(D)$-coverings of $F \cap S_a(D)$ consisting of $\{U_j\}$ and $\{V_j\}$ with the properties

(i) if $c^{k+1}d(D) < d(U_j) \le c^k d(D)$, then there exists some $b \in \mathcal{I}_{k-i+\tau}$ such that
$$\frac{h(d(U_j))}{\alpha_k(U_j)} \le \frac{h(d(S_b(D')))}{\alpha_k(S_b(D'))}; \tag{8}$$

(ii) $\{V_j\}$ satisfies
$$\sum_j h(d(V_j)) < \varepsilon;\ \ \sum_j \alpha_{i-\tau}(V_j) < \varepsilon. \tag{9}$$

Now assume $\{B_j\} \in \mathcal{U}_i$ such that there exists a j_0 satisfying

$$\frac{h(d(B_{j_0}))}{\alpha_k(B_{j_0})} > \frac{h(d(S_b(D')))}{\alpha_k(S_b(D'))}$$

for some $b \in \mathcal{I}_{k-i+\tau}$ with $c^{k+1}d(D) < d(B_{j_0}) \le c^k d(D)$. Then by taking $i_* = k$ in Lemma 5 (a), there exists an efficient $S_b(D')$-substitution $\{B'_j\}$ of $B_{j_0} \cap D$. In particular, we may choose a $c^i d(D)$-covering $\{E_j\}$ of $(F \cap S_a(D) \cap B_{j_0}) \setminus \bigcup B'_j$ such that $\sum h(d(E_j)) < \varepsilon/2^{j_0}$ and $\sum \alpha_{i-\tau}(E_j) < \varepsilon/2^{j_0}$ (cf. Remark 1). We now replace the B_{j_0} in the covering $\{B_j\}$ by $\{B'_j\} \cup \{E_j\}$.

As described above, we may ultimately transform the given covering $\{B_j\} \in \mathcal{U}_i$ into some $\{U_j\} \cup \{V_j\} \in \mathcal{U}_i^\varepsilon$. Therefore, we have

$$\begin{aligned}
\inf\Big\{\sum_j h(d(U_j)) : \{U_j\} \cup \{V_j\} \in \mathcal{U}_i^\varepsilon\Big\} \\
\le \inf\Big\{\sum_j h(d(U_j)) : \{U_j\} \in \mathcal{U}_i\Big\} \\
\le \inf\Big\{\sum_j h(d(U_j)) + \sum_j h(d(V_j)) : \{U_j\} \cup \{V_j\} \in \mathcal{U}_i^\varepsilon\Big\}.
\end{aligned} \tag{10}$$

The first inequality in (10) follows from the above consideration, and the fact $\mathcal{U}_i^\varepsilon \subset \mathcal{U}_i$ implies the second inequality in (10).

Now, let $\{U_j\} \cup \{V_j\} \in \mathcal{U}_i^\varepsilon$ be given such that there exists a sequence (k_j), $k_j \ge i$, satisfying $c^{k_j+1}d(D) < d(U_j) \le c^{k_j}d(D)$. By (8), Lemma 1 (a) and (7) in order, we have for some $b \in \mathcal{I}_{k_j-i+\tau}$

$$\begin{aligned}
h(d(U_j)) &\le \alpha_{k_j}(U_j)\frac{h(d(S_b(D')))}{\alpha_{k_j}(S_b(D'))} \\
&= \alpha_{i-\tau}(U_j)\frac{h(d(S_b(D')))}{\alpha_{i-\tau}(S_b(D'))} \\
&= \alpha_{i-\tau}(U_j)\frac{h(d(D'))}{\alpha_{i-\tau}(D')} \\
&\le \alpha_{i-\tau}(U_j)(\Phi_{i-\tau}(D) + \varepsilon).
\end{aligned}$$

Since $\inf\{\sum \alpha_{i-\tau}(U_j) : \{U_j\} \cup \{V_j\} \in \mathcal{U}_i^\varepsilon\} \le 1$, condition (9), together with the above inequality, implies

$$\begin{aligned}
&\inf\Big\{\sum_j h(d(U_j)) + \sum_j h(d(V_j)) : \{U_j\} \cup \{V_j\} \in \mathcal{U}_i^\varepsilon\Big\} \\
&\le \inf\Big\{\sum_j \alpha_{i-\tau}(U_j) : \{U_j\} \cup \{V_j\} \in \mathcal{U}_i^\varepsilon\Big\} \cdot \big(\Phi_{i-\tau}(D) + \varepsilon\big) + \varepsilon \\
&\le \Phi_{i-\tau}(D) + 2\varepsilon.
\end{aligned}$$

Since $\varepsilon > 0$ may be sufficiently small, we obtain

$$\inf\Big\{\sum_j h(d(U_j)) + \sum_j h(d(V_j)) : \{U_j\} \cup \{V_j\} \in \mathcal{U}_i^\varepsilon\Big\} \le \Phi_{i-\tau}(D). \tag{11}$$

On the other hand, if $\{U_j\} \cup \{V_j\}$ belongs to $\mathcal{U}_i^\varepsilon$, then for some k with $c^{k+1}d(D) < d(U_j) \le c^k d(D)$

$$\frac{h(d(U_j))}{\alpha_k(U_j)} \ge \Phi_k(D). \tag{12}$$

Thus, it follows from (12), Lemma 1 (a) and Lemma 6 that

$$h(d(U_j)) \ge \alpha_k(U_j)\Phi_k(D) = \alpha_{i-\tau}(U_j)\Phi_{i-\tau}(D).$$

Hence, as $\sum \alpha_{i-\tau}(U_j) \ge 1 - \varepsilon$ by (9), we get

$$\sum_j h(d(U_j)) \ge \Phi_{i-\tau}(D) \sum_j \alpha_{i-\tau}(U_j) \ge \Phi_{i-\tau}(D)(1-\varepsilon)$$

and so

$$\inf\Big\{\sum_j h(d(U_j)) : \{U_j\} \cup \{V_j\} \in \mathcal{U}_i^\varepsilon\Big\} \ge \Phi_{i-\tau}(D). \tag{13}$$

The assertion of the lemma follows from (10), (11) and (13). □

Now, we shall prove the main result of this paper.

Theorem 8. *Suppose F is the self-similar set under $S_1, \ldots, S_m$ with common ratio c and let τ be the index of F. Let $h \in \mathcal{H}_m^c$. Suppose $i\,(\ge \tau)$ is a positive integer such that* (2) *holds for all $0 < t \le c^{i-\tau}d(D)$. Then for any $a \in \mathcal{I}_{i-\tau}$ $\mu^h(F) = m^i\Phi_i(S_a(D))$.*

PROOF. Let j $(> 2\tau)$ be a sufficiently large integer for which (2) holds for all $0 < t \le c^{j-2\tau}d(D)$. By (3), we get

$$\min\{\rho(S_b(D), S_{b'}(D)) : b, b' \in \mathcal{I}_{j-\tau};\, b \ne b'\} \ge c^j d(D).$$

Hence, by Lemma 7, we have

$$\begin{aligned}
&\inf\Big\{\sum_k h(d(U_k)) : \{U_k\} \text{ is a } c^j d(D)\text{-covering of } F\Big\} \\
&\quad = \sum_{b\in\mathcal{I}_{j-\tau}} \inf\Big\{\sum_k h(d(U_k)) : \{U_k\} \text{ is a } c^j d(D)\text{-covering of } F \cap S_b(D)\Big\} \\
&\quad = m^{j-\tau}\Phi_{j-\tau}(D).
\end{aligned}$$

Thus, it follows from Lemma 6 and Lemma 4 that

$$\begin{aligned}\mu^h(F) &= \lim_{j\to\infty} \inf\left\{\sum_k h(d(U_k)) : \{U_k\} \text{ is a } c^j d(D)\text{-covering of } F\right\}\\ &= \lim_{j\to\infty} m^{j-\tau}\Phi_{j-\tau}(D)\\ &= m^i\Phi_i(D)\\ &= m^i\Phi_i(S_a(D)).\end{aligned}$$ □

4 Applications

First, we introduce the definition of the uniform Cantor set. Let m (≥ 2) be a given integer, and we choose positive real numbers c and d such that $mc + (m-1)d = 1$. In this section, let $D = [0,1]$ and define the similarities $S_i : D \to D$ $(i = 1, \dots, m)$ by $S_i(x) = (i-1)(c+d) + cx$. Then c is the common ratio of the similarities $S_1, \dots, S_m$ and the family of those similarities is disjoint. The self-similar set F under the S_i's is called a uniform Cantor set.

Let C be a compact subset of $\mathbb{R}$ and let $t > 0$ be given. The t-entropy of C is defined by

$$E(C,t) = \min\{n \in \mathbb{N} : \{U_1, \dots, U_n\} \text{ is a } t\text{-covering of } C\}.$$

Mycielski [8] and Kahnert [7] have considered the Hausdorff function

$$h_C(t) = \begin{cases} 1/E(C,t) & \text{for } t > 0,\\ 0 & \text{for } t = 0\end{cases}$$

to construct an invariant Hausdorff measure μ^{h_C}. From now on, we write μ^C instead of μ^{h_C}. For some interesting properties of such a measure. See [3, 4, 8, 9].

It is easy to see $\mu^D(F) = 0$. Obviously, the Cantor set F can be covered by m^n intervals of the length c^n for every positive integer n. Since $h_D(c^n) \leq c^n$, (1) implies $\mu^D(F) \leq \lim_{n\to\infty} m^n c^n = 0$.

Analogously, we can easily prove $\mu^F(D) = \infty$: Let $\{U_i\}$ be a c^n-covering of D. Then $\sum d(U_i) \geq 1$. Let n_i $(\geq n)$ be an integer with $c^{n_i+1} < d(U_i) \leq c^{n_i}$ for every i. Then

$$h_F(d(U_i)) \geq m^{-(n_i+1)} = c^{n_i} m^{-1} (mc)^{-n_i} > d(U_i) m^{-1} (mc)^{-n}$$

and hence

$$\sum_i h_F(d(U_i)) \geq m^{-1}(mc)^{-n} \sum_i d(U_i) \geq m^{-1}(mc)^{-n}.$$

Therefore, (1) implies $\mu^F(D) \geq \lim_{n\to\infty} m^{-1}(mc)^{-n} = \infty$.

Similarly, we can show $\mu^D(D) = 1$ and $\mu^F(F) \leq 1$, but it is virtually impossible to determine the exact value of $\mu^F(F)$ by using formula (1). However, we can use Theorem 8 to evaluate the lower and upper bounds for the set $\{\mu^F(F) : F \text{ is a uniform Cantor set}\}$ as well as the exact values of $\mu^F(F)$ for many special cases. Moreover, Theorem 8 might provide us with a possibility to evaluate the value of $\mu^F(F)$ within a given error.

Remark 2. Let C be a compact subset of $\mathbb{R}$ with a positive Lebesgue measure. Then we can also prove that $\mu^C(C) = 1$.

In the following lemma, we prove $h_F \in \mathcal{H}_m^c$.

Lemma 9. $h_F \in \mathcal{H}_m^c$.

PROOF. Let $n = E(F,t)$. Suppose $\{U_i\}_{i=1,\ldots,n}$ is a t-covering of F. Then $\{S_j(U_i)\}_{i=1,\ldots,n}$ is a ct-covering of $F \cap S_j(D)$ for any $j = 1,\ldots,m$. Hence, we obtain $E(F,ct) \leq mE(F,t)$.

On the other hand, let $n = E(F \cap S_j(D), ct)$ and suppose $\{U_i\}_{i=1,\ldots,n}$ is a ct-covering of $F \cap S_j(D)$ for some $j = 1,\ldots,m$. Then $\{S_j^{-1}(U_i)\}_{i=1,\ldots,n}$ is a t-covering of F. Hence, if $0 < t < d/c$, then since $ct < d$

$$E(F,t) \leq E(F \cap S_j(D), ct) = \frac{1}{m}E(F,ct). \qquad \square$$

Let τ be the index of F. According to Theorem 8 and Lemma 9, the value of $\mu^F(F)$ can be evaluated by the formula

$$\mu^F(F) = m^i \cdot \inf\left\{\frac{h_F(d(I))}{\alpha_i(I)} : I \subset [0, c^{i-\tau}];\ c^{i+1} < d(I) \leq c^i\right\} \tag{14}$$

where i $(> \tau)$ is a sufficiently large integer.

Theorem 10. *For any uniform Cantor set F $1/2 \leq \mu^F(F) < 1$.*

PROOF. (a) First, assume that I $(\subset [0, c^{i-\tau}])$ is an interval with

$$(k+1)c^{i+1} + kc^i d < d(I) \leq (k+1)(c^{i+1} + c^i d)$$

for some $k \in \{0, 1, \ldots, m-2\}$. By considering the structure of F and the fact $E(S_a(D), d(I)) \leq [(m+k)/(k+1)]$ $(a \in \mathcal{I}_i)$, where $[x]$ denotes the greatest integer which does not exceed x, we conclude

$$E(F, d(I)) \leq m^i \left[\frac{m+k}{k+1}\right] \quad \text{and} \quad \alpha_i(I) \leq \frac{k+1}{m}.$$

Hence, the fact $m \leq (k+1)[(m+k)/(k+1)] \leq m+k \leq 2m-2$ implies

$$m^i \frac{h_F(d(I))}{\alpha_i(I)} \geq \frac{m}{(k+1)[(m+k)/(k+1)]} > \frac{1}{2}. \tag{15}$$

Now, assume that I $(\subset [0, c^{i-\tau}])$ is an interval with

$$k(c^{i+1} + c^i d) < d(I) \leq (k+1)c^{i+1} + kc^i d$$

for some $k \in \{1, 2, \ldots, m-1\}$. As in the previous case, we obtain

$$E(F, d(I)) \leq m^i \left[\frac{m+k-1}{k}\right], \quad \alpha_i(I) \leq \frac{k+1}{m},$$

and hence the inequality

$$m^i \frac{h_F(d(I))}{\alpha_i(I)} \geq \frac{m}{(k+1)[(m+k-1)/k]} \geq \frac{1}{2} \tag{16}$$

follows from the fact $\frac{m+k-1}{k} \leq \frac{2m}{k+1}$ which can be easily proved under the assumption $m \geq k+1$.

Altogether, Theorem 8, together with (15) and (16), implies $\mu^F(F) \geq 1/2$.

(b) Consider an interval $I = [0, c^i - c^{2i}]$. By (3) we have

$$\min\{\rho(S_b(D), S_{b'}(D)) : b, b' \in \mathcal{I}_{i-\tau};\ b \neq b'\} \geq c^i.$$

Therefore, if $a \in \mathcal{I}_{i-\tau}$, then

$$E(F, d(I)) = m^{i-\tau} E(F \cap S_a(D), d(I)) \geq m^{i-\tau}\,(m^\tau + 1) \tag{17}$$

and

$$\alpha_i(I) = \frac{m^i - 1}{m^i}. \tag{18}$$

Hence, by (14), (17) and (18), we obtain

$$\mu^F(F) \leq m^i \frac{h_F(d(I))}{\alpha_i(I)} \leq \frac{m^\tau}{m^\tau + 1} \frac{m^i}{m^i - 1} < 1$$

because i can be arbitrarily large. □

As we can see in the following theorem, $1/2$ is the best possible estimation of the lower bound for $\mu^F(F)$ in Theorem 10.

Theorem 11. *Let τ be the index of the uniform Cantor set F.*

(a) If $\tau = 0$, then $\mu^F(F) = 1/2$.

(b) $\mu^F(F) \to 1$ as $\tau \to \infty$.

PROOF. (a) Let i, k be positive integers. Choose an interval $I = [0, c^i - c^{i+k}]$. Clearly, we obtain $h_F(d(I)) = \frac{1}{2m^i}$, $\alpha_i(I) = 1 - \frac{1}{m^k}$ and hence

$$m^i \frac{h_F(d(I))}{\alpha_i(I)} = \frac{1}{2} \frac{m^k}{m^k - 1}. \tag{19}$$

By letting $k \to \infty$ in (19) and considering (14) and Theorem 10, we conclude $\mu^F(F) = 1/2$.

(b) Let τ and i ($> \tau$) be sufficiently large integers. Suppose I ($\subset [0, c^{i-\tau}]$) is an interval for which there exists an $n \in \mathbb{N} \cup \{\infty\}$ such that $\ell_n(k_j) - c^{i+n-1}d \le d(I) \le \ell_n(k_j)$ with

$$\ell_n(k_j) = \sum_{j=1}^{\infty} k_j \left(c^{i+j} + c^{i+j-1}d\right)$$

where $k_j \in \{0, 1, \ldots, m-1\}$, $k_1 k_n \neq 0$ and $k_j = 0$ for any $j > n$ (for the case of $n \in \mathbb{N}$) and we follow the convention $c^{i+\infty-1} = 0$ (for $n = \infty$). In view of (3) we have

$$E(F, d(I)) \le m^{i-\tau} \left(\frac{c^{i-\tau}}{\ell_n(k_j) - c^{i+n-1}d} + 1 \right) \le m^{i-\tau} \frac{c^{-\tau} + 1}{k_1 c + k_2 c^2 + \cdots},$$

$$\alpha_i(I) \le \frac{k_1}{m} + \frac{k_2}{m^2} + \cdots$$

and hence

$$\begin{aligned} m^i \frac{h_F(d(I))}{\alpha_i(I)} &\ge \frac{(mc)^\tau (k_1 c + k_2 c^2 + \cdots)}{1 + c^\tau} \left(\frac{k_1}{m} + \frac{k_2}{m^2} + \cdots \right)^{-1} \\ &\ge \frac{(mc)^\tau}{1 + c^\tau} g(\tau) \end{aligned}$$

where $g(\tau) \to 1$ as $\tau \to \infty$ ($c \to 1/m$ as $\tau \to \infty$). By (14) we get

$$\mu^F(F) \ge \frac{(mc)^\tau}{1 + c^\tau} g(\tau). \tag{20}$$

From $mc + (m-1)d = 1$ and $c^{\tau+1} \le d < c^\tau$ (see Definition 5) it follows that $(mc)^\tau \to 1$ as $\tau \to \infty$. Consequently, by Theorem 10 and (20), we conclude that $\mu^F(F) \to 1$ as $\tau \to \infty$. □

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