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SIERPIŃSKI–ZYGMUND FUNCTIONS THAT HAVE THE CANTOR INTERMEDIATE VALUE PROPERTY

Abstract

We construct (in ZFC) an example of Sierpiński–Zygmund function having the Cantor intermediate value property and observe that every such function does not have the strong Cantor intermediate value property, which solves the problem of R. Gibson [8, Question 2]. Moreover we prove that both families: $SCIVP$ functions and $CIVP \setminus SCIVP$ functions are $2^{\mathfrak{c}}$ dense in the uniform closure of the class of $CIVP$ functions. We show also that if the real line $\mathbb{R}$ is not a union of less than continuum many its meager subsets, then there exists an almost continuous Sierpiński–Zygmund function having the Cantor intermediate value property. Because such a function does not have the strong Cantor intermediate value property, it is not extendable. This solves another problem of Gibson [8, Question 3].

1 Introduction

Our terminology is standard. We shall consider only real-valued functions of one real variable. No distinction is made between a function and its graph. The family of all functions from a set X into Y will be denoted by Y^X. Symbol $\mathrm{card}\,(X)$ will stand for the cardinality of a set X. The cardinality of

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$\mathbb{R}$ is denoted by $\mathfrak{c}$. If A is a planar set, we denote its x-projection by $\operatorname{dom}(A)$. For $f, g \in \mathbb{R}^{\mathbb{R}}$ the notation $[f = g]$ means the set $\{x \in \mathbb{R}\colon f(x) = g(x)\}$.

Recall also the following definitions.

- $f\colon \mathbb{R} \to \mathbb{R}$ is of *Sierpiński-Zygmund type* (shortly, $f \in SZ$, or f is of S-Z type) if its restriction $f|M$ is discontinuous for each set $M \subset \mathbb{R}$ with $\operatorname{card}(M) = \mathfrak{c}$ [19].
- $f\colon \mathbb{R} \to \mathbb{R}$ *has a perfect road* at $x \in \mathbb{R}$ if there exists a perfect set C such that x is a bilaterally limit point of C and $f|C$ is continuous at x. We say that f is of *perfect road type* (shortly, $f \in PR$) if f has a perfect road at each point [15].
- $f\colon \mathbb{R} \to \mathbb{R}$ is said to be *almost continuous in the sense of Stallings* (shortly, $f \in AC$) if each open subset of the plane containing f contains also a continuous function $g\colon \mathbb{R} \to \mathbb{R}$ [20].
- $F\colon \mathbb{R} \times [0,1] \to \mathbb{R}$ is *connectivity* if the graph of its restriction $F|X$ is connected in $\mathbb{R}^3$ for every connected $X \subset \mathbb{R} \times [0,1]$.
- $f\colon \mathbb{R} \to \mathbb{R}$ is *extendable* (shortly, $f \in Ext$) if there is a connectivity function $F\colon \mathbb{R} \times [0,1] \to \mathbb{R}$ such that $F(x,0) = f(x)$ for every $x \in \mathbb{R}$.
- $f\colon \mathbb{R} \to \mathbb{R}$ has the *Cantor intermediate value property* (CIVP), if for every $x, y \in \mathbb{R}$ and for each Cantor set K between $f(x)$ and $f(y)$ there is a Cantor set C between x and y such that $f(C) \subset K$ [10].
- $f\colon \mathbb{R} \to \mathbb{R}$ has the *strong Cantor intermediate value property* (SCIVP), if for every $x, y \in \mathbb{R}$ and for each Cantor set K between $f(x)$ and $f(y)$ there is a Cantor set C between x and y such that $f(C) \subset K$ and $f|C$ is continuous [17].
- $f\colon \mathbb{R} \to \mathbb{R}$ has the *weak Cantor intermediate value property* (WCIVP), if for every $x, y \in \mathbb{R}$ with $f(x) \neq f(y)$ there exists a Cantor set C between x and y such that $f(C)$ is between $f(x)$ and $f(y)$ [11].
- $f\colon \mathbb{R} \to \mathbb{R}$ is a *peripherally continuous* (shortly, $f \in PC$) if for every $x \in \mathbb{R}$ there are sequences $a_n \nearrow x$ and $b_n \searrow x$ such that $\lim_{n\to\infty} f(a_n) = \lim_{n\to\infty} f(b_n) = f(x)$.

The relationships between those classes were discussed in many papers. (See [8] or the survey [9].) In particular, the following implications hold.

$$\begin{tikzcd} & AC \arrow[rr] & & D \arrow[dr] & \\ Ext \arrow[ur] \arrow[dr] & & & & PC \\ & SCIVP \arrow[r] & CIVP \arrow[r] & PR \arrow[ur] & \end{tikzcd}$$

At Banach Center in Warsaw, in 1989, Jerry Gibson gave a talk in which he posed several problems, in particular:

Problem 1. [8, Question 2] *Does $CIVP \Longrightarrow SCIVP$?*

and

Problem 2. [8, Question 3] *Does $AC + CIVP \Longrightarrow Ext$?*

In this paper we solve both those problems in the negative. To see it, we shall construct an example of an S-Z function with the Cantor intermediate value property. This generalizes the rezult of Darji from [7]. Because no S-Z function has the strong Cantor intermediate value property, $CIVP \setminus SCIVP \neq \emptyset$. Moreover we generalize Theorem 1 from [1] by showing that under the assumption that the real line is not the union of less than continuum many meager sets (which is somewhat weaker than CH or the Martin's Axiom MA [18]) there exists an almost continuous S-Z function having the CIVP. (On the other hand, there is a model of ZFC in which there is no Darboux S-Z function [1]. Thus, the additional set theoretical assumptions are necessary in the result mentioned above.) Again, since $SZ \cap SCIVP = \emptyset$ and $Ext \subset SCIVP$ [17], we obtain $AC + CIVP \neq Ext$.

2 S-Z functions having the CIVP

In our constructions we will use the following easy and well-known lemmas.

Lemma 1. [19, 12] *Suppose $U \subset \mathbb{R}$ and $f\colon U \to \mathbb{R}$ is continuous. Then there exists a G_δ set M containing U and a continuous function $g\colon M \to \mathbb{R}$ such that $g|U = f$.*

Let $\mathcal{U}_0$ be the class of all functions f such that for every interval J the set $f(J)$ is dense in the interval $[\inf_J f, \sup_J f]$ [3].

Lemma 2. [2, Lemma 3.1] *Let $J = (a, b)$, $f \in \mathcal{U}_0 \cap WCIVP$, $A = f^{-1}(J)$ and denote by $\{I_m\}_{m=1}^{\infty}$ the set of all intervals having rational endpoints for which $I_m \cap A \neq \emptyset$. If $A \neq \emptyset$ then there exists a sequence of pairwise disjoint Cantor sets $\{K_m\}_{m=1}^{\infty}$ such that $K_m \subset A \cap I_m$ for $m \in \mathbb{N}$.*

Theorem 1. *There exists a Sierpiński–Zygmund function having the CIVP.*

Proof. Let $\{x_\alpha\colon \alpha < \mathfrak{c}\}$ be a one-to-one enumeration of $\mathbb{R}$, $\{I_n\colon n < \omega\}$ be a sequence of all open intervals with rational end-points, $\{g_\alpha\colon \alpha < \mathfrak{c}\}$ be an enumeration of all continuous functions defined on G_δ subsets of $\mathbb{R}$ and $\{C_\alpha\colon \alpha < \mathfrak{c}\}$ be an enumeration of all Cantor sets (i.e., non-empty compact perfect nowhere dense subsets of the line). It is well-known (and easy to prove) that there exists a family $\{K_{n,\alpha}\colon\ n < \omega,\ \alpha < \mathfrak{c}\}$ of pairwise disjoint Cantor sets such that $K_{n,\alpha} \subset I_n$ for each $n < \omega$ and $\alpha < \mathfrak{c}$. (See, e.g., [2].)

Now, define the values $f(x_\alpha)$ of function f by induction on $\alpha < \mathfrak{c}$ as follows.

(a) $f(x_\alpha) \in C_\beta \setminus \{g_\gamma(x_\alpha)\colon \gamma \le \alpha\}$ provided $x_\alpha \in \bigcup_{n<\omega} K_{n,\beta}$.

(b) $f(x_\alpha) \in \mathbb{R} \setminus \{g_\gamma(x_\alpha)\colon \gamma \le \alpha\}$ otherwise.

We will show that f has the desired properties.

To prove that $f \in CIVP$ fix $x, y \in \mathbb{R}$ and a Cantor set C between $f(x)$ and $f(y)$. There exist $n < \omega$ and $\beta < \mathfrak{c}$ such that $I_n \subset (x, y)$ and $C = C_\beta$. Then $K_{n,\beta} \subset I_n$ and, by (a), $f(K_{n,\beta}) \subset C_\beta$. Thus f has the CIVP.

To prove that $f \in SZ$, by Lemma 1 it is enough to show that $\operatorname{card}([f = g_\beta]) < \mathfrak{c}$ for each $\beta < \mathfrak{c}$. But $[f = g_\beta] \subset \{x_\alpha\colon \alpha < \beta\})$, so $\operatorname{card}([f = g_\beta]) < \mathfrak{c}$. Hence, $f \in SZ$. □

Since $SZ \cap SCIVP = \emptyset$, we obtain the following

Corollary 1. *$CIVP \ne SCIVP$.*

Moreover, we shall prove in the next theorem that both sets $CIVP \setminus SCIVP$ and $SCIVP$ are dense in the uniform closure of the class $CIVP$. Recall that this closure is equal to the class $\mathcal{U} \cap WCIVP$ [2]. Here $\mathcal{U}$ denote the uniform limit of the class of Darboux functions, i.e., the class of all functions f such that for every interval J and every set A of cardinality less than $\mathfrak{c}$, the set $f(J \setminus A)$ is dense in the interval $[\inf_J f, \sup_J f]$ [3]. (Note also that $\mathcal{U} \subset \mathcal{U}_0$ [3] and $\mathcal{U} \cap WCIVP = \mathcal{U} \cap PR$.)

Theorem 2. *For every $\varepsilon > 0$ and each $h \in \mathcal{U} \cap WCIVP$ the sets $\{f \in SCIVP\colon ||h - f|| < \varepsilon\}$ and $\{k \in CIVP \setminus SCIVP\colon ||h - k|| < \varepsilon\}$ have cardinality equal to $2^{\mathfrak{c}}$.*

Proof. Let $\{x_\alpha\colon \alpha < \mathfrak{c}\}$ be a one-to-one enumeration of $\mathbb{R}$ and $\{g_\alpha\colon \alpha < \mathfrak{c}\}$ be an enumeration of all continuous functions defined on G_δ subsets of $\mathbb{R}$. Choose a sequence $\{J_n\}_n$ of half open intervals, each of length ε, such that $\operatorname{rng}(h) \subset \operatorname{int} \bigcup J_n$ and $\operatorname{int} J_n \cap \operatorname{rng}(h) \ne \emptyset$. For every n let $\{r_{n,\beta}\}_{\beta<\mathfrak{c}}$ be a net of all points of J_n and $\{C_{n,\alpha}\colon \alpha < \mathfrak{c}\}$ be an enumeration of all Cantor sets contained in J_n.

Set $A_n = h^{-1}(\operatorname{int} J_n)$. Let $\{I_{n,m}\}_m$ be a sequence of all open intervals with rational end-points such that $I_{n,m} \cap A_n \neq \emptyset$. By Lemma 2, for every n there exists a sequence $\{K_{n,m}\}_m$ of pairwise disjoint Cantor sets such that $K_{n,m} \subset I_{n,m} \cap A_n$. Moreover, we can require that $\operatorname{card}(A_n \setminus \bigcup_m K_{n,m}) = \mathfrak{c}$. Decompose each $K_{n,m}$ into $\mathfrak{c}$ many Cantor sets $\{K_{n,m,\alpha}\}_{\alpha<\mathfrak{c}}$.

Let $\mathcal{F}$ and $\mathcal{K}$ be families of functions such that

(a) $f(x_\alpha) = r_{n,\beta}$ and $k(x_\alpha) \in C_{n,\beta} \setminus \{g_\gamma(x_\alpha)\colon \gamma \leq \alpha\}$ provided $f \in \mathcal{F}$, $k \in \mathcal{K}$ and $x_\alpha \in \bigcup_{m=1}^\infty K_{n,m,\beta}$.

(b) $f(x_\alpha) \in J_n$ and $k(x_\alpha) \in J_n \setminus \{g_\gamma(x_\alpha)\colon \gamma \leq \alpha\}$ if $f \in \mathcal{F}$, $k \in \mathcal{K}$ and $x_\alpha \in h^{-1}(J_n) \setminus \bigcup_{m=1}^\infty K_{n,m}$.

Note that for every $x \in \bigcup_{n,m} K_{n,m}$ the value $k(x)$ can be any element from the set of size $\mathfrak{c}$, so $\operatorname{card}(\mathcal{K}) = 2^{\mathfrak{c}}$. Similarly, for every $x \in h^{-1}(J_n) \setminus \bigcup_m K_{n,m}$, $f(x)$ can be any element from the interval J_n, so $\operatorname{card}(\mathcal{F}) = 2^{\mathfrak{c}}$.

Observe that for each $x \in \mathbb{R}$, $f \in \mathcal{F}$ and $k \in \mathcal{K}$, if $h(x) \in J_n$ then $f(x), k(x) \in J_n$. Therefore $||f - h|| < \varepsilon$ and $||k - h|| < \varepsilon$.

To prove that $\mathcal{F} \subset SCIVP$ and $\mathcal{K} \subset CIVP$ fix $f \in \mathcal{F}$, $k \in \mathcal{K}$, $x, y \in \mathbb{R}$, and Cantor sets C^1 between $f(x)$ and $f(y)$ and C^2 between $k(x)$ and $k(y)$, respectively. We can assume that $C^1 \subset J_{n_1}$ and $C^2 \subset J_{n_2}$ for some n_1, n_2. Let $r \in C^1$. Then there exist $\beta_1, \beta_2 < \mathfrak{c}$ such that $r = r_{n_1,\beta_1}$, $C^2 = C_{n_2,\beta_2}$ and $m_1, m_2 < \omega$ such that $I_{n_1,m_1} \cup I_{n_2,m_2} \subset (x, y)$ and $I_{n_1,m_1} \cap A_{n_1} \neq \emptyset \neq I_{n_2,m_2} \cap A_{n_2}$. Then $f|K_{n_1,m_1,\beta_1} = r_{n_1,\beta_1} = r \in C^1$ and consequently, $f \in SCIVP$.

Now, for $k \in \mathcal{K}$ observe that $k(K_{n_2,m_2,\beta_2}) \subset C_{n_2,\beta_2}$ and $[k = g_\beta] \subset \{x_\alpha\colon \alpha < \beta\}$ for each $\beta < \mathfrak{c}$, so $\operatorname{card}([k = g_\beta]) < \mathfrak{c}$ and, by Lemma 1, $k \in SZ$. Therefore, $k \in CIVP \setminus SCIVP$. $\square$

3 An almost continuous S-Z function having the CIVP

Recall that it is consistent with ZFC that no SZ function $h\colon \mathbb{R} \to \mathbb{R}$ is almost continuous. In fact, this happens in the iterated perfect set model, where there is no SZ function $h\colon \mathbb{R} \to \mathbb{R}$ with the Darboux property [1]. Thus, in this section we shall work with the additional set theoretical assumption. In our construction we will use some ideas from [6], [14] and [1]. We shall need also the following lemma, basic in the theory of almost continuous maps from $\mathbb{R}$ into $\mathbb{R}$.

Lemma 3. [13] *If $f\colon \mathbb{R} \to \mathbb{R}$ intersects every closed set $K \subset \mathbb{R}^2$ with the domain being a non-degenerate interval, then it is almost continuous.*

Theorem 3. *Assume that the real line is not a union of less than $\mathfrak{c}$ many meager sets. Then there exists an almost continuous Sierpiński–Zygmund function which has the CIVP.*

PROOF. For $A \subset \mathbb{R}$ we denote $L(A) = A \times \mathbb{R}$. Let $\{x_\alpha\colon \alpha < \mathfrak{c}\}$ be a one-to-one enumeration of $\mathbb{R}$, $\{I_n\colon n < \omega\}$ be a sequence of all open intervals with rational end-points, $\{g_\alpha\colon \alpha < \mathfrak{c}\}$ be an enumeration of all continuous functions defined on G_δ subsets of $\mathbb{R}$ and $\{P_\alpha\colon \alpha < \mathfrak{c}\}$ be an enumeration of all Cantor sets. Moreover, let $\{K_{n,\beta,\gamma}\colon\ n < \omega,\ \beta, \gamma < \mathfrak{c}\}$ be a family of pairwise disjoint Cantor sets such that $K_{n,\beta,\gamma} \subset I_n$ for each $n < \omega$ and $\beta, \gamma < \mathfrak{c}$. (See, e.g., [2].) Let $\varphi\colon \mathfrak{c} \to \omega \times \mathfrak{c}$ be a bijection and $\varphi = (\varphi_1, \varphi_2)$.

Choose, by induction on $\alpha < \mathfrak{c}$, a sequence $\langle\langle C_\alpha, D_\alpha\rangle\colon \alpha < \mathfrak{c}\rangle$ such that for every $\alpha < \mathfrak{c}$

(1) $D_\alpha \subset \operatorname{dom}(g_\alpha) \setminus \bigcup_{\beta<\alpha}(C_\beta \cup D_\beta)$ is an at most countable set such that $g_\alpha|D_\alpha$ is a dense subset of $g_\alpha \setminus \bigcup_{\beta<\alpha}(g_\beta \cup L(C_\beta \cup D_\beta))$;

(2) if $\varphi(\alpha) = (n, \beta)$ then $C_\alpha = K_{n,\beta,\gamma}$ for some $\gamma < \mathfrak{c}$ with $C_\alpha \cap \bigcup_{\delta\le\alpha} D_\delta = \emptyset$.

The choice as in (2) can be made, since the set $\bigcup_{\delta\le\alpha} D_\delta$ has cardinality less than continuum, and there is continuum many pairwise disjoint sets $K_{n,\beta,\gamma}$, $\gamma < \mathfrak{c}$.

Now, define the values $f(x_\alpha)$ of function f by induction on $\alpha < \mathfrak{c}$ as follows.

(a) $f(x_\alpha) = g_\beta(x_\alpha)$ provided $x_\alpha \in D_\beta$ for some $\beta < \mathfrak{c}$.

(b) $f(x_\alpha) \in P_\beta \setminus \{g_\delta(x_\alpha)\colon \delta \le \alpha\}$ provided $x_\alpha \in C_\nu$ and $\varphi_2(\nu) = \beta$.

(c) $f(x_\alpha) \in \mathbb{R} \setminus \{g_\gamma(x_\alpha)\colon \gamma \le \alpha\}$ otherwise.

We will show that f has the desired properties.

To verify that f has the CIVP fix a Cantor set $P \subset \mathbb{R}$ and an interval $I \subset \mathbb{R}$. There exist $n < \omega$ and $\beta < \mathfrak{c}$ such that $I_n \subset I$ and $P = P_\beta$. Let $\alpha = \varphi^{-1}(n, \beta)$. Then $C_\alpha \subset I$ and $f(C_\alpha) \subset P$, so $f \in CIVP$.

The proof that $f \in SZ \cap AC$ is the same as in [1]. To prove that $f \in SZ$, by Lemma 1 it is enough to show that $\operatorname{card}([f = g_\beta]) < \mathfrak{c}$ for each $\beta < \mathfrak{c}$. But $[f = g_\beta] \subset \bigcup_{\alpha\le\beta} D_\alpha \cup \{x_\alpha\colon \alpha < \beta\})$, so $\operatorname{card}([f = g_\beta]) < \mathfrak{c}$. Hence, $f \in SZ$.

To verify that f is almost continuous choose a closed set $F \subset \mathbb{R}^2$ with the domain being a non-degenerate interval. By Lemma 3, it is enough to show that $f \cap F \neq \emptyset$. To see this, note that there exist a non-degenerate interval $J \subset \operatorname{dom}(F)$ and an upper semicontinuous function $h\colon J \to \mathbb{R}$ such that $h \subset F$. (See [14, Lemma 1].) Thus there exists an $\alpha_0 < \mathfrak{c}$ such that $g_{\alpha_0} = h|C(h)$, where $C(h)$ denotes the set of all points at which h is continuous. Then $\operatorname{dom} g_{\alpha_0}$ is residual in J and $g_{\alpha_0} \subset F$. In particular, if S is the set of all $\alpha < \mathfrak{c}$ such that $\operatorname{dom}(g_\alpha \cap F)$ is residual in some non-degenerate interval I then $S \neq \emptyset$.

Let $\alpha = \min S$ and I be a non-degenerate interval such that $\operatorname{dom}(g_\alpha \cap F)$ is residual in I. But F is closed and g_α is continuous. So, $g_\alpha|I \subset F$. Moreover, by the minimality of α, for each $\beta < \alpha$ the set $I \cap [g_\beta = g_\alpha] \subset \operatorname{dom}(g_\beta \cap F)$ is nowhere dense in I. Consequently,

$$I \cap \operatorname{dom}\Big[g_\alpha \setminus \bigcup_{\beta<\alpha} (g_\beta \cup L(C_\beta \cup D_\beta))\Big] =$$

$$(I \cap \operatorname{dom}(g_\alpha)) \setminus \bigcup_{\beta<\alpha} \Big(I \cap ([g_\beta = g_\alpha] \cup C_\beta \cup D_\beta)\Big) \neq \emptyset,$$

since, by our set theoretic assumption, I cannot be cover by less than $\mathfrak{c}$ many meager sets. Thus, by (1), $I \cap D_\alpha \neq \emptyset$. Let $x \in I \cap D_\alpha$. Then, by (a), $\langle x, f(x)\rangle = \langle x, g_\alpha(x)\rangle \in f \cap F$. $\square$

Because no Sierpiński-Zygmund function has the SCIVP, we obtain the following corollary.

Corollary 2. *Assume that the real line is not a union of less than* $\mathfrak{c}$ *many meager sets. Then there exists an almost continuous function with the CIVP but without the SCIVP.*

Moreover, because every extendable function has the SCIVP, we have the following result

Corollary 3. *Assume that the real line is not a union of less than* $\mathfrak{c}$ *many meager sets. Then*

$$Ext \neq AC + CIVP.$$

Nevertheless, note that the problem whether $Ext = AC + SCIVP$ remains open. (See [8, Question 4] or [17, Question].) We do not know also whether an example of almost continuous function with the CIVP but without the SCIVP can be constructed in ZFC.

4 Final remarks

The following new results concerning the problems above were obtained after this paper was written.

- Krzysztof Ciesielski constructed in ZFC an example of an additive function in the class $AC + CIVP \setminus SCIVP$ [4].
- Harvey Rosen proved that CH implies the inequality $AC + SCIVP \setminus Ext \neq \emptyset$ [16]. The same result was recently obtained in ZFC by Krzysztof Ciesielski and Andrzej Rosłanowski [5].

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