Real Analysis Exchange
Vol. 24(2), 1998/9, pp. 837–842

Togo Nishiura, Department of Mathematics, Wayne State University, MI 48202, Department of Mathematics and Computer Science, Dickinson College, Carlisle, PA 17013; e-mail: nishiura@dickinson.edu

MEASURE-PRESERVING MAPS OF $\mathbb{R}^n$

Abstract

An elementary proof is given of the existence of a measure-preserving bijection of $\mathbb{R}^n$ that maps a preassigned Borel set with Lebesgue measure 1 onto the unit cube. The proof requires the use of only the Vitali Covering Theorem, translations and elementary properties of infinite sets.

This short note provides an elementary, real analytic proof of the existence stated in the abstract. The statement is part of the folklore surrounding measure-preserving bijections of $\mathbb{R}^n$. It can be inferred from very general theorems on the isomorphisms of σ-algebras in complete separable metric spaces. (See, for example, Chapter 15 of [3].) The present proof requires only the use of the usual Vitali Covering Theorem. Lebesgue measure on $\mathbb{R}^n$ will be denoted by λ.

Theorem 1. *Let $Q = [0,1]^n$ and let B be a Borel set in $\mathbb{R}^n$ with $\lambda(B) = 1$. Then there exists a measure-preserving bijection $H\colon \mathbb{R}^n \to \mathbb{R}^n$ such that*

$$\lambda(Q \setminus H[B]) = 0$$

and H and H^{-1} are Borel measurable. Moreover, H may be chosen so that it is the identity map on $\mathbb{R}^n \setminus (Q \cup B)$.

Sharper theorems will be discussed at the end of the note (see Theorems 4 and 6 below).

The next lemma is key to the proof of Theorem 1.

Lemma 2. *Let $\varepsilon > 0$ and $Q = [0,1]^n$. If E and F are Borel sets in $\mathbb{R}^n$ such that $E \subset Q$, $Q \cap F = \emptyset$ and $\lambda(E) = \lambda(F)$, then there are Borel sets U and V and there is a measure-preserving bijection*

$$H\colon U \cup V \to U \cup V$$

Key Words: measure-preserving maps, Borel measurable sets, Vitali Covering Theorem
Mathematical Reviews subject classification: 28A05, 28D05, 28E99
Received by the editors June 1, 1998

such that

$$U \subset E, \quad V \subset F, \quad H[U] = V, \quad H[V] = U, \text{ and } \quad \lambda(F \setminus V) < \varepsilon.$$

Moreover, H and H^{-1} may be assumed to be Borel measurable.

PROOF. As $U = V = \emptyset$ easily disposes the case $\lambda(F) < \varepsilon$, the contrary case will be assumed. The proof will use Vitali coverings consisting of cubes whose side-lengths are 2^{-k}, $k = 1, 2, \ldots$, and that do not intersect the boundary of Q. (The interior of a set A will be denoted by A^{o}.)

First, consider the Vitali covering of the set $Q^{\mathrm{o}} \cap E$ formed by the cubes described above. Since $\lambda(E) \leq 1$, the Vitali Covering Theorem yields finitely many disjoint cubes I_i, $i = 1, 2, \ldots, n_E$, contained in Q^{o} with the properties

$$\lambda\big(E \setminus \textstyle\bigcup_{i=1}^{n_E} I_i\big) < \varepsilon/4 \quad \text{and} \quad \lambda\big(\textstyle\bigcup_{i=1}^{n_E} I_i \setminus E\big) < \varepsilon/4.$$

Second, consider the Vitali covering of the set F formed by the cubes described above. As $\lambda(F) = \lambda(E)$, the Vitali Covering Theorem yields finitely many disjoint cubes J_j, $j = 1, 2, \ldots, n_F$, contained in $\mathbb{R}^n \setminus Q$ with the properties

$$\lambda\big(F \setminus \textstyle\bigcup_{j=1}^{n_F} J_j\big) < \varepsilon/4 \quad \text{and} \quad \lambda\big(\textstyle\bigcup_{j=1}^{n_F} J_j \setminus F\big) < \varepsilon/4.$$

Let k_0 be a positive integer such that 2^{-k_0} is less than the side-lengths of the cubes I_i, $i = 1, 2, \ldots, n_E$, and the cubes J_j, $j = 1, 2, \ldots, n_F$. Now subdivide the cubes I_i, $i = 1, 2, \ldots, n_E$, into nonoverlapping cubes K_k, $k = 1, 2, \ldots, N_E$, with side-lengths *exactly* equal to 2^{-k_0}, and also subdivide the cubes J_j, $j = 1, 2, \ldots, n_F$, into nonoverlapping cubes L_l, $l = 1, 2, \ldots, N_F$, with side-lengths *exactly* equal to 2^{-k_0}. Let $N = \min\{ N_E, N_F \}$ and let

$$(K_p, L_p), \quad p = 1, 2, \ldots, N,$$

be pairings of these newly formed cubes. Clearly there may be some cubes K_k or L_l that are not paired. Let us show

$$\lambda\big(\textstyle\bigcup_{k>N} K_k\big) < \varepsilon/2.$$

If $N = N_E$, then $\bigcup_{k>N} K_k = \emptyset$. So assume $N < N_E$. Then

$$\begin{aligned}
\lambda\big(\textstyle\bigcup_{k>N} K_k\big) &= \lambda\big(\textstyle\bigcup_{k=1}^{N_E} K_k\big) - \lambda\big(\textstyle\bigcup_{l=1}^{N_F} L_l\big) \\
&= \lambda\big(\textstyle\bigcup_{i=1}^{n_E} I_i\big) - \lambda\big(\textstyle\bigcup_{j=1}^{n_F} J_j\big) \\
&< \big(\lambda(E) + \varepsilon/4\big) - \big(\lambda(F) - \varepsilon/4\big) \\
&= \varepsilon/2
\end{aligned}$$

because $\lambda(E) = \lambda(F)$.

Next let (K_p, L_p) be any of the paired cubes. Denote by T_p the translation of K_p onto L_p. Define

$$E_p = ({K_p}^{\mathrm{o}} \cap E) \cap {T_p}^{-1}[{L_p}^{\mathrm{o}} \cap F] \quad \text{and} \quad F_p = T_p[{K_p}^{\mathrm{o}} \cap E] \cap ({L_p}^{\mathrm{o}} \cap F).$$

Clearly, as T_p is one-to-one,

$$T_p[E_p] = F_p \quad \text{and} \quad {T_p}^{-1}[F_p] = E_p,$$

and, as T_p is measure-preserving,

$$\lambda(E_p) \geq \lambda({K_p}^{\mathrm{o}} \cap E) - \lambda({L_p}^{\mathrm{o}} \setminus F).$$

Finally define

$$U = \textstyle\bigcup_{p=1}^{N} E_p \quad \text{and} \quad V = \bigcup_{p=1}^{N} F_p,$$

and define $H\colon U \cup V \to U \cup V$ by

$$H(x) = \begin{cases} T_p(x) & \text{if} \quad x \in E_p,\ p = 1, 2, \ldots, N, \\ {T_p}^{-1}(x) & \text{if} \quad x \in F_p,\ p = 1, 2, \ldots, N. \end{cases}$$

Only $\lambda(F \setminus V) < \varepsilon$ remains to be verified. To this end we have

$$\begin{aligned} \lambda(F \setminus V) &= \lambda\big(F \setminus \textstyle\bigcup_{p=1}^{N} F_p\big) \\ &= \lambda(F) - \textstyle\sum_{p=1}^{N} \lambda(F_p) \\ &= \lambda(E) - \textstyle\sum_{p=1}^{N} \lambda(E_p) \\ &\leq \lambda(E) - \textstyle\sum_{p=1}^{N} \lambda(K_p \cap E) + \sum_{p=1}^{N} \lambda(L_p \setminus F) \\ &\leq \lambda(E) - \textstyle\sum_{p=1}^{N_E} \lambda(K_p \cap E) + \sum_{p>N} \lambda(K_p) + \sum_{p=1}^{N_F} \lambda(L_p \setminus F) \\ &< \lambda\big(E \setminus \textstyle\bigcup_{i=1}^{n_E} I_i\big) + \varepsilon/2 + \lambda\big(\bigcup_{j=1}^{n_F} J_j \setminus F\big) \\ &< \varepsilon/4 + \varepsilon/2 + \varepsilon/4 \\ &= \varepsilon. \end{aligned}$$

The lemma is proved. Observe that H is Borel measurable on the Borel set $U \cup V$ and that H^2 is the identity map, whence H^{-1} is Borel measurable. $\square$

Proof of Theorem 1. We repeatedly apply the lemma.

Let $E_1 = Q \setminus B$ and $F_1 = B \setminus Q$ and $\varepsilon_1 = 1/2$ in the lemma. We get sets U_1 and V_1 and a map $H_1\colon U_1 \cup V_1 \to U_1 \cup V_1$ such that H_1 is measure-preserving and $\lambda(F_1 \setminus V_1) < 1/2$, where $U_1 \subset E_1$, $V_1 \subset F_1$, $H_1[U_1] = V_1$ and $H_1[V_1] = U_1$.

Continuing inductively, we have sets E_k, F_k, U_k and V_k and measure-preserving maps $H_k\colon U_k \cup V_k \to U_k \cup V_k$ such that $\lambda(F_k \setminus V_k) < 2^{-k}$, where

$$U_k \subset E_k, \quad V_k \subset F_k,$$
$$E_{k+1} = E_k \setminus U_k = E_1 \setminus \bigcup_{j=1}^{k} U_j, \quad F_{k+1} = F_k \setminus V_k = F_1 \setminus \bigcup_{j=1}^{k} V_j,$$

and

$$H_k[U_k] = V_k, \quad H_k[V_k] = U_k.$$

Let

$$E_\infty = E_1 \setminus \bigcup_{j=1}^{\infty} U_j \quad \text{and} \quad F_\infty = F_1 \setminus \bigcup_{j=1}^{\infty} V_j.$$

Then $\lambda(E_\infty) = 0$ and $\lambda(F_\infty) = 0$ hold. We define H in the obvious way. That is, for each k, $H(x) = H_k(x)$ if $x \in U_k \cup V_k$, and $H(x) = x$ otherwise. Clearly H is a measure-preserving bijection of $\mathbb{R}^n$ onto $\mathbb{R}^n$. Finally,

$$\begin{aligned} H[B] &= H\big[(Q \cap B) \cup F_\infty \cup \textstyle\bigcup_{k=1}^{\infty} V_k\big] \\ &= (Q \cap B) \cup F_\infty \cup \textstyle\bigcup_{k=1}^{\infty} U_k \\ &= (Q \cap B) \cup F_\infty \cup \big((Q \setminus B) \setminus E_\infty\big) \\ &= (Q \setminus E_\infty) \cup F_\infty, \end{aligned}$$

whence $\lambda(Q \setminus H[B]) = 0$, and the theorem is proved. □

For a simple application let us turn to the assertion stated in the abstract.

Theorem 3. *Let $Q = [0,1]^n$ and let B be a Borel set in $\mathbb{R}^n$ with $\lambda(B) = 1$. Then there is a measure-preserving bijection $H\colon \mathbb{R}^n \to \mathbb{R}^n$ such that $H[B] = Q$. Moreover, H may be assumed to be the identity map on the complement of $B \cup Q \cup Z$, where Z is a set of Lebesgue measure 0.*

PROOF. Let H be given by Theorem 1. Define

$$A = Q \setminus H[B] \quad \text{and} \quad C = B \setminus H^{-1}[Q].$$

Let Z_{A0}, Z_{A1}, Z_{C0} and Z_{C1} be mutually disjoint, nonempty perfect sets such that

$$Z_{A0} \cup Z_{A1} \subset Q \cap H[B] \quad \text{and} \quad (Z_{C0} \cup Z_{C1}) \cap (Q \cup B) = \emptyset.$$

Note that the sets $Z_{A0} \cup A$, $A \cup Z_{C0}$, $Z_{C1} \cup C$ and $C \cup Z_{A1}$ are Borel sets with cardinality $\mathfrak{c} = 2^{\aleph_0}$. Using the cardinality only, we select one-to-one correspondences

$$h_1\colon Z_{A0} \to Z_{A0} \cup A, \quad h_2\colon A \cup Z_{C0} \to Z_{C0},$$
$$h_3\colon Z_{C1} \to Z_{C1} \cup C, \quad h_4\colon C \cup Z_{A1} \to Z_{A1}.$$

These maps will define a one-to-one map of $(Z_{A0} \cup A \cup Z_{C0}) \cup (Z_{C1} \cup C \cup Z_{A1})$ onto itself. Let h be the extension of this map to all of $\mathbb{R}^n$ by means of the identity on the complement of this set. The map $h \circ H$ satisfies $Q = h \circ H[B]$. All that remains is to require $\lambda(Z_{A0}) = \lambda(Z_{A1}) = \lambda(Z_{C0}) = \lambda(Z_{C1}) = 0$ to complete the proof. $\square$

Theorem 3 can be sharpened to include the Borel measurability of the bijections H and H^{-1}. We give here a straightforward proof that relies on an elementary topological property due to Menger (Theorem 5 below).

Theorem 4. *Let $Q = [0,1]^n$ and let B be a Borel set in $\mathbb{R}^n$ with $\lambda(B) = 1$. Then there is a measure-preserving bijection $H\colon \mathbb{R}^n \to \mathbb{R}^n$ such that $H[B] = Q$ and both H and H^{-1} are Borel measurable. Moreover, H may be assumed to be the identity map on the complement of $B \cup Q \cup Z$, where Z is a Borel set of Lebesgue measure 0.*

The main idea of the proof is found in the proof of Theorem 3. The proof will use the following topological property of 0-dimensional subsets of separable metrizable spaces which may not be found in the "toolbox" of every analyst. (A separable metrizable space is **0-dimensional** if each point has arbitrarily small neighborhoods with empty boundaries.)

Theorem 5 (Menger)**.** *Every 0-dimensional separable metrizable space is topologically embeddable in the Cantor ternary set.*

(See [1, page 26] for Menger's theorem.) Another useful fact is that the Cantor ternary set can be topologically embedded in every nonempty perfect subset of $\mathbb{R}^n$.

Let us turn to the first step of the proof. Observe that every nonempty Borel subset of $\mathbb{R}^n$ can be written as the union of at most countably many Borel sets whose dimensions are 0. This is quite obvious since, for each positive integer k, the subset of $\mathbb{R}^k$ that consists of all points $(x_1, x_2, \ldots, x_k)$ with every coordinate being irrational is a 0-dimensional Borel set (see [2, Example II.6, page 11]). As the Cantor ternary set $\mathcal{C}$ contains countably many (indeed, $2^{\aleph_0}$ many) mutually disjoint topological copies of $\mathcal{C}$, it now follows that, for each nonempty Borel subset Y_0 of $\mathbb{R}^n$, there is an injection φ_0 of Y_0 into $\mathcal{C}$ such that φ_0 and ${\varphi_0}^{-1}$ are Borel measurable. Let us denote the image of this injection by Y_1. We may assume that Y_1 is contained in the interval $[2/3, 1]$. For each positive integer ν define the subset $Y_{\nu+1} = 3^{-1} Y_\nu$ of $\mathcal{C}$. In the set $Y_0 \cup \mathcal{C}$ we form the space $Y = Y_0 \cup Y_1 \cup Y_2 \cup \cdots \cup Y_\nu \cup \cdots$, where Y is given the obvious metric topology. Then the injection $\varphi\colon Y \to Y_1 \cup Y_2 \cup \cdots \cup Y_\nu \cup \cdots$ given by

$$\varphi | Y_0 = \varphi_0 \quad \text{and} \quad \varphi(y) = 3^{-1} y \ \text{ for } y \in Y_\nu,\ \nu = 1, 2, \ldots,$$

is onto and is such that φ and φ^{-1} are Borel measurable. The reader should be able to complete the proof.

Remarks. Using the general results alluded to at the beginning of this note, we can establish the following more general statement.

Theorem 6. *Let X be a complete separable metric space and let μ be a nonatomic, σ-finite, Borel regular measure on X. If A and B are Borel subsets of X with $0 < \mu(A) = \mu(B) < \mu(X)$ then there is a bijection of X onto itself such that $H[B] = A$, H and H^{-1} are Borel measurable, and H is the identity map on the complement of $A \cup B \cup Z$ where Z is a Borel set with $\mu(Z) = 0$.*

Clearly we must deal with the sets $A \setminus B$ and $B \setminus A$, just as in the proof of Theorem 1. When $\mu(A \setminus B) > 0$ holds, the measure μ restricted to $A \setminus B$ is "essentially the same as Lebesgue measure on the interval $[0, \mu(A \setminus B)]$" and analogously for $B \setminus A$. (In passing, we note that the machinery of Borel isomorphisms of uncountable Borel subsets of complete separable metric spaces is used in the proof of the last assertion. For a good reference see [3], in particular, Theorem 16 on page 409.) So, there is a bijection H of $A \cup B$ onto itself so that H and H^{-1} are Borel measurable and $\mu(A \setminus H[B]) = 0$ (that is, the analogue of Theorem 1). The remainder of proof follows the pattern of that for Theorem 4. Here, we will need 2 facts. The first is that every Borel set in X is Borel isomorphic to a Borel subset of the Cantor ternary set $\mathcal{C}$. The second is that $\mathcal{C}$ is topologically embeddable into any uncountable Borel subset of X.

Of course, the point of the first of the proofs given in this note is that the additional information of μ being Lebesgue measure on $\mathbb{R}^n$ permits an elementary, real analytic proof. The author wishes to thank Professors Rae Shortt of Wesleyan University and Bertram Schreiber of Wayne State University for pointing out the reference to the third edition of [3] used here and Professor Daniel Waterman of Syracuse University for bringing the folklore theorem to the author's attention.

References

[1] R. Engelking, *Dimension theory*, Elsevier/North Holland, Amsterdam, 1978.

[2] W. Hurewicz and H. Wallman, *Dimension theory*, Princeton University Press, Princeton, 1948.

[3] H.L. Royden, *Real analysis*, 3rd ed., Macmillan, New York, 1988.