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J. Ewert and S. P. Ponomarev, Institute of Mathematics, Pedagogical University, 76-200 Slupsk, Poland. e-mail: `im-pap@admin.wsp.slupsk.pl`, `stapon@admin.wsp.slupsk.pl`

ON THE GENERALIZED LINDELÖF PROPERTY

Abstract

We consider two types of topological spaces which generalize the notion of a Lindelöf space. The invariance of these classes under some operation is discussed. We also show that the density topology in $\mathbb{R}^n$ is an example of both generalizations.

1 Preliminaries and Basic Properties

By (X,T), or simply X, we denote a topological space on which, unless otherwise stated, no separation axioms are assumed. Thus we define compactness and paracompactness without T_2 and the Lindelöfness without T_3 (cf. [15]). We will also consider some other covering properties of topological spaces, so we recall the corresponding definitions.

A family $\mathcal{U}$ of sets in X is said to be locally countable if for each $x \in X$ there is a neighborhood W of x such that $\operatorname{card}\{U \in \mathcal{U} : U \cap W \neq \emptyset\} \leq \aleph_0$.

A topological space X is said to be:

- almost compact if for each open cover $\mathcal{U}$ of X there is a finite family $\{U_1, U_2, \dots, U_n\} \subset \mathcal{U}$ whose union is dense in X, [6, p.239];

- almost paracompact if for each open cover $\mathcal{U}$ of X there is a locally finite family $\mathcal{V}$ of open subsets of X which refines $\mathcal{U}$ and whose union is dense in X [17];

- para-Lindelöf if for each open cover $\mathcal{U}$ of X there is a locally countable open cover $\mathcal{V}$ which refines $\mathcal{U}$, [14].

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In regular spaces the compactness and the almost compactness (resp. paracompactness and almost paracompactness) coincide [6, 17]. Examples of an almost compact (almost paracompact) space which is not compact (paracompact) are given in [1,p.270] and [8]. Furthermore, each T_3 Lindelöf space is paracompact and a T_2 paracompact space is T_4 [15, p.90, 85].

A topological space X will be called:

- almost Lindelöf if each open cover $\mathcal{U}$ of X contains at most countable family $\{U_n : n \in \mathbb{N}\} \subset \mathcal{U}$ whose union is dense in X;

- almost para-Lindelöf if for each open cover $\mathcal{U}$ of X there is a locally countable family $\mathcal{V}$ of open subsets of X which refines $\mathcal{U}$ and whose union is dense in X.

Then we have the following relations between the classes of spaces mentioned.

$$
\begin{tikzcd}
\text{compact} \arrow[rr] \arrow[dr] \arrow[dd] & & \text{Linfelöf} \arrow[dr] \arrow[dd] & \\
& \text{paracompact} \arrow[dd] & & \text{para-Lindelöf} \arrow[dd] \\
\text{almost compact} \arrow[rr] \arrow[dr] & & \text{almost Lindelöf} \arrow[dr] & \\
& \text{almost paracompact} \arrow[rr] & & \text{almost para-Lindelöf}
\end{tikzcd}
$$

All inclusions (depicted by arrows) are proper in this diagram, which follows in part from previous remarks. Examples illustrating other cases will be given in the sequel.

For a subset A in a topological space X the symbols $\operatorname{cl} A, \operatorname{int} A$ will be used to denote the closure and the interior of A respectively.

Example 1.1. (a) Each separable space is almost Lindelöf.
(b) Each uncountable discrete space is almost para-Lindelöf but is not almost Lindelöf.

Example 1.2. The Niemytzki plane is separable; so it is almost Lindelöf. Furthermore, it is T_3 and not T_4. Therefore it is neither almost compact, nor almost paracompact nor Lindelöf.

Example 1.3. Let $\mathbf{m}$ be the Banach space of all bounded real sequences with the usual norm $\|x\| = \sup_{n\in\mathbb{N}} |t_n|$, where $x = (t_1, t_2, \ldots)$. Since $\mathbf{m}$ is paracompact, it is also almost para-Lindelöf. We claim that $\mathbf{m}$ is not almost Lindelöf. Suppose that this is not the case. By $B(x,r)$ we denote the open ball with center x and of radius $r > 0$. Then from the open cover $\mathcal{U} = \{B(x, 1/6) : x \in \mathbf{m}\}$ we can choose a countable family $\{B(x_n, 1/6) : n \in \mathbb{N}\}$ with $\operatorname{cl} \bigcup_{n=1}^{\infty} B(x_n, 1/6) = \mathbf{m}$. Now by A we denote the set of all $x \in \mathbf{m}$, $x = (t_1, t_2, \ldots)$, such that $t_j \in \{0, 1\}$ for $j \in \mathbb{N}$. The set A is uncountable and $\|x - y\| = 1$ for all $x, y \in A$, $x \neq y$. If $x \in A$, then $B(x, 1/6) \cap B(x_k, 1/6) \neq \emptyset$ for some $k \geq 1$; hence $x \in B(x_k, 1/3)$. Thus for some $k \geq 1$ we have $\operatorname{card}(A \cap B(x_k, 1/3)) \geq \aleph_0$. It follows that $\|x - y\| < 2/3$ for all $x, y \in A \cap B(x_k, 1/3)$, and this contradiction finishes the proof.

It follows from Examples 1.2 and 1.3 that almost Lindelöfness and almost paracompactness are independent properties, but we have the following.

Theorem 1.4. *If X is an almost paracompact (almost para-Lindelöf) space in which every locally finite (locally countable) family of non-empty open sets is countable, then X is almost Lindelöf.*

Proof. Let $\mathcal{U} = \{U_s : s \in S\}$ be an open cover of X. There exists a locally finite (locally countable) family $\mathcal{V}$ of open sets which refines $\mathcal{U}$ and $\bigcup\{V : V \in \mathcal{V}\}$ is dense in X. It follows from our assumptions that $\mathcal{V}$ is countable; i.e., $\mathcal{V} = \{V_n : n \in \mathbb{N}\}$. For each $n = 1, 2, \ldots$ we fix $U_{s(n)} \in \mathcal{U}$ with $V_n \subset U_{s(n)}$; hence $X = \operatorname{cl} \bigcup_{n=1}^{\infty} U_{s(n)}$. □

Lemma 1.5. *If X is an almost Lindelöf space, then every locally countable family of non-empty open sets is at most countable.*

Proof. Let $\mathcal{U}$ be a locally countable family of non-empty open sets in X. For each $x \in X$ we fix a neighborhood V_x which intersects at most countably many sets belonging to $\mathcal{U}$. The open cover $\{V_x : x \in X\}$ contains a countable family $\{V_{x_n} : n \in \mathbb{N}\}$ such that $\bigcup_{n=1}^{\infty} V_{x_n}$ is dense. Since each set from $\mathcal{U}$ intersects some V_{x_n}, it follows that $\operatorname{card} \mathcal{U} \leq \aleph_0$. □

Lemma 1.5 leads to the following two corollaries.

Corollary 1.6. *A topological space X is a Lindelöf space if and only if it is almost Lindelöf and para-Lindelöf.* □

Corollary 1.7. *[7, Coroll. 5.1.26]. Every separable paracompact space is a Lindelöf space.*

Theorem 1.8. *If an almost para-Lindelöf space X contains a dense almost Lindelöf subspace A, then X is almost Lindelöf.*

Proof. Let $\mathcal{U} = \{U_j : j \in J\}$ be an open cover of X. Then there is a locally countable family $\mathcal{V} = \{V_s : s \in S\}$ of open sets which refines $\mathcal{U}$ and $\mathrm{cl}\bigcup_{s\in S} V_s = X$. Since A is dense, $\{A \cap V_s : s \in S\}$ is a locally countable family of non-empty open sets in the almost Lindelöf space A. According to Lemma 1.5, we have $\mathrm{card}\, S \leq \aleph_0$. Thus

$$X = \mathrm{cl}\, A = \mathrm{cl}(\bigcup_{s\in S} A \cap V_s) \subset \mathrm{cl} \bigcup_{s\in S} V_s.$$

For each $s \in S$ we choose $j(s) \in J$ such that $V_s \subset U_{j(s)}$; hence $X = \mathrm{cl}(\bigcup_{s\in S} U_{j(s)})$ which completes the proof. □

Theorem 1.9. *If X is an almost Lindelöf (para-Lindelöf, almost para-Lindelöf) space, then each open-closed subspace M in X is almost Lindelöf (para-Lindelöf, almost para-Lindelöf).*

Proof. Assume that X is almost para-Lindelöf. (For an almost Lindelöf or a para-Lindelöf space the proof is analogous.) Let $\mathcal{U} = \{U_s : s \in S\}$ be an open in M cover of M; then $U_s = M \cap V_s$, where V_s are open in X. For the open cover $\{X \setminus M\} \cup \{V_s : s \in S\}$ of X there is a locally countable family of open sets $\{W_j : j \in J\}$ which refines this cover and such that $\mathrm{cl}\bigcup_{j\in J} W_j = X$. Hence $\mathcal{V} = \{W_j \cap M : j \in J\}$ is a locally countable in M family of open sets in M which refines $\mathcal{U}$. Furthermore,

$$M = \mathrm{cl}(M \cap X) = \mathrm{cl}(M \cap \mathrm{cl}(\bigcup_{j\in J} W_j)) = \mathrm{cl}(\bigcup_{j\in J} M \cap W_j) \subset M,$$

which completes the proof. □

As an immediate consequence of the above theorem, we obtain the following theorem.

Theorem 1.10. *Let $\bigoplus_{s\in S} X_s$ be the topological sum of a family $\{X_s : s \in S\}$ of topological spaces such that $X_s \cap X_{s'} = \emptyset$ for $s, s' \in S,\ s \neq s'$.*

(a) The sum $\bigoplus_{s\in S} X_s$ is almost Lindelöf (para-Lindelöf) if and only if all X_s are almost Lindelöf (para-Lindelöf) spaces and the set S is countable.

(b) The space $\bigoplus_{s\in S} X_s$ is almost para-Lindelöf (almost paracompact) if and only if all spaces X_s are almost para-Lindelöf (almost paracompact).

Example 1.11. Let X denote the Niemytzki plane and $\mathbf{m}$ the Banach space of all bounded sequences of reals. It follows from Examples 1.2, 1.3 and Theorem 1.10 that the space $X \oplus \mathbf{m}$ is almost para-Lindelöf, but it is neither para-Lindelöf nor almost Lindelöf, nor almost paracompact.

A topological space is said to be a P-space if the intersection of any countable family of open sets is again an open set [14].

Lemma 1.12. *[14, Lemma 4.4]. If $\mathcal{A}$ is a locally countable family of subsets in a P-space X, then* $\mathrm{cl}(\bigcup\{A : A \in \mathcal{A}\}) = \bigcup\{\mathrm{cl}\, A : A \in \mathcal{A}\}$.

Theorem 1.13. *Let X be a T_3 P-space.*
(a) If X is almost Lindelöf, then it is a Lindelöf T_4 space.
(b) If X is almost para-Lindelöf, then it is paracompact.

Proof. (a) Let $\mathcal{U}$ be an open cover of X. For each point $x \in X$ we fix $U_x \in \mathcal{U}$ and an open set W_x such that $x \in W_x \subset \mathrm{cl}\, W_x \subset U_x$. Then we can choose a countable family $\{W_{x_n} : n \in \mathbb{N}\}$ for which $\mathrm{cl}(\bigcup_{n=1}^{\infty} W_{x_n}) = X$. But according to Lemma 1.12 we have $\mathrm{cl}(\bigcup_{n=1}^{\infty} W_{x_n}) = \bigcup_{n=1}^{\infty} \mathrm{cl}\, W_{x_n}$. Hence $\{U_{x_n} : n \in \mathbb{N}\}$ is a countable subcover chosen from $\mathcal{U}$. As it is stated in [14, Prop. 4.2], each Hausdorff Lindelöf P-space is normal; so item (a) is proved.
(b) Let $\mathcal{U}$ be an open cover of X. For each $x \in X$ we fix $U_x \in \mathcal{U}$ and an open set W_x satisfying $x \in W_x \subset \mathrm{cl}\, W_x \subset U_x$. Then there exists a locally countable family of open sets $\mathcal{V}_1$ which refines $\{W_x : x \in X\}$ and such that $\mathrm{cl}(\bigcup\{V : V \in \mathcal{V}_1\}) = X$. From Lemma 1.12 we have $\mathrm{cl}(\bigcup\{V : V \in \mathcal{V}_1\}) = \bigcup\{\mathrm{cl}\, V : V \in \mathcal{V}_1\}$. Thus $\mathcal{V} = \{\mathrm{cl}\, V : V \in \mathcal{V}_1\}$ refines $\mathcal{U}$. So we have shown that for each open cover $\mathcal{U}$ there is a closed locally countable cover which refines $\mathcal{U}$. By [14, Th. 4.3], in each T_3 P-space the last property is equivalent to paracompactness. □

Corollary 1.14. *[14]. Let X be a T_3 P-space. Then X is para-Lindelöf if and only if it is paracompact.*

Finally we will give an example of a space which is not almost para-Lindelöf. Given a cover $\mathcal{U}$ of X and a set $M \subset X$, let

$$\mathrm{St}(M,\mathcal{U}) = \bigcup\{U \in \mathcal{U} : U \cap M \neq \emptyset\}.$$

A cover $\mathcal{U}$ is said to be a star refinement of another cover $\mathcal{V}$ if the family $\{\mathrm{St}(U,\mathcal{U})\} : U \in \mathcal{U}\}$ is a refinement of $\mathcal{V}$.

Example 1.15. Let $X = \{p = (x, y) \in \mathbb{R}^2 : x \geq 0, y \geq 0\} \setminus \{(0, 0)\}$ and $U_x = \{x\} \times [0, \infty)$, $W_y = [0, \infty) \times \{y\}$ for all $x > 0$, $y > 0$. For each $p = (x, y) \in X$ write

$$\mathcal{B}(p) = \begin{cases} \{\{p\}\} & \text{if } x > 0,\ y > 0 \\ \{U_x \setminus L : \mathrm{card}\, L \leq \aleph_0,\ p \notin L\} & \text{if } x > 0,\ y = 0 \\ \{W_y \setminus L : \mathrm{card}\, L \leq \aleph_0,\ p \notin L\} & \text{if } x = 0,\ y > 0. \end{cases}$$

Then $\mathcal{B} = \bigcup\{\mathcal{B}(p) : p \in X\}$ is a base of a Hausdorff topology τ on X. The base $\mathcal{B}$ consists of open-closed subsets; so (X, τ) is a Tychonoff space. Let $V_n,\ n \in \mathbb{N}$, be open subsets of (X, τ) and $p = (x, y) \in \bigcap_{n=1}^{\infty} V_n$. If $x > 0, y > 0$, then $\{p\}$ is an open set and thus $p \in \operatorname{int} \bigcap_{n=1}^{\infty} V_n$. If $x = 0, y > 0$, then for each $n \in \mathbb{N}$ there is at most countable set L_n such that $p \notin L_n$ and $W_y \setminus L_n \subset V_n$. This implies $W_y \setminus \bigcup_{n=1}^{\infty} L_n \subset \bigcap_{n=1}^{\infty} V_n$ and consequently $p \in \operatorname{int} \bigcap_{n=1}^{\infty} V_n$. In case $x > 0,\ y = 0$ we use the analogous argument. Thus we have shown that (X, τ) is a P-space. Now we take the open cover $\mathcal{U} = \{U_x : x > 0\} \cup \{W_y : y > 0\}$ and let $\mathcal{A}$ be an open refinement of $\mathcal{U}$. For each $p_x = (x, 0), p^y = (0, y), x \neq 0, y \neq 0$, we fix neighborhoods $U_x \setminus L_x, W_y \setminus L_y$ contained in some fixed sets belonging to $\mathcal{U}$. Then we can choose a point $p_0 = (x_0, y_0)$ such that $p_0 \in (U_{x_0} \setminus L_{x_0}) \cap (W_{y_0} \setminus L_{y_0})$ and $A_1, A_2 \in \mathcal{A}$ with $U_{x_0} \setminus L_{x_0} \subset A_1$ and $W_{y_0} \setminus L_{y_0} \subset A_2$. Therefore $(U_{x_0} \setminus L_{x_0}) \cup (W_{y_0} \setminus L_{y_0}) \subset \operatorname{St}(A_1, \mathcal{A})$, thus $\operatorname{St}(A_1, \mathcal{A})$ is not contained in any set from $\mathcal{U}$. So we have shown that the open cover $\mathcal{U}$ has no any open star refinement, whence by [7, Th. 5.1.12] the space (X, τ) is not paracompact. Applying Theorem 1.13(b) we obtain that (X, τ) is not almost para-Lindelöf either.

2 Invariance under Mappings

Let X and Y be topological spaces and $F : X \to Y$ a multivalued map. For any sets $A \subset X, B \subset Y$ we will write as usually $F(A) = \bigcup_{x \in A} F(x),\ F^+(B) = \{x \in X : F(x) \subset B\}$, and $F^-(B) = \{x \in X : F(x) \cap B \neq \emptyset\}$.

A subset A of a topological space X is said to be:

- semi-open, if $A \subset \operatorname{cl}(\operatorname{int} A)$;
- semi-closed, if $X \setminus A$ is semi-open.

The union (intersection) of any family of semi-open (semi-closed) sets is semi-open (semi-closed). The intersection of all semi-closed sets containing A is called the semi-closure of A; we will denote it by $\operatorname{scl} A$ [3,4,12]. For each set A we have $\operatorname{int}(\operatorname{cl} A) = \operatorname{int}(\operatorname{scl} A)$, [3,4]. A multivalued map $F : X \to Y$ is said to be:

- semi-open, if for each open set $U \subset X$ the set $F(U)$ is semi-open;
- closed, if for each closed set $A \subset X$ the set $F(A)$ is closed.

In the sequel we will use the following well known characterization of closed multivalued maps.

Lemma 2.1. *A multivalued map $F : X \to Y$ is closed if and only if for each point $y \in Y$ and each open set U containing $F^{-}(y)$ there exists a neighborhood V of y such that $F^{-}(V) \subset U$.*

A multivalued map $F : X \to Y$ is said to be:

- upper semicontinuous, if for each open set $V \subset Y$ the set $F^{+}(V)$ is open;
- lower quasicontinuous, if for each open set $V \subset Y$ the set $F^{-}(V)$ is semi-open [9].

If F is single-valued, then upper semicontinuity means continuity of F. Moreover, in this case lower quasicontinuity coincides with quasicontinuity [11, 12].

Lemma 2.2. *A multivalued map $F : X \to Y$ is lower quasicontinuous if and only if $F(\operatorname{scl} A) \subset \operatorname{cl} F(A)$ for each set $A \subset X$.*

We omit the standard proof.

Theorem 2.3. *Let X and Y be topological spaces and let F be an upper semicontinuous and lower quasicontinuous multivalued map from X onto Y such that $F(x)$ has the Lindelöf property for each $x \in X$.*
(a) If X is almost Lindelöf, then Y is also.
(b) Assume,in addition, that F is semi-open closed and $F^{-}(y)$ has the Lindelöf property for each $y \in Y$. Then X is almost para-Lindelöf if and only if Y is almost para-Lindelöf.

Proof. (a) Let $\mathcal{V}$ be an open cover of Y. For $x \in X$ we choose a countable cover $\mathcal{V}_x \subset \mathcal{V}$ of $F(x)$ and we set $V_x = \bigcup\{V : V \in \mathcal{V}_x\}$. Since F is upper semicontinuous, the family $\{F^{+}(V_x) : x \in X\}$ forms an open cover of X. Now we choose a countable family $\{F^{+}(V_{x_n}) : n \in \mathbb{N}\}$ such that $\operatorname{cl}(\bigcup_{n=1}^{\infty} F^{+}(V_{x_n})) = X$. Hence we obtain

$$X = \operatorname{int}(\operatorname{cl}(\bigcup_{n=1}^{\infty} F^{+}(V_{x_n}))) = \operatorname{int}(\operatorname{scl}(\bigcup_{n=1}^{\infty} F^{+}(V_{x_n})));$$

so $X = \operatorname{scl}(\bigcup_{n=1}^{\infty} F^{+}(V_{x_n}))$. This fact and Lemma 2.2 give

$$\begin{aligned} Y = F(X) =& F(\operatorname{scl}(\bigcup_{n=1}^{\infty} F^{+}(V_{x_n}))) \subset \operatorname{cl}(F(\bigcup_{n=1}^{\infty} F^{+}(V_{x_n}))) \\ & \subset \operatorname{cl}(\bigcup_{n=1}^{\infty} V_{x_n}) = \operatorname{cl}(\bigcup\{V : V \in \mathcal{V}_{x_n},\ n \in \mathbb{N}\}). \end{aligned}$$

Thus $\mathcal{V}_1 = \{V : V \in \mathcal{V}_{x_n},\ n \in \mathbb{N}\}$ is a countable subfamily whose union is dense and (a) is thus proved.
(b) Suppose that X is almost para-Lindelöf and let $\mathcal{V}$ be an open cover of Y. For each $x \in X$ we choose a countable cover $\mathcal{V}_x$ of $F(x)$, $\mathcal{V}_x \subset \mathcal{V}$, and we set $V_x = \bigcup\{V : V \in \mathcal{V}_x\}$. Since F is upper semicontinuous, $\mathcal{U} = \{F^+(V_x) : x \in X\}$ is an open cover of X. Then there exists a locally countable open refinement $\mathcal{U}_1 = \{U_j : j \in J\}$ of $\mathcal{U}$ such that $\mathrm{cl}(\bigcup_{j\in J} U_j) = X$. For each $j \in J$ we fix a point $x_j \in X$ for which $U_j \subset F^+(V_{x_j})$. Then $F(U_j) \subset V_{x_j}$ and $F(U_j)$ is semi-open. Hence $\mathcal{U}_2 = \{\mathrm{int}\, F(U_j) \cap V : j \in J,\ V \in \mathcal{V}_{x_j}\}$ is an open refinement of $\mathcal{V}$. We show that $\mathcal{U}_2$ is locally countable. Let $y \in Y$. Since the set $F^-(y)$ has the Lindelöf property and $\mathcal{U}_1$ is locally countable we can choose an open set $A \subset X$ such that $F^-(y) \subset A$ and the set $J_0 = \{j \in J : A \cap U_j \neq \emptyset\}$ is at most countable. By Lemma 2.1 there is a neighborhood W of y such that $F^-(W) \subset A$. This implies $\{j \in J : F(U_j) \cap W \neq \emptyset\}\} \subset J_0$, so W intersects at most countably many sets $F(U_j)$. Furthermore, each family $\mathcal{V}_{x_j}$ being countable, W intersects at most countably many sets belonging to $\mathcal{U}_2$. We have thus shown that $\mathcal{U}_2$ is locally countable. Since the union of the family $\mathcal{U}_1$ is dense we have

$$X = \mathrm{int}(\mathrm{cl}(\bigcup_{j\in J} U_j)) = \mathrm{int}(\mathrm{scl}(\bigcup_{j\in J} U_j))$$

and this implies $X = \mathrm{scl}(\bigcup_{j\in J} U_j)$. Applying Lemma 2.2 we obtain

$$Y = F(X) = F(\mathrm{scl}(\bigcup_{j\in J} U_j)) \subset \mathrm{cl}(F(\bigcup_{j\in J} U_j))$$

which gives $Y = \mathrm{cl}(\bigcup_{j\in J} F(U_j))$. The sets $F(U_j)$ are semi-open; so the last equality implies $\mathrm{cl}(\bigcup_{j\in J} \mathrm{int}\, F(U_j)) = Y$. On the other hand we have

$$\begin{aligned}
\mathrm{cl}(\bigcup_{j\in J} \bigcup_{V\in\mathcal{V}_{x_j}} \mathrm{int}\, F(U_j) \cap V) &= \mathrm{cl}(\bigcup_{j\in J} (\mathrm{int}\, F(U_j) \cap \bigcup_{V\in\mathcal{V}_{x_j}} V)) \\
&= \mathrm{cl}(\bigcup_{j\in J} \mathrm{int}\, F(U_j) \cap V_{x_j}) \\
&= \mathrm{cl}(\bigcup_{j\in J} \mathrm{int}\, F(U_j)).
\end{aligned}$$

This implies $\mathrm{cl}(\bigcup_{j\in J} \bigcup_{V\in\mathcal{V}_{x_j}} \mathrm{int}\, F(U_j) \cap V) = Y$. Thus we have shown that $\mathcal{U}_2$ is a locally countable open refinement of $\mathcal{V}$ whose union is dense and this means that Y is almost para-Lindelöf.

Conversely, assume that Y is almost para-Lindelöf. Consider the multivalued map $G : Y \to X$ defined by $G(y) = F^-(y)$. Then for any sets $A \subset X, B \subset Y$ the equalities $G^-(A) = F(A)$ and $G(B) = F^-(B)$ hold. Hence G is upper semicontinuous, lower quasicontinuous, semi-open and closed. Furthermore, the sets $G(y),\ G^-(x)$ have the Lindelöf property for all $x \in X,\ y \in Y$. Thus according to the proved part, X is almost para-Lindelöf. □

Corollary 2.4. *Let X, Y be topological spaces and a continuous surjection $f : X \to Y$ be given.*
(a) If X is almost Lindelöf, then so is Y.
(b) Assume, in addition, that f is semi-open, closed and $f^{-1}(y)$ has the Lindelöf property for each $y \in Y$. Then X is almost Lindelöf if Y is.

Corollary 2.5. *Let $f : X \to Y$ be a continuous semi-open, closed surjection such that $f^{-1}(y)$ has the Lindelöf property for each $y \in Y$. Then X is almost para-Lindelöf if and only if Y is.*

Lemma 2.6. *[14, Th. 2.1]. A topological space X is a P-space if and only if for each Lindelöf space Y the projection $p : X \times Y \to X$ is a closed map.*

Theorem 2.7. *For a P-space X the following conditions are equivalent:*
(a) X is almost para-Lindelöf;
(b) for each Lindelöf space Y the product $X \times Y$ is almost para-Lindelöf.

Proof. According to Lemma 2.6, the projection $p : X \times Y \to X$ is a continuous open closed map such that $p^{-1}(y)$ has the Lindelöf property for each $y \in X$. Thus the conclusion follows from Theorem 2.3. □

3 Hashimoto Topologies and Almost Para-Lindelöfness

Let P be an ideal of subsets in a topological space (X, T). For any set $A \subset X$ we write

$$D_P(A) = \{x \in X : U \cap A \notin P \text{ for each neighborhood } U \text{ of } x\}.$$

If the following condition $(*)$ is satisfied

$$A \in P \iff D_P(A) = \emptyset \iff A \cap D_P(A) = \emptyset, \qquad (*)$$

then the family $T(P) = \{U \setminus H : U \in T, H \in P\}$ is a topology on X [10]. The $T(P)$ closure of any set A will be denoted $\operatorname{cl}_P A$.

Lemma 3.1. *[10]. If P is any ideal satisfying $(*)$ and $T \cap P = \{\emptyset\}$, then for every $U \in T,\ H \in P$, the equalities $\operatorname{cl}_P(U \setminus H) = \operatorname{cl}(U \setminus H) = \operatorname{cl} U = \operatorname{cl}_P U$ hold.*

Theorem 3.2. *Let P be an ideal of subsets in a topological space (X,T) satisfying $(*)$ and $T \cap P = \{\emptyset\}$. Then:*
(a) (X,T) is an almost Lindelöf space if and only if $(X,T(P))$ is almost Lindelöf.
(b) (X,T) is an almost para-Lindelöf space if and only if $(X,T(P))$ is almost para-Lindelöf.

Proof. (b) Assume that (X,T) is almost para-Lindelöf and let $\mathcal{U}$ be a $T(P)$-open cover of X; $\mathcal{U} = \{U_j \setminus H_j : j \in J\}$ where $U_j \in T, H_j \in P$ for each $j \in J$. The family $\mathcal{U}_1 = \{U_j : j \in J\}$ is a T-open cover of X; so there exists a T-open T-locally countable refinement $\mathcal{V}_1 = \{V_s : s \in S\}$ of $\mathcal{U}_1$ such that $\mathrm{cl}(\bigcup_{s\in S} V_s) = X$. For each $s \in S$ we fix $j_s \in J$ with $V_s \subset U_{j_s}$. Then $\mathcal{V} = \{V_s \setminus H_{j_s} : s \in S\}$ is a $T(P)$-open $T(P)$-locally countable refinement of $\mathcal{U}$. Let $x \in X$ and let $W \setminus H$ be a $T(P)$-open neighborhood of x. Then $W \cap V_s \neq \emptyset$ for some $s \in S$. Thus $(W \setminus H) \cap (V_s \setminus H_{j_s}) = (W \cap V_s) \setminus (H \cup H_{j_s}) \neq \emptyset$. So we have shown that $\bigcup_{s\in S}(V_s \setminus H_{j_s})$ is $T(P)$-dense, and consequently $(X,T(P))$ is almost para-Lindelöf.

Conversely, suppose that $(X,T(P))$ is almost para-Lindelöf and let $\mathcal{U} = \{U_j;\ j \in J\}$ be a T-open cover of X. The assumptions and the condition $T \subset T(P)$ imply the existence of a $T(P)$-open $T(P)$-locally countable refinement $\mathcal{V}_1 = \{V_s \setminus H_s : s \in S\}, V_s \in T, H_s \in P$, of $\mathcal{U}$ such that $\bigcup_{s\in S}(V_s \setminus H_s)$ is $T(P)$-dense. For each $s \in S$ we fix $j_s \in J$ for which $V_s \setminus H_s \subset U_{j_s}$. Then $\mathcal{V} = \{V_s \cap U_{j_s} : s \in S\}$ is a T-open T-locally countable refinement of $\mathcal{U}$ and $\bigcup_{s\in S}(V_s \cap U_{j_s})$ is T-dense. Hence (X,T) is almost para-Lindelöf.

The proof of item (a) is quite analogous; so we omit it. □

Given any topological space (X,T), we denote by $SO(X,T)$ the class of all its semi-open subsets. The equality $SO(X,T) = SO(X,T_1)$ induces the equivalence relation on the family of all topologies on X. The class $[T]$ of all topologies equivalent to T has the finest one T_α. This is exactly the topology $T(P)$, where P is the ideal of all T-nowhere dense sets in X, [2, 5]. Thus as a consequence of Theorem 3.2 we have the following corollary.

Corollary 3.3. *If a topological space (X,T) is almost Lindelöf (almost para-Lindelöf), then all topologies belonging to the equivalence class $[T]$ have the same property.*

A bijective map from (X,T) onto (Y,τ) is said to be a semi-homeomorphism if for any semi-open sets $A \subset X, B \subset Y$ the sets $f(A)$ and $f^{-1}(B)$ are semi-open [2, 5].

Theorem 3.4. *Almost Lindelöfness and almost para-Lindelöfness are invariant under semi-homeomorphisms.*

Proof. If $f : (X, T) \to (Y, \tau)$ is a semi-homeomorphism, then $f : (X, T_\alpha) \to (Y, \tau_\alpha)$ is a homeomorphism, [2. 5]. Thus the proof is a consequence of Theorem 3.2 and Corollary 2.4. $\square$

4 Covering Properties of the Density Topology

The result of this section is closely related to the notion of the almost para-Lindelöfness discussed above.

By τ we mean the standard topology in $\mathbb{R}^n$, whereas the ordinary density topology in $\mathbb{R}^n$ will be denoted by τ_d, [13, p.167]. The symbol $\mathrm{cl}_d\, E$ will stand for the closure of $E \subset \mathbb{R}^n$ in the topology τ_d. The Lebesgue measure of $E \subset \mathbb{R}^n$ will be denoted by $|E|$. The topological space $(\mathbb{R}^n, \tau_d)$ is completely regular (This follows from [13]: (a) on p.198 and Coroll. 3.13.), but it is not normal. (This is a consequence of Th. 6.23; Ex. 6.24; Coroll. 2.14 and Th. 2.1 in [13].) On the other hand, as it was mentioned in Section 1, each T_3 Lindelöf space is a paracompact T_4 space. Thus $(\mathbb{R}^n, \tau_d)$ is neither Lindelöf nor paracompact. We will prove the following covering property of $(\mathbb{R}^n, \tau_d)$ which is stronger than the almost Lindelöfness.

Theorem 4.1. *The space $(\mathbb{R}^n, \tau_d)$ has the following property:*
For each τ_d-open cover $\mathcal{U}$ of $\mathbb{R}^n$ there exists a τ_d-open refinement $\mathcal{W}$ of $\mathcal{U}$ such that
(i) the elements of $\mathcal{W}$ are pairwise disjoint (Hence $\mathcal{W}$ is at most countable.);
(ii) $\mathrm{cl}_d(\bigcup\{W : W \in \mathcal{W}\}) = \mathbb{R}^n$.

Proof. Let $\mathcal{U}$ be a τ_d-open cover of $\mathbb{R}^n$. The construction of $\mathcal{W}$ will be carried out inductively.

1st step. For each $x \in \mathbb{R}^n$ we fix an $E(x) \in \mathcal{U}$ with $x \in E(x)$. For each $x \in \mathbb{R}^n$ we let $\mathcal{V}(x)$ be the family of all τ-closed cubes $Q(x)$, centered at x, with edges parallel to the coordinate axes, and such that

$$|Q(x) \cap E(x)| > \frac{2}{3}|Q(x)| \text{ for each } \quad Q(x) \in \mathcal{V}(x). \tag{1}$$

Let $\mathcal{V} = \bigcup_{x \in \mathbb{R}^n} \mathcal{V}(x)$. Since each x is a point of density of the corresponding $E(x) \in \mathcal{V}(x)$ we have $\inf\{\operatorname{diam} Q(x) : Q(x) \in \mathcal{V}(x)\} = 0$ for each x; so we have that $\mathcal{V}$ is a Vitali cover of $\mathbb{R}^n$. By the Vitali covering theorem there exists a family of pairwise disjoint cubes $\{Q(x_j) : j \in \mathbb{N}\}$ extracted from $\mathcal{V}$ so that $|\mathbb{R}^n \setminus \bigcup_{j \in \mathbb{N}} Q(x_j)| = 0$. By (1) there is a corresponding family $\{E(x_j) : j \in \mathbb{N}\} \subset \mathcal{U}$ such that

$$|Q(x_j) \cap E(x_j)| > \frac{2}{3}|Q(x_j)| \tag{2}$$

for all $j \in \mathbb{N}$. Aiming at a "uniformization" of our notations in the sequel, we will write $\{Q_{i_1} : i_1 \in \mathbb{N}\}$ and $\{E_{i_1} : i_1 \in \mathbb{N}\}$ instead of $\{Q(x_j) : j \in \mathbb{N}\}$ and $\{E(x_j) : j \in \mathbb{N}\}$, respectively. So by (2) there exist τ-open sets G_{i_1} such that $\operatorname{int} Q_{i_1} \setminus E_{i_1} \subset G_{i_1} \subset \operatorname{int} Q_{i_1}$ and $|G_{i_1}| < \frac{1}{3}|Q_{i_1}|$ for $i_1 \in \mathbb{N}$.
To summarize, on the first step we have constructed:

- a family of τ-closed pairwise disjoint cubes $\{Q_{i_1} : i_1 \in \mathbb{N}\}$;
- a family of τ_d-open sets $\{E_{i_1} : i_1 \in \mathbb{N}\} \subset \mathcal{U}$;
- a family of τ-open sets $\{G_{i_1} : i_1 \in \mathbb{N}\}$;

so that the following conditions are satisfied for each $i_1 \in \mathbb{N}$:

$$\begin{aligned} \operatorname{int} Q_{i_1} \setminus E_{i_1} \subset G_{i_1} &\subset \operatorname{int} Q_{i_1}, \\ |G_{i_1}| &< \frac{1}{3}|Q_{i_1}|, \\ |\mathbb{R}^n \setminus \bigcup_{i_1 \in \mathbb{N}} Q_{i_1}| &= 0, \\ |Q_{i_1} \cap E_{i_1}| &> \frac{2}{3}|Q_{i_1}|. \end{aligned}$$

2nd step. We will define the second induction step, since the first one is not entirely "typical". We proceed just like on the first step but this time the Vitali cover will be constructed for each $G_{i_1}, i_1 \in \mathbb{N}$. Namely, given any $x \in G_{i_1}$, we fix a set $E_{i_1}(x) \in \mathcal{U},\ x \in E_{i_1}$. Since x is a density point of $E_{i_1}(x)$, there is a family $\mathcal{V}_{i_1}(x)$ consisting of τ-closed cubes $Q_{i_1}(x)$ centered at x, with edges parallel to the coordinate axes, such that $x \in Q_{i_1}(x) \subset G_{i_1}$ and

$$|Q_{i_1}(x) \cap E_{i_1}(x)| > \frac{2}{3}|Q_{i_1}(x)| \tag{3}$$

for each $Q_{i_1}(x) \in \mathcal{V}_{i_1}(x)$. (One should carefully distinguish between Q_{i_1} and $Q_{i_1}(x)$ as well as between E_{i_1} and $E_{i_1}(x)$. This warning extends to similar notation with multiindexes to come.) As in the first step, for each $x \in G_{i_1}$, we have $\inf\{\operatorname{diam} Q_{i_1}(x) : Q_{i_1}(x) \in \mathcal{V}_{i_1}(x)\} = 0$, whence it follows that

$$\mathcal{V}_{i_1} = \bigcup \{\mathcal{V}_{i_1}(x) : x \in G_{i_1}\} \tag{4}$$

forms a Vitali cover of G_{i_1}. Therefore there exists a family $\{Q_{i_1}(x_j) : j \in \mathbb{N}\}$ of pairwise disjoint τ-closed cubes from $\mathcal{V}_{i_1}$ such that $|G_{i_1} \setminus \bigcup_{j \in \mathbb{N}} Q_{i_1}(x_j)| = 0$ and

$$|Q_{i_1}(x_j) \cap E_{i_1}(x_j)| > \frac{2}{3}|Q_{i_1}(x_j)| \tag{5}$$

for all $i_1, j \in \mathbb{N}$, where, of course, $E_{i_1}(x_j) \in \mathcal{U}$ is a set in (3) with $x = x_j$.

Keeping in mind the uniformization of our notations, we will write again $Q_{i_1i_2}$ and $E_{i_1i_2}$ instead of $Q_{i_1}(x_j)$ and $E_{i_1}(x_j)$ respectively. In view of (5) we can fix, in each $\operatorname{int} Q_{i_1i_2}$, a τ-open set $G_{i_1i_2}$ so that we have

$$\operatorname{int} Q_{i_1i_2} \setminus E_{i_1i_2} \subset G_{i_1i_2} \subset \operatorname{int} Q_{i_1i_2}$$

and $|G_{i_1i_2}| < \frac{1}{3}|Q_{i_1i_2}|$.

Thus on the second induction step we have constructed for each $i_1 \in \mathbb{N}$:

- a family of τ-closed pairwise disjoint cubes $\{Q_{i_1i_2} : i_2 \in \mathbb{N}\}$;
- a family of τ_d-open sets $\{E_{i_1i_2} : i_2 \in \mathbb{N}\} \subset \mathcal{U}$;
- a family of τ-open sets $\{G_{i_1i_2} : i_2 \in \mathbb{N}\}$;

so that the following conditions are satisfied for all $i_1, i_2 \in \mathbb{N}$

$$\begin{aligned} \operatorname{int} Q_{i_1i_2} \setminus E_{i_1i_2} \subset G_{i_1i_2} &\subset \operatorname{int} Q_{i_1i_2}; \\ |G_{i_1i_2}| &< \frac{1}{3}|Q_{i_1i_2}|; \\ |G_{i_1}| &= \sum_{i_2} |Q_{i_1i_2}|; \\ |Q_{i_1i_2} \cap E_{i_1i_2}| &> \frac{2}{3}|Q_{i_1i_2}|; \\ Q_{i_1i_2} &\subset G_{i_1}. \end{aligned}$$

Now we are in a position to pass to the general case. Namely, suppose that for a natural $k > 2$ we have already constructed:

(I). The families $\{Q_{i_1}\}$, $\{Q_{i_1i_2}\}$, . . . , $\{Q_{i_1\ldots i_k}\}$; each family

$$\{Q_{i_1\ldots i_s}\} = \{Q_{i_1\ldots i_s} : (i_1 \ldots i_s) \in \mathbb{N}^s\},$$

$1 \le s \le k$, being composed of pairwise disjoint τ-closed cubes.

(II). The families of τ_d-open sets $\{E_{i_1}\}, \{E_{i_1i_2}\}, \ldots, \{E_{i_1\ldots i_k}\}$, each family

$$\{E_{i_1\ldots i_s}\} = \{E_{i_1\ldots i_s} : (i_1 \ldots i_s) \in \mathbb{N}^s\},$$

$1 \le s \le k$, being composed of some elements of the cover $\mathcal{U}$.

(III). The families of τ-open sets $\{G_{i_1}\}$, $\{G_{i_1i_2}\}$, . . . , $\{G_{i_1\ldots i_k}\}$, so that the following conditions are satisfied:

(IV). $\operatorname{int} Q_{i_1\ldots i_s} \setminus E_{i_1\ldots i_s} \subset G_{i_1\ldots i_s} \subset \operatorname{int} Q_{i_1\ldots i_s}$, $1 \le s \le k$;

(V). $|G_{i_1\ldots i_s}| < \frac{1}{3}|Q_{i_1\ldots i_s}|, 1 \le s \le k$;

(**VI**).$|G_{i_1\dots i_s}| = \sum_{i_{s+1}} |Q_{i_1\dots i_s i_{s+1}}|, 1 \le s \le k-1$;
(**VII**).$|Q_{i_1\dots i_s} \cap E_{i_1\dots i_s}| > \frac{2}{3}|Q_{i_1\dots i_s}|$, $1 \le s \le k$;
(**VIII**).$Q_{i_1\dots i_s i_{s+1}} \subset G_{i_1\dots i_s}$, $1 \le s \le k-1$.

Observe that from (IV), (VIII) one gets immediately

$$Q_{i_1\dots i_s i_{s+1}} \subset \operatorname{int} Q_{i_1\dots i_s}.$$

Now, to define the families $\{Q_{i_1\dots i_{k+1}}\}$, $\{E_{i_1\dots i_{k+1}}\}$, $\{G_{i_i\dots i_{k+1}}\}$, as before we construct a Vitali cover of each set $G_{i_1\dots i_k} \subset \operatorname{int} Q_{i_1\dots i_k}$, $(i_1 \dots i_k) \in \mathbb{N}^k$, repeating, in fact, the argument we used for G_{i_1} on step two. As a result, we obtain:

- a family of pairwise disjoint τ-closed cubes $\{Q_{i_1\dots i_k i_{k+1}}\}$, $Q_{i_1\dots i_k i_{k+1}} \subset G_{i_1\dots i_k}$;
- a family of τ_d-open sets $\{E_{i_1\dots i_k i_{k+1}}\} \subset \mathcal{U}$;
- a family of τ-open sets $\{G_{i_1\dots i_k i_{k+1}}\}$, with $G_{i_1\dots i_k i_{k+1}} \subset \operatorname{int} Q_{i_1\dots i_k i_{k+1}}$;

so that the properties (IV), (V), (VII) are satisfied for $s = k+1$ and the properties (VI), (VIII) are satisfied for $s = k$. The verification of these facts is immediate, and we omit it to avoid quite unnecessary repetitions.

We have thus obtained inductively the three sequences of families of sets:

$$\begin{array}{l} \{Q_{i_1}\},\ \{Q_{i_1 i_2}\},\ \{Q_{i_1\dots i_k}\},\dots \\ \{E_{i_1}\},\ \{E_{i_1 i_2}\},\ \{E_{i_1\dots i_k}\},\dots \\ \{G_{i_1}\},\ \ \{G_{i_1 i_2}\}, \{G_{i_1\dots i_k}\},\dots \end{array}$$

satisfying the conditions (IV)–(VIII) for all $k \in \mathbb{N}$.

Given a Lebesgue measurable set $A \subset \mathbb{R}^n$, we let $D(A)$ be the set of all density points of A belonging to A. It is clear that $D(A)$ is a τ_d-open set.

Let us consider the following families of sets

$$\{D((\operatorname{int} Q_{i_1\dots i_k} \cap E_{i_1\dots i_k}) \setminus G_{i_1\dots i_k}) : (i_1 \dots i_k) \in \mathbb{N}^k\},\quad k \in \mathbb{N}. \tag{6}$$

Evidently, elements of (6) are τ_d-open sets, each of them being contained in some element of $\mathcal{U}$, so

$$\mathcal{W} = \bigcup_{k=1}^{\infty} \{\{D((\operatorname{int} Q_{i_1\dots i_k} \cap E_{i_1\dots i_k}) \setminus G_{i_1\dots i_k}) : (i_1 \dots i_k) \in \mathbb{N}^k\}\}$$

is a refinement of $\mathcal{U}$. It is also clear that each family (6) is composed of pairwise disjoint sets (for the cubes $\{Q_{i_1 \dots i_k}\}$ are pairwise disjoint whenever k is fixed). Next we shall verify that all elements of $\mathcal{W}$ are pairwise disjoint. It is sufficient, in view of the previous remark, to show that sets belonging to different families (6) are disjoint. To this end we take two multiindexes $(i_1 \dots i_k)$, $(j_1 \dots j_m)$, $m > k$, and we check that

$$D((\operatorname{int} Q_{i_1 \dots i_k} \cap E_{i_1 \dots i_k}) \setminus G_{i_1 \dots i_k}) \cap D((\operatorname{int} Q_{j_1 \dots j_m} \cap E_{j_1 \dots j_m}) \setminus G_{j_1 \dots j_m}) = \emptyset. \quad (7)$$

There are two cases to consider. At first let $(i_1 \dots i_k) \neq (j_1 \dots j_k)$. Then obviously

$$Q_{i_1 \dots i_k} \cap Q_{j_1 \dots j_k} = \emptyset; \quad (8)$$

but since $Q_{j_1 \dots j_k \dots j_m} \subset Q_{j_1 \dots j_k}$, the relation (8) yields $Q_{i_1 \dots i_k} \cap Q_{j_1 \dots j_m} = \emptyset$, whence (7) follows. In the second case suppose that $(i_1 \dots i_k) = (j_1 \dots j_k)$. Since $m > k$ we get by (IV), (VIII) that

$$Q_{j_1 \dots j_{m-1} j_m} \subset G_{j_1 \dots j_{m-1}} \subset G_{j_1 \dots j_{m-2}} \subset \dots \subset G_{j_1 \dots j_k} \subset G_{i_1 \dots i_k},$$

whence $Q_{j_1 \dots j_m} \cap E_{j_1 \dots j_m} \setminus G_{j_1 \dots j_m} \subset G_{i_1 \dots i_k}$ which again immediately implies (7).

So the family $\mathcal{W}$ is composed of pairwise disjoint τ_d-open sets and forms a refinement of $\mathcal{U}$. It remains to show that

$$\operatorname{cl}_d\left(\bigcup\{W : W \in \mathcal{W}\}\right) = \mathbb{R}^n. \quad (9)$$

Since sets of measure zero are τ_d-nowhere dense, we easily observe that the proof of (9) reduces to showing that for each fixed $i_1 \in \mathbb{N}$ we have

$$|Q_{i_1} \setminus \bigcup_{k=1}^{\infty} \bigcup_{(i_1 \dots i_k) \in \mathbb{N}^k} ((\operatorname{int} Q_{i_1 \dots i_k} \cap E_{i_1 \dots i_k}) \setminus G_{i_1 \dots i_k}| = 0$$

or, "more explicitly", what amounts to the same,

$$\begin{aligned} &|Q_{i_1} \setminus \{((\operatorname{int} Q_{i_1} \cap E_{i_1}) \setminus G_{i_1}) \cup \bigcup_{i_2} ((\operatorname{int} Q_{i_1 i_2} \cap E_{i_1 i_2}) \setminus G_{i_1 i_2}) \cup \\ &\dots \cup \bigcup_{i_2 \dots i_k} ((\operatorname{int} Q_{i_1 i_2 \dots i_k} \cap E_{i_1 i_2 \dots i_k}) \setminus G_{i_1 i_2 \dots i_k}) \cup \dots | = 0. \end{aligned} \quad (10)$$

Note that "D" is omitted here. Now we shall prove (10). First we write the almost trivial equality

$$\operatorname{int} Q_{i_1} = ((\operatorname{int} Q_{i_1} \cap E_{i_1}) \setminus G_{i_1}) \cup G_{i_1}. \quad (11)$$

In view of the Vitali cover of G_{i_1} we may write

$$G_{i_1} = \bigcup_{i_2} Q_{i_1 i_2} \cup H_{i_1}, \quad |H_{i_1}| = 0 \tag{12}$$

where all terms in (12) are disjoint. By substituting (12) into the last term of (11) we get

$$\operatorname{int} Q_{i_1} = ((\operatorname{int} Q_{i_1} \cap E_{i_1}) \setminus G_{i_1}) \cup \bigcup_{i_2} \operatorname{int} Q_{i_1 i_2} \cup H_{i_1}. \tag{13}$$

Applying the decomposition of type (11) for $\operatorname{int} Q_{i_1 i_2}$ we obtain

$$\operatorname{int} Q_{i_1 i_2} = ((Q_{i_1 i_2} \cap E_{i_1 i_2}) \setminus G_{i_1 i_2}) \cup G_{i_1 i_2}$$

and after substituting it into (13) we get

$$\begin{aligned} \operatorname{int} Q_{i_1} =& ((\operatorname{int} Q_{i_1} \cap E_{i_1}) \setminus G_{i_1}) \cup \bigcup_{i_2} ((\operatorname{int} Q_{i_1 i_2} \cap E_{i_1 i_2}) \setminus G_{i_1 i_2}) \\ & \cup \bigcup_{i_2} G_{i_1 i_2} \cup H_{i_1}. \end{aligned} \tag{14}$$

Next we repeat the procedure applying (VI) to $G_{i_1 i_2}$ which yields

$$G_{i_1 i_2} = \bigcup_{i_3} Q_{i_1 i_2 i_3} \cup H_{i_1 i_2}, \ |H_{i_1 i_2}| = 0, \tag{15}$$

all the terms in (15) being, of course, disjoint. Again, rewriting $\operatorname{int} Q_{i_1 i_2 i_3}$ in the form (11) we get

$$\operatorname{int} Q_{i_1 i_2 i_3} = ((\operatorname{int} Q_{i_1 i_2 i_3} \cap E_{i_1 i_2 i_3}) \setminus G_{i_1 i_2 i_3}) \cup G_{i_1 i_2 i_3}$$

which, in view of (15), permits to rewrite (14) as

$$\begin{aligned} \operatorname{int} Q_{i_1} &= ((\operatorname{int} Q_{i_1} \cap E_{i_1}) \setminus G_{i_1}) \cup \bigcup_{i_2} ((\operatorname{int} Q_{i_1 i_2} \cap E_{i_1 i_2}) \setminus G_{i_1 i_2}) \\ \cup \bigcup_{i_2 i_3} & ((\operatorname{int} Q_{i_1 i_2 i_3} \cap E_{i_1 i_2 i_3}) \setminus G_{i_1 i_2 i_3}) \cup \bigcup_{i_2 i_3} G_{i_1 i_2 i_3} \cup \bigcup_{i_2} H_{i_1 i_2} \cup H_{i_1}. \end{aligned}$$

Now it is clear that by the evident induction procedure, we have for each $k \in \mathbb{N}, \ k > 2$,

$$\begin{aligned} \operatorname{int} Q_{i_1} = A_{i_1} \cup \bigcup_{i_2} A_{i_1 i_2} \cup \cdots \cup \bigcup_{i_2 \ldots i_k} A_{i_1 \ldots i_k} \cup \bigcup_{i_2 \ldots i_k} G_{i_1 \ldots i_k} \\ \cup H_{i_1} \cup \bigcup_{i_2} H_{i_1 i_2} \cup \cdots \cup \bigcup_{i_2 \ldots i_{k-1}} H_{i_1 \ldots i_{k-1}}, \end{aligned} \tag{16}$$

where

$$A_{i_1\ldots i_s} = ((\operatorname{int} Q_{i_1\ldots i_s} \cap E_{i_1\ldots i_s}) \setminus G_{i_1\ldots i_s}),\ 1 \le s \le k,$$

and all $H_{i_1\ldots i_s}$, $1 \le s \le k-1$, are null sets. From (16) it follows that the proof of (10) amounts to showing that

$$\lim_{k\to\infty} |\bigcup_{i_2\ldots i_k} G_{i_1\ldots i_k}| = 0. \tag{17}$$

The required estimate (17) could be easily carried out using properties (V), (VI) consecutively:

$$\begin{aligned} |\bigcup_{i_2\ldots i_k} G_{i_1\ldots i_k}| &\le \sum_{i_2\ldots i_k} |G_{i_1\ldots i_k}| \le 3^{-1} \sum_{i_2\ldots i_k} |Q_{i_1\ldots i_k}| \\ &= 3^{-1} \sum_{i_2\ldots i_{k-1}} |G_{i_1\ldots i_{k-1}}| \le 3^{-2} \sum_{i_2\ldots i_{k-1}} |Q_{i_1\ldots i_{k-1}}| \\ &= 3^{-2} \sum_{i_2\ldots i_{k-2}} |G_{i_1\ldots i_{k-2}}| \le \cdots \\ &\le 3^{2-k} \sum_{i_2} |G_{i_1 i_2}| \le 3^{1-k} \sum_{i_2} |Q_{i_1 i_2}| = 3^{1-k}|G_{i_1}| \le 3^{-k}|Q_{i_1}| \end{aligned}$$

whence we get (17). Finally, (10) and therefore (9), follow from (16) and (17), thereby completing the whole proof. □

Corollary 4.2. $(\mathbb{R}^n, \tau_d)$ *is an almost Lindelöf space.*

References

[1] R. A. Aleksandrian, E. A. Mirzahanian, *General Topology* (in Russian), Moscow, (1979).

[2] S. G. Crossley, *A note on semitopological properties*, Proc. Amer. Math. Soc., **72** (1978), 409–412.

[3] S. G. Crossley, S. K. Hildebrand, *Semi-closure*, Texas J. Sci., **22** (1971), 99–112.

[4] S. G. Crossley, S .K. Hildebrand, *Semi-closed sets and semicontinuity in topological spaces*, Texas J. Sci., **22** (1971), 123–126.

[5] S. G. Crossley, S. K. Hildebrand, *Semi-topological properties*, Fundam. Math., **74** (1972), 233–254.

[6] Á. Császár, *General topology*, Budapest, (1978).

[7] R. Engelking, *General Topology*, Warszawa, PWN, (1977).

[8] J. Ewert, *Invariance of almost paracompactness under some operations*, Tatra Mount. Math. Publ., **19** (2000), 97–103.

[9] J. Ewert, T. Lipski, *Quasicontinuous multivalued mappings*, Math. Slovaca, **33** (1983), 69–74.

[10] H. Hashimoto, *On the $*$-topology and its applications*, Fundam. Math., **91** (1976), 5–10.

[11] S. Kempisty, *Sur les fonctions quasicontinues*, Fundam. Math., **19** (1932), 184–197.

[12] N. Levine, *Semi-open sets and semi-continuity in topological spaces*, Amer. Math. Monthly, **70** (1963), 36–41.

[13] J. Lukeš, J. Malý, L. Zajíček, *Fine topology methods in real analysis and potential theory*, Lecture Notes in Math., Springer Verlag, Berlin Heidelberg (1986).

[14] A. K. Misra, *A topological view of P-spaces*, General Top. Appl., **2** (1972), 349–362.

[15] J. Nagata, *Modern general topology*, Amsterdam (1968).

[16] J. C. Oxtoby, *Measure and category*, Springer Verlag, New York, Heidelberg, Berlin (1971).

[17] M. K. Singal, S. P. Arya, *On* **m**-*paracompact spaces*, Math. Ann., **181** (1969), 119–133.