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THOMSON'S VARIATIONAL MEASURE AND NONABSOLUTELY CONVERGENT INTEGRALS

Abstract

In 1987 Jarník and Kurzweil [11] proved the following result: *A function $F : [a, b] \to \mathbb{R}$ is AC^*G on $[a, b]$ if and only if μ_F^* (Thomson's variational measure) is absolutely continuous on $[a, b]$ and F is derivable a.e. on $[a, b]$*. But condition "F is derivable *a.e.* on $[a, b]$" is superfluous, as it was shown in [3]. In this paper we shall improve this result (from where we obtain an answer to a question of Faure [9]). Then using Faure's definition for a Kurzweil-Henstock-Stieltjes integral with respect to a function ω, we give corresponding definitions for: a Denjoy*-Stieltjes integral with respect to ω, a Ward-Perron-Stieltjes integral with respect to ω, a Henstock-Stieltjes variational integral with respect to ω, and we show that the four integrals are equivalent.

1 Introduction

Throughout the paper we shall use Thomson's variational measure μ_F^* for a function F (see Definition 2.4).

In 1987, Jarník and Kurzweil proved the following result [11] (see 3.19, p. 656):

Theorem A. *A function $F : [a, b] \to \mathbb{R}$ is AC^*G on $[a, b]$ if and only if μ_F^* is absolutely continuous and F is derivable a.e. on $[a, b]$.*

Key Words: Thomson's variational measure, VB^*, VB, VB^*G, AC^*, VB^*G, the Kurzweil-Henstock-Stieltjes integral, the Denjoy*-Stieltjes integral.
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Almost three years later, P. Y. Lee proved the same theorem [13] (see Theorem 4, p. 757), without any reference to the paper of Jarník and Kurzweil. A variant of Theorem A is presented by W. F. Pfeffer in [15] (see Theorem 6.4.4, p. 115), and he mentioned neither Jarník and Kurzweil's theorem, nor P. Y. Lee's result. Not knowing the paper of Jarník and Kurzweil, in 1994 [3], we improved Theorem A (giving credit to P. Y. Lee), showing that the condition "F is derivable *a.e.* on $[a,b]$" is superfluous:

Theorem B. ([3], Corollary 1, (i), (vii) or [4], Corollary 2.27.1, (i), (vii)). *A function $F : [a,b] \to \mathbb{R}$ is AC^*G on $[a,b]$ if and only if μ_F^* is absolutely continuous.*

In proving Theorem B, otherwise than Jarník and Kurzweil, P. Y. Lee and W. F. Pfeffer, we haven't used the Kurzweil-Henstock theory. In 1996, using the Kurzweil-Henstock theory, Bongiorno, Di Piazza and Skvortsov also proved Theorem B without mentioning [3] (see Theorems 3 and 4 of [1]).

Using Theorem B and a result of Thomson (see Theorem 3.1), we can easily deduce the following theorem:

Theorem C. *Let $F : [a,b] \to \mathbb{R}$ be a function such that μ_F^* is absolutely continuous. Then μ_F^* is σ-finite on $[a,b]$.*

In this paper we shall improve Theorem B and Theorem C (see Theorem 5.1 and Theorem 3.2), and then use these results to answer to a question of Faure [9]. In performing this task we shall use many definitions and results of Faure's paper [9].

Using Faure's definition for a Kurzweil-Henstock-Stieltjes integral with respect to a function ω, we give corresponding definitions for: a Denjoy*-Stieltjes integral with respect to ω, a Ward-Perron-Stieltjes integral with respect to ω, a Henstock-Stieltjes variational integral with respect to ω, and we show that the four integrals are equivalent.

2 Notations, Definitions and Preliminary Results

We denote by $m^*(X)$ the outer measure of the set X and by $m(A)$ the Lebesgue measure of A, whenever $A \subseteq \mathbb{R}$ is Lebesgue measurable. For the definitions of VB, VB^* and AC^*, see [16]. Let $\langle x, y\rangle$ denote the closed interval with the endpoints x and y. We denote by $\mathcal{P}(E) = \{X : X \subseteq E\}$ whenever $E \subseteq \mathbb{R}$. Let $C[a,b] = \{F : [a,b] \to \mathbb{R} : F \text{ is continuous on } [a,b]\}$ and $\mathcal{B}or(X) = \{A \subset X : A \text{ is a Borel set}\}$. We denote by $\mathcal{O}(F; X)$ the oscillation of the function F on the set X. Let C_f denote the set of continuity points of the function f.

Definition 2.1. Let $F : [a,b] \to \mathbb{R}$, and let P be a closed subset of [a,b], $c = \inf(P)$, $d = \sup(P)$. Let $F_P : [c,d] \to \mathbb{R}$ be defined as follows: $F_P(x) = F(x)$, $x \in P$ and F_P is linear on each $[c_k, d_k]$, where $\{(c_k, d_k)\}_{k\geq 1}$ are the intervals contiguous to P.

Definition 2.2. ([17]). A sequence $\{E_n\}$ of sets whose union is E is called an E-form with parts E_n. If, in addition, each part E_n is closed in E (i.e. $E_n = \overline{E}_n \cap E$) then the E-form is said to be closed. An expanding E-form is called an E-chain.

Definition 2.3. Let $f : [a,b] \to \mathbb{R}$ and $E \subseteq [a,b]$. f is said to be VB^*G (respectively AC^*G) on E if there is an E-form $\{E_n\}$ such that f is VB^* (respectively AC^*) on each E_n. Note that AC^*G here differs from the definitions given in [16], because f is not supposed to be continuous.

Definition 2.4. Let $E \subset \mathbb{R}$, $\delta : E \to (0,+\infty)$,

$$\beta^*(E;\delta) = \Big\{ \big(\langle x,y\rangle, x\big) \,:\, x \in E,\ y \in \big(x-\delta(x), x+\delta(x)\big) \Big\}.$$

The finite set $\pi = \big\{\big(\langle x_i,y_i\rangle, x_i\big)\big\}_{i=1}^n \subset \beta^*(E;\delta)$ is said to be a partition if the $\{\langle x_i,y_i\rangle\}_{i=1}^n$ is a set of nonoverlapping closed intervals. Let $f : \mathbb{R} \to \mathbb{R}$,

$$V_\delta^*(f;E) = \sup\Big\{ \sum_{(\langle x,y\rangle,x)\in\pi} \big|f(y)-f(x)\big| \,:\, \pi \subset \beta^*(E;\delta) \text{ is a partition} \Big\},$$

and

$$\mu_f^*(E) = \inf_\delta V_\delta^*(f;E).$$

Note that this μ_f^* is the same as that of Thomson [19, p. 186], and it is also identical with Thomson's $\mathcal{S}_o$-μ_F of [18] and Faure's m_F [9].

Definition 2.5. Let X be a nonempty set and $\mathcal{P}(X) = \{E : E \subseteq X\}$. Let $\alpha : \mathcal{P}(X) \to [0,+\infty]$ be a set function with $\alpha(\emptyset) = 0$. α is said to be σ-finite on E if there exists a sequence $\{E_i\}_i$ of sets such that $E \subset \cup_i E_i$ and $\alpha(E_i) \neq +\infty$ for each i.

Definition 2.6. A function $\alpha : \mathcal{P}(E) \to \overline{\mathbb{R}}$ is said to be absolutely continuous on $E \subseteq \mathbb{R}$ if $\alpha(Z) = 0$ whenever $Z \subseteq E$ and $m^*(Z) = 0$.

Definition 2.7. [9] Let $F, \omega : [a,b] \to \mathbb{R}$, $\omega \in VB^*G$ and $\omega \in C[a,b]$.

- F is called ω-Lipschitzian on a set $E \subset [a,b]$ or LZ_ω on E, if there exists $C > 0$ such that $\mu_F^*(A) \leq C \cdot \mu_\omega^*(A)$ for every subset $A \subseteq E$. The function F is called generalized ω-Lipschitzian or $LZ_\omega G$, if there exists an $[a,b]$-form $\{E_n\}$ such that F is ω-Lipschitzian on each E_n.

- Similarly, F is called ω-absolutely continuous on a set E, or AC_ω on E, if for any $\epsilon > 0$ there exists $\delta > 0$ such that $A \subseteq E$ and $\mu_\omega^*(A) < \delta$ imply $\mu_F^*(A) < \epsilon$. And it is called generalized ω-absolutely continuous, or $AC_\omega G$, if there exists an $[a,b]$-form $\{E_n\}$ such that F is ω-absolutely continuous on each E_n. If in addition each set E_n is closed then we say that $F \in [AC_\omega G]$.

- One says that F is ω-variational normal or shortly ω-normal, if $\mu_\omega^*(A) = 0$ implies $\mu_F^*(A) = 0$.

Definition 2.8. Let μ be a positive measure defined on a σ-algebra $\mathcal{A}$ of X. A real measure ν defined on $\mathcal{A}$ is absolutely continuous with respect to μ (shortly $\nu \ll \mu$) if $\nu(A) = 0$ whenever $\mu(A) = 0$ and $A \in \mathcal{A}$.

Remark 2.1. If F is ω-normal then the restrictions of the outer measures μ_F^* and μ_ω^* on a σ-algebra $\mathcal{A}$ satisfy $\mu_F^* \ll \mu_\omega^*$.

Proposition 2.1. [16, p. 31] *If μ is a positive measure on a σ-algebra $\mathcal{A}$ of X and ν is a finite positive measure on $\mathcal{A}$ then $\nu \ll \mu$ if and only if for $\epsilon > 0$ there is a $\delta > 0$ such that $\nu(A) < \epsilon$ whenever $\mu(A) < \delta$.*

3 An Extension of Theorem C

Lemma 3.1. *Let $F : [a,b] \to \mathbb{R}$, $Q \subset [a,b]$ a compact set and $\mu^* : \mathcal{P}(Q) \to [0, +\infty]$ an outer measure such that for every compact subset S of Q, there exists a G_δ-set $Z \subset S$ with $\overline{Z} = S$ and $\mu^*(Z) = 0$. Then the following assertions are equivalent:*

(i) $F \in VB^*G$ on Q;

(ii) each closed subset S of Q contains a portion on which $F \in VB^*$;

(iii) $F \in VB^*G$ on Z whenever Z is a G_δ-subset of Q and $\mu^*(Z) = 0$.

Proof. (i) $\Leftrightarrow$ (ii) See Theorem 9.1 of [16, p. 233] (F needs not to be continuous on Q, because $F \in VB^*$ on $A \subset Q$ implies that $F \in VB^*$ on $\overline{A}$, see Theorem 7.1 of [16, p. 229]).

(i) $\Rightarrow$ (iii) This is obvious.

(iii) $\Rightarrow$ (ii) Let S be a closed subset of Q (so S is compact). Then there is a G_δ-set $Z \subset S$, with $\overline{Z} = S$ and $\mu^*(Z) = 0$ (see the condition on μ^*). By (iii) $F \in VB^*G$ on Z, so there exists a Z-form $\{Z_i\}$ such that $F \in VB^*$ on each Z_i. Then $F \in VB^*$ on each $\overline{Z}_i$ (see Theorem 7.1 of [16, p. 229]). By

Baire's Category Theorem [16, p. 54], there is an open interval I such that $\emptyset \neq I \cap Z \subset \overline{Z}_{i_o}$ for some i_o. But

$$\emptyset \neq I \cap S = I \cap \overline{Z} \subset \overline{I \cap Z} \subset \overline{Z}_{i_o}\,.$$

Indeed, let $x \in I \cap \overline{Z}$ and let V_x be a neighborhood of x. Then $I \cap V_x$ is a neighborhood of $x \in \overline{Z}$ too, so $V_x \cap I \cap Z \neq \emptyset$. Hence $x \in \overline{I \cap Z}$ and the above relation is proved. It follows that $F \in VB^*$ on $I \cap S$. □

Lemma 3.2. (Lemma 4.2 of [9]) *Let $\omega : \mathbb{R} \to \mathbb{R}$, $\omega \in C[a,b]$, $\omega(x) = \omega(a)$ for $x < a$, $\omega(x) = \omega(b)$ for $x > b$, and $E \subset [a,b]$. If $\mu_\omega^*(E) \neq +\infty$ the function $V : \mathbb{R} \to [0,+\infty)$,*

$$V(x) = \begin{cases} 0 & \text{if } x \in (-\infty, a] \\ \mu_\omega^*\big(E \cap [a,x]\big) & \text{if } x \in (a,+\infty) \end{cases}$$

is continuous, increasing and bounded on $\mathbb{R}$.

Moreover, if $x, y \in [a,b]$, $x < y$ then:

$$V(y)-V(x) = \mu_\omega^*(E\cap[x,y]) = \mu_\omega^*(E\cap(x,y)) = \mu_\omega^*(E\cap[x,y)) = \mu_\omega^*(E\cap(x,y])\,.$$

PROOF. That V is continuous follows by Lemma 4.2 of [9], and that V is increasing and bounded is evident. □

Lemma 3.3. *Let $\omega : \mathbb{R} \to \mathbb{R}$ be a continuous function, $\omega(x) = \omega(a)$ for $x < a$, $\omega(x) = \omega(b)$ for $x > b$, and let $S \subset [a,b]$ be a G_δ-set with $\mu_\omega^*(S) \neq +\infty$. Then there is a null G_δ-set $Z \subset S$ such that $\overline{Z} \supset S$ and $\mu_\omega^*(Z) = 0$.*

PROOF. Let d be the usual distance on $\mathbb{R}$ (i.e., $d(x,y) = |x-y|$ for $x, y \in \mathbb{R}$). Since $(\mathbb{R}, d)$ is separable, it follows that (S,d) is also a separable metric space (see for example [2, Theorem 12]). Thus there is a countable set $Z_1 = \{x_1, x_2, \ldots\} \subset S$ such that $\overline{Z}_1 \cap S = S$. Let V be the function defined in Lemma 3.2, with $E = S$. Let $j \in \mathbb{N}$. For each x_i let a_{ji}, b_{ji} be such that $x \in (a_{ji}, b_{ji})$,

$$V(b_{ji}) - V(a_{ji}) < \frac{1}{2^{j+i}} \quad \text{and} \quad (b_{ji} - a_{ji}) < \frac{1}{2^{j+i}}$$

(this is possible because V is continuous and increasing). Let

$$G_j = S \cap \big(\cup_{i=1}^\infty (a_{ji}, b_{ji})\big) \quad \text{and} \quad Z = \cap_{j=1}^\infty G_j\,.$$

Then Z is a G_δ-subset of S that contains Z_1. Hence $\overline{Z} \supset S$ and

$$\mu_\omega^*(Z) \le \mu_\omega^*(G_j) \le \sum_{i=1}^{\infty} \mu_\omega^*\big((a_{ji}, b_{ji}) \cap S\big) =$$

$$= \sum_{i=1}^{\infty}\big(V(b_{ji}) - V(a_{ji})\big) < \sum_{i=1}^{\infty} \frac{1}{2^{j+i}} = \frac{1}{2^j} \quad \text{for all } j \in \mathbb{N}\,.$$

Thus $\mu_\omega^*(Z) = 0$. Clearly Z is a null set. □

Theorem 3.1 (Thomson)**.** [18, p. 94]. *Let $F : [a,b] \to \mathbb{R}$, $A \subset [a,b]$. If F is continuous at each point of A then $F \in VB^*G$ on A if and only if μ_F^* is σ-finite on A.*

Lemma 3.4. *Let $F : [a,b] \to \mathbb{R}$, $P = \overline{P} \subset [a,b]$, $F \in VB^*$ on P, $F \in C[a,b]$. Then $\mu_F^*(P) \le 2V^*(F;P)$.*

Proof. We shall use Thomson's technique of [18, p. 94]. Let $A = \{x \in P : x$ is an isolated point of P at one side at least$\}$. By [16, p. 260], A is a countable set. Since $F \in C[a,b]$, $\mu_F^*(A) = 0$. Let $\delta : P \setminus A \to (0, +\infty)$. Let $\pi = \big\{(\langle x_i, y_i\rangle, x_i)\big\}_{i=1}^{p} \subset \beta^*(P \setminus A; \delta)$ be a partition. Split π into

$$\pi_1 = \big\{([x_i, y_i], x_i)\big\}_{i=1}^{m} \quad \text{and} \quad \pi_2 = \big\{([y_i, x_i], x_i)\big\}_{i=m+1}^{p}\,.$$

In both cases we label the intervals from the left to the right. Let $c = \inf P$, $d = \sup P$, $y_m^* \in [y_m, d) \cap (P \setminus A)$ and $x_{m+1}^* \in (c, y_{m+1}] \cap (P \setminus A)$. Then we have

$$\sum_{\pi}\big|F(y_i) - F(x_i)\big| \le \sum_{i=1}^{m-1} \mathcal{O}\big(F; [x_i, x_{i+1}]\big) + \mathcal{O}\big(F; [x_m, y_m^*]\big)$$

$$+\mathcal{O}\big(F; [x_{m+1}^*, x_{m+1}]\big) + \sum_{i=m+1}^{p-1} \mathcal{O}\big(F; [x_i, x_{i+1}]\big) < 2V^*(F;P)\,.$$

Thus $V_\delta^*(F; P\setminus A) \le 2V^*(F;P)$. It follows that $\mu_F^*(P) \le \mu_F^*(P\setminus A) + \mu_F^*(A) \le 2V^*(F;P)$. □

Theorem 3.2 (An extension of Theorem C)**.** *Let $F : [a,b] \to \mathbb{R}$ be ω-normal, where $\omega \in C[a,b]$ is a VB^*G function. Then $F \in C[a,b]$ and F is VB^*G on $[a,b]$ (or equivalently μ_F^* is σ-finite on $[a,b]$, see Theorem 3.1).*

Proof. Since ω is continuous at $x \in [a,b]$, we have that $\mu^*_\omega(\{x\}) = 0$, so F being ω-normal, $\mu^*_F(\{x\}) = 0$. It follows that F is continuous at x, so on $[a,b]$. Since ω is VB^*G on $[a,b]$, by Theorem 7.1 of [16, p. 229], there exists a sequence $\{Q_n\}$ of compact sets such that $[a,b] = \cup_n Q_n$ and ω is VB^* on each Q_n. By Lemma 3.4, $\mu^*_\omega(Q_n) \neq +\infty$. Fix some n and let S be a compact subset of Q_n. Then $\mu^*_\omega(S) \neq +\infty$, so by Lemma 3.3, there is a G_δ-set $Z \subset S$, with $\overline{Z} = S$ and $\mu^*_\omega(Z) = 0$. Thus $(\mu^*_\omega)_{|\mathcal{P}(Q_n)}$ satisfies the condition of Lemma 3.1. Let Y be a subset of Q_n such that $\mu^*_\omega(Y) = 0$. Since F is ω-normal, $\mu^*_F(Y) = 0$, and by Theorem 3.1, F is VB^*G on Y. It follows that F is VB^*G on each Q_n (see Lemma 3.1). Hence F is VB^*G on $[a,b]$. □

4 An Answer to a Question of Faure

Lemma 4.1 (Thomson)**.** (A particular case of Theorem 43.1 of [18], p. 101). *Let $F : [a,b] \to \mathbb{R}$ and $E \subseteq [a,b]$. Then $m^*(F(E)) \le \mu^*_F(E)$.*

From this lemma we obtain immediately the following corollary.

Corollary 4.1 (Faure)**.** (Lemma 5.1 of [9]). *Let $F : [a,b] \to \mathbb{R}$ and $E \subseteq [a,b]$ with $\mu^*_F(E) = 0$. Then $m(F(E)) = 0$.*

Lemma 4.2. *Let $f : [a,b] \to \mathbb{R}$, $E \subseteq [a,b]$ and $A \subseteq \{x \in E : f$ is continuous at $x\}$. If $f \in VB^*G$ on E then $m^*(f(A)) = 0$ if and only if $\mu^*_f(A) = 0$.*

Proof. Since $\mathcal{S}_o$-μ_f and μ^*_f are identical, the assertion follows immediately by Theorem 8 of [5] (which is an extension of Thomson's Corollary 43.4 of [18, p. 103]). □

Theorem 4.1. *Let $F : [a,b] \to \mathbb{R}$ and $A \subseteq [a,b]$. The following assertions are equivalent:*

(i) $\mu^*_F(E) = 0$;

(ii) F is continuous at each point of E, $m(F(E)) = 0$ and $\mu^*_F(E) \neq +\infty$;

(iii) F is continuous at each point of E, $m(F(E)) = 0$ and μ^*_F is σ-finite;

(iv) F is continuous at each point of E, $m(F(E)) = 0$ and F is VB^*G on E.

Proof. (i) ⇒ (ii) That F is continuous at each point of E and $\mu^*_F(E) \neq +\infty$ is obvious. By Corollary 4.1 we also have that $m(F(E)) = 0$.

(ii) ⇒ (iii) This is evident.

(iii) ⇔ (iv) See Theorem 3.1.

(iv) ⇒ (i) See Lemma 4.2. □

Remark 4.1. Theorem 4.1, (i) $\Leftrightarrow$ (ii) is in fact Proposition 5.3 of Faure [9, p. 121] (our proof is different).

Example. C. A. Faure asked if in Theorem 4.1 (ii), "$\mu_F^*(E) \neq +\infty$" can be replaced by "$F \notin VB^*G$ but F is derivable *a.e.* on E". The answer is no.

Proof. Let C be the Cantor ternary set. We say that $(a_{11}, b_{11}) = \left(\frac{1}{3}, \frac{2}{3}\right)$ is an open interval from the first step, $(a_{21}, b_{21}) = \left(\frac{1}{9}, \frac{2}{9}\right)$ and $(a_{22}, b_{22}) = \left(\frac{7}{9}, \frac{8}{9}\right)$ are the two intervals from the second step. In general the 2^{n-1} open intervals of length $\frac{1}{3^n}$ contiguous to C are said to be the intervals from the step n. We denote them from the left to the right as $\{(a_{ni}, b_{ni})\}_{i=1}^{2^{n-1}}$. Let $c_{ni} = \dfrac{a_{ni} + b_{ni}}{2}$ and let $[a'_{ni}, b'_{ni}]$ be an interval contained in (a_{ni}, b_{ni}) centered in c_{ni}. Let $F : [0,1] \to [0,1]$,

$$F(x) = \begin{cases} 0 & \text{if } x \in C \\ \frac{1}{2^{n-1}} & \text{if } x \in \cup_{i=1}^{2^{n-1}} [a'_{ni}, b'_{ni}] \\ \text{linear} & \text{on } [a_{ni}, a'_{ni}] \text{ and } [b'_{ni}, b_{ni}]\,. \end{cases}$$

Then we have:

(i) $F \in C[0,1]$;

(ii) F is derivable *a.e.* on $[0,1]$;

(iii) $F'(x) = 0$ *a.e.* on $E = C \cup \left(\cup_{n=1}^{\infty} \cup_{i=1}^{2^{n-1}} (a'_{ni}, b'_{ni})\right)$;

(iv) $m(F(E)) = 0$;

(v) $F \notin VB^*G$ on C (so on E), or equivalently (see Theorem 3.1), μ_F^* is not σ-finite on C (so on E).

$\square$

5 An Extension of Theorem B

Lemma 5.1. *Let $F : [a,b] \to \mathbb{R}$, $F \in C[a,b]$, $E \subset [a,b]$. If $\mu_F^*(E) < +\infty$ then there is a $F_{\sigma\delta}$-set H such that $E \subset H$ and $\mu_F^*(H) = \mu_F^*(E)$.*

Proof. For $\epsilon > 0$ there is a $\delta_\epsilon : E \to (0, +\infty)$ such that $V_{\delta_\epsilon}^*(F; E) < \mu_F^*(E) + \epsilon/2$. Let $E_n^\epsilon = \{x \in E : \delta_\epsilon(x) > 1/n\}$. Then $E = \cup_{n=1}^{\infty} E_n^\epsilon$ and $\{E_n^\epsilon\}_n$ is an expanding sequence of sets. Let

$$\pi = \left\{ \left(\langle x_i, y_i \rangle, x_i\right) \right\}_{i=1}^{p} \subset \beta\left(\overline{E_n^\epsilon}; \frac{1}{2n}\right).$$

Since $F \in C[a,b]$, for each i one can choose $x_i^* \in E_n^\epsilon$ such that

$$|x_i^* - x_i| < \frac{1}{2n} \quad \text{and} \quad \big|F(x_i^*) - F(x_i)\big| < \frac{\epsilon}{2^{i+1}}\,.$$

1) If $y_i < x_i = x_j < y_j$, then one chooses $x_i^* = x_j^* \in (y_i, y_j) \cap E_n^\epsilon$.

2) If $x_i \neq x_j$ for all $i \neq j$, then one chooses x_i^* such that $|x_i^* - x_i| < \frac{1}{2}\delta(x_i, C_i)$ where $C_i = \cup_{j\neq i}\langle x_j, y_j\rangle$.

Since $|y_i - x_i| < \frac{1}{2n}$, it follows that $|x_i^* - y_i| < \frac{1}{n}$, so

$$\big(\langle x_i^*, y_i\rangle, x_i^*\big) \in \beta\Big(E_n^\epsilon; \frac{1}{n}\Big) \subset \beta(E_n^\epsilon; \delta_\epsilon) \subset \beta(E; \delta_\epsilon)\,.$$

We obtain that

$$\begin{aligned}\sum_{i=1}^{p}\big|F(y_i) - F(x_i)\big| &\le \sum_{i=1}^{p}\big|F(x_i) - F(x_i^*)\big| + \sum_{i=1}^{p}\big|F(y_i) - F(x_i^*)\big| \\ &< \frac{\epsilon}{2} + V_{\delta_\epsilon}(F; E) < \epsilon + \mu_F^*(E)\,.\end{aligned}$$

Hence

$$\mu_F^*(\overline{E_n^\epsilon}) \le V^*_{\frac{1}{2n}}(F; \overline{E_n^\epsilon}) < \epsilon + \mu_F^*(E)\,.$$

Let $H^\epsilon = \cup_{n=1}^\infty \overline{E_n^\epsilon}$. Since any Borelian subset of $[a,b]$ is μ_F^* measurable and $\{\overline{E_n^\epsilon}\}_{n=1}^\infty$ is an E-chain (so $\{\overline{E_n^\epsilon}\}_{n=1}^\infty$ is an expanding sequence of sets), we have

$$\mu_F^*(H^\epsilon) = \lim_{n\to\infty}\mu_F^*(\overline{E_n^\epsilon}) \le \epsilon + \mu_F^*(E)\,.$$

Let $H = \cap_{k=1}^\infty H^{\frac{1}{k}}$. Clearly $E \subset H$ and H is of $F_{\sigma\delta}$-type. We have

$$\mu_F^*(E) \le \mu_F^*(H) \le \mu_F^*(H^{\frac{1}{k}}) \le \frac{1}{k} + \mu_F^*(E)\,,$$

for all $k = 1, 2, \ldots$. Thus $\mu_F^*(E) = \mu_F^*(H)$. □

Remark 5.1. That μ_F^* in Lemma 5.1 is Borel regular was pointed out (without proof) by Thomson in [18, p. 43]. In fact we can prove even more, see Lemma 5.4.

Lemma 5.2. [7, Corollary 5] *Let $f, g : [a,b] \to \mathbb{R}$, $E \subseteq [a,b]$. If $f, g \in VB^*$ on E and $f = g$ on E, then*

$$\mu_f^*(E \cap C_f \cap C_g) = \mu_g^*(E \cap C_f \cap C_g)\,.$$

Particularly, $\mu_f^*(E \cap C_f) = \mu_{\tilde f}^*(E \cap C_f)$, where $\tilde f = f_{\overline{E}\cup\{a,b\}}$ (see Definition 2.1 for the function f_P).

Lemma 5.3. [7, Lemma 5] *Let $f : [a,b] \to \mathbb{R}$ and $E \subseteq [a,b]$. If $f \in VB$ on $[a,b]$ then $\mu_f^*(E \cap C_f) = m^*(V_f(E \cap C_f))$, where $V_f(x) = V(f;[a,x])$).*

Lemma 5.4. *Let $F : [a,b] \to \mathbb{R}$, $F \in C[a,b]$. If F is VB^* on $P = \overline{P} \subset [a,b]$, then for every $E \subset P$ there is a G_δ-set $H \subset P$ such that $\mu_F^*(E) = \mu_F^*(H)$.*

Proof. Note that $\mathcal{S}_o$-$\mu_F \equiv \mu_F^*$ and let $\tilde{F} = F_{P\cup\{a,b\}}$. By Lemma 5.2, $\mu_F^*(X) = \mu_{\tilde{F}}(X)$ for all $X \subset P$, and by Lemma 5.3, $\mu_{\tilde{F}}^*(X) = m^*(V_{\tilde{F}}(X))$ for all $X \subset [a,b]$. Thus

$$\mu_F^*(X) = m^*\big(V_{\tilde{F}}(X)\big) \quad \text{for all } X \subset P\,. \tag{1}$$

Let G be a G_δ-set such that $V_{\tilde{F}}(E) \subset G$ and $m^*(V_{\tilde{F}}(E)) = m(G)$, and let $H = P \cap V_{\tilde{F}}^{-1}(G)$. Then H is a G_δ-set (because $V_{\tilde{F}}$ is a continuous function, so $V_{\tilde{F}}^{-1}(G)$ is a G_δ-set). Clearly $E \subset H$ and by (1) we have

$$\mu_F^*(E) \le \mu_F^*(H) = m^*\big(V_{\tilde{F}}(H)\big) \le m^*(G) = m(G) = m^*\big(V_{\tilde{F}}(E)\big) = \mu_F^*(E)\,,$$

Thus $\mu_F^*(E) = \mu_F^*(H)$. □

Theorem 5.1. *Let $F, \omega : [a,b] \to \mathbb{R}$, $\omega \in VB^*G$ and $\omega \in C[a,b]$. The following assertions are equivalent:*

(i) $F \in LZ_\omega G$;

(ii) F is $AC_\omega G$;

(iii) F is ω-normal.

(iv) There is a closed $[a,b]$-form $\{E_n\}$ such that $\omega, F \in VB^*$ on each E_n and F is AC_ω on each E_n.

(v) There is an $[a,b]$-form $\{E_n\}$ with each E_n a Borel set, such that F is LZ_ω on each E_n.

(vi) $F \in C[a,b]$, F is N_ω and F is VB^*G on $[a,b]$.

Proof. (i) ⇒ (ii) ⇒ (iii) See Lemma 4.3 of [9].

(iii) ⇒ (iv) By Theorem 3.2, F is VB^*G on $[a,b]$, and by Theorem 7.1 of [16, p. 229], there is a sequence of closed sets $\{E_n\}$ with $\cup_n E_n = [a,b]$ such that $\omega, F \in VB^*$ on each E_n. Then $\mu_F^*(E_n) < +\infty$ and $\mu_\omega^*(E_n) < +\infty$ for each n (see Lemma 3.4). Since $\mu^*_{F|\mathcal{B}or(E_n)}$ is a positive finite measure, by Proposition 2.1, it follows that for $\epsilon > 0$ there is a $\delta = \delta(\epsilon, E_n) > 0$ such that $\mu_F^*(A) < \epsilon$ whenever A is a Borel subset of E_n and $\mu_\omega^*(A) < \delta$. But $(\mu_\omega^*)_{|\mathcal{P}(E_n)}$

and $(\mu_F^*)_{|\mathcal{P}(E_n)}$ are both Borel regular (see Lemma 5.1), so for $A \subset E_n$ with $\mu_\omega^*(A) < \delta$, we have $\mu_F^*(A) < \epsilon$ (because there exists $A^* \subset \mathcal{B}or(E_n)$ such that $A \subset A^*$ and $\mu_\omega^*(A) = \mu_\omega^*(A^*)$, $\mu_F^*(A) = \mu_F^*(A^*)$). Thus F is AC_ω on each E_n, so F is $[AC_\omega G]$ on $[a, b]$.

(iv) $\Rightarrow$ (v) By Lemma 3.4 and Proposition 2.1 we have

$$(\mu_F^*)_{|\mathcal{B}or(E_n)} \ll (\mu_\omega^*)_{|\mathcal{B}or(E_n)} \,.$$

Hence, by the Radon-Nikodym Theorem, it follows that there is a Borel measurable function $f_n : E_n \to [0, +\infty)$ such that

$$\mu_F^*(A) = \int_A f_n \, d\mu_\omega^* \,, \text{ whenever } A \in \mathcal{B}or(E_n) \,.$$

Let $E_{nk} = \{x \in E_n : f_n(x) < k\}$. Then $\{E_{nk}\}_k$ is an E_n-chain of Borel sets. Let $A \subset E_{nk}$. Since $(\mu_\omega^*)_{|\mathcal{P}(E_{nk})}$ and $(\mu_F^*)_{|\mathcal{P}(E_{nk})}$ are both Borel regular (see Lemma 5.1), there exists a Borel set $A^* \subset E_{nk}$ such that

$$\mu_F^*(A) = \mu_F^*(A^*) = \int_{A^*} f_n \, d\mu_\omega^* \le k \cdot \mu_\omega^*(A^*) = k \cdot \mu_\omega^*(A) \,.$$

(v) $\Rightarrow$ (i) This is evident.

(iii) $\Rightarrow$ (vi) Clearly $F \in C[a, b]$, and by Theorem 3.2, F is VB^*G on $[a, b]$. Let Z with $m(\omega(Z)) = 0$. By Theorem 4.1, (i), (iv), it follows that $\mu_\omega^*(Z) = 0$. Since F is ω-normal, $\mu_F^*(Z) = 0$. Again by Theorem 4.1, (i), (iv), we obtain that $m(F(Z)) = 0$. Thus $F \in N_\omega$.

(vi) $\Rightarrow$ (iii) Let Z with $\mu_\omega^*(Z) = 0$. By Theorem 4.1, (i), (iv), we have $m(\omega(Z)) = 0$. Since $F \in N_\omega$, it follows that $m(F(Z)) = 0$. Again by Theorem 4.1, (i), (iv), we obtain that $\mu_F^*(Z) = 0$, so F is ω-normal. □

Remark 5.2. Theorem 5.1 was proved by Faure in [9, Theorem 4.7], but in (iii) F is assumed to be VB^*G. As we can see from Theorem 3.2, F being VB^*G is superfluous. Also, our proof is different from that of Faure.

6 The Equivalence of the Integrals KHS, $\mathcal{D}^*$S, $\mathcal{V}$ and $\mathcal{W}$ with Respect to ω

Definition 6.1. Let $\delta : [a, b] \to (0, +\infty)$ and $E \subset [a, b]$. Let

$$\beta_\delta^o[E] = \Big\{ \big([y, z]; x\big) \,:\, x \in E \text{ and } x \in [y, z] \subset \big(x - \delta(x), x + \delta(x)\big) \Big\} \,.$$

Let π be a finite set of pairs $\{[c_i, d_i]; t_i) \in \beta_\delta^o[E]$, such that $\{[c_i, d_i]\}_i$ is a set of nonoverlapping nondegenerate closed intervals, and let $\sigma(\pi) = \cup_i [c_i, d_i]$. We denote by $\mathcal{P}^\circ(E;\delta)$ the collection of all π defined as above. Let $f,\omega:[a,b]\to\mathbb{R}$, and let

$$\sigma(f;\omega;\pi)=\sum_i f(t_i)\big(\omega(d_i)-\omega(c_i)\big)\,,\quad S(f;\pi)=\sum_i\big(f(d_i)-f(c_i)\big)\,,$$

for $\pi\in\mathcal{P}^\circ(E;\delta)$. If $E=[a,b]$ and $\sigma(\pi)=[a,b]$ then we denote the collection of all these π by $\mathcal{P}_1^\circ([a,b];\delta)$.

Remark 6.1. Recall that $D^o[E]=\{\beta_\delta^o[E]\,:\,\delta:[a,b]\to(0,+\infty)\}$ is called the ordinary derivation basis on the set E (see for example [4, p. 87]).

Definition 6.2. [9]. Let $f,\omega:[a,b]\to\mathbb{R}$. f is said to be Kurzweil-Henstock-Stieltjes integrable (short KHS-integrable) on $[a,b]$ with respect to ω, if there exists a real number I with the following property: for $\epsilon>0$ there exists $\delta:[a,b]\to(0,+\infty)$ such that $|\sigma(f;\omega;\pi)-I|<\epsilon$, whenever $\pi\in\mathcal{P}_1^\circ([a,b];\delta)$. Then $(KHS)\int_a^b f(t)\,d\omega(t)=I$.

Remark 6.2. In the above definition, the real number I is unique (the proof is similar to that in Remark 5.4.2 of [4]).

Definition 6.3. ([8, p. 415]) Let $\omega,F:[a,b]\to\mathbb{R}$, ω strictly increasing on $[a,b]$. We define the lower and upper derivatives of F with respect to ω at a point $x\in[a,b]$ as follows:

$$\underline{D}_\omega F(x)=\liminf_{y\to x}\frac{F(y)-F(x)}{\omega(y)-\omega(x)}\quad\text{and}\quad\overline{D}_\omega F(x)=\limsup_{y\to x}\frac{F(y)-F(x)}{\omega(y)-\omega(x)}\,.$$

F is said to be derivable with respect to ω at x if $\underline{D}_\omega F(x)=\overline{D}_\omega F(x)\in\mathbb{R}$. The derivative with respect to ω of F at x will be their common value and will be denoted by $F'_\omega(x)$.

Lemma 6.1. *Let $f,\omega:[a,b]\to\mathbb{R}$ be (KHS)-integrable on $[a,b]$ with respect to ω, and let $F(x)=(KHS)\int_a^x f(t)\,d\omega(t)$. Then F is derivable with respect to ω and $F'_\omega=f$ on $[a,b]$, except on a set Z with $\mu_\omega^*(Z)=0$.*

Proof. This is Corollary 4.8 of [9]. □

Lemma 6.2. *Let $f,\omega:[a,b]\to\mathbb{R}$, and let $E\subset[a,b]$ with $\mu_\omega^*(E)=0$ such that $f(x)=0$ for $x\in[a,b]\setminus E$. Then f is (KHS)-integrable with respect to ω on $[a,b]$, and its integral is 0.*

Proof. This is a particular case of Proposition 2.9 in [9]. □

Corollary 6.1. *Let $f, g, \omega : [a, b] \to \mathbb{R}$. If f is (KHS)-integrable with respect to ω on $[a, b]$, and $f = g$ except on a set E with $\mu^*_\omega(E) = 0$, then g is also (KHS)-integrable with respect to ω on $[a, b]$ and the two integrals are equal.*

PROOF. The proof follows from Lemma 6.2 and the linearity of the integral. $\square$

Definition 6.4. Let $f, \omega : [a, b] \to \mathbb{R}$, $\omega \in VB^*G$ on $[a, b]$, $\omega \in C[a, b]$. f is said to be Denjoy*-Stieltjes integrable (short $\mathcal{D}^*S$-integrable) with respect to ω on $[a, b]$ if there is a ω-normal function $F : [a, b] \to \mathbb{R}$ such that $F'_\omega = f$ on $[a, b]$, except on a set E with $\mu^*_\omega(E) = 0$. We write $(\mathcal{D}^*S) \int_a^b f(t)\, d\omega(t) = F(b) - F(a)$, and we say that F is an indefinite $\mathcal{D}^*S$-integral of f.

Lemma 6.3. *The $\mathcal{D}^*S$ integral is well-defined. Moreover, let $f, \omega : [a, b] \to \mathbb{R}$. If f is $(\mathcal{D}^*S)$-integrable with respect to ω on $[a, b]$, then f is (KHS)-integrable with respect to ω on $[a, b]$, and the two integrals are equal.*

PROOF. Let F be an indefinite $\mathcal{D}^*S$ integral of f. Then $F'_\omega = f$ on $[a, b]$ except on a set Z with $\mu^*_\omega(Z) = 0$. Since F is ω-normal, it follows that $\mu^*_F(Z) = 0$. Let $f_o : [a, b] \to \mathbb{R}$,

$$f_o(x) = \begin{cases} f(x) & \text{if } x \in [a, b] \setminus Z \\ 0 & \text{if } x \in Z\,. \end{cases}$$

By [9, Proposition 4.5], f_o is (KHS)-integrable with respect to ω on $[a, b]$, and

$$F(x) - F(a) = (KHS) \int_a^x f_o(t)\, d\omega(t)\,.$$

So the $\mathcal{D}^*S$ integral of f is well defined. By Corollary 6.1 it follows that f is (KHS)-integrable with respect to ω on $[a, b]$ and the two integrals are equal. $\square$

Definition 6.5. Let $f, \omega : [a, b] \to \mathbb{R}$.

- We define the following class of majorants: $\overline{\mathcal{W}}(f) = \{M : [a, b] \to \mathbb{R} : M(a) = 0$; there exists $\delta : [a, b] \to (0, \infty)$ such that $M(z) - M(y) > f(x)\big(\omega(z) - \omega(y)\big)$, whenever $x \in [y, z] \subset (x - \delta(x), x + \delta(x))\}$;
- We define the following class of minorants: $\underline{\mathcal{W}}(f) = \{m : [a, b] \to \mathbb{R} : -m \in \overline{\mathcal{W}}(-f)\}$.

- If $\overline{\mathcal{W}} \neq \emptyset$ then we denote by $\overline{J}(b)$ the lower bound of all $M(b)$, $M \in \overline{\mathcal{W}}(f)$. If $\underline{\mathcal{W}}(f) \neq \emptyset$ then we denote by $\underline{J}(b)$ the upper bound of all $m(b)$, $m \in \underline{\mathcal{W}}(f)$.

- We say that f has a $(\mathcal{W})$-integral with respect to ω on $[a,b]$, if $\overline{\mathcal{W}}(f) \times \underline{\mathcal{W}}(f) \neq \emptyset$ and $\overline{J}(b) = \underline{J}(b) = (\mathcal{W}) \int_a^b f(t)\, d\omega(t)$.

Definition 6.6. Let $f, \omega : [a,b] \to \mathbb{R}$.

- f is said to be $(\mathcal{V})$-integrable with respect to ω on $[a,b]$, if there exists $H : [a,b] \to \mathbb{R}$ such that for every $\epsilon > 0$ there exist $\delta : [a,b] \to (0,+\infty)$ and $G : [a,b] \to \mathbb{R}$ with the following properties: $G(a) = 0$, $G(b) < \epsilon$, G is increasing on $[a,b]$ and $|H(z) - H(y) - f(x)(\omega(z) - \omega(y))| < G(z) - G(y)$, whenever $x \in [y,z] \subset (x - \delta(x), x + \delta(x))$.

- H is called the $(\mathcal{V})$-indefinite integral of f with respect to ω on $[a,b]$, and $(\mathcal{V}) \int_a^b f(t)\, d\omega(t) = H(b) - H(a)$.

- Clearly the $(\mathcal{V})$-integral is well defined.

Theorem 6.1. *Let $f, \omega : [a,b] \to \mathbb{R}$, $\omega \in VB^*G$ and $\omega \in C[a,b]$. Then f is (KHS)-integrable with respect to ω on $[a,b]$ if and only if f is $(\mathcal{D}^*S)$-integrable with respect to ω on $[a,b]$ and the two integrals are equal.*

Proof. "$\Rightarrow$" The proof follows by Theorem 4.7 and Corollary 4.8 of [9, p. 120].

"$\Leftarrow$" See Lemma 6.1. □

Remark 6.3. Let $f, \omega : [a,b] \to \mathbb{R}$. The following assertions are equivalent:

- f is (KHS)-integrable with respect to ω on $[a,b]$;

- f is $(\mathcal{D}^*S)$-integrable with respect to ω on $[a,b]$;

- f is $(\mathcal{V})$-integrable with respect to ω on $[a,b]$;

- f is $(\mathcal{W})$-integrable with respect to ω on $[a,b]$;

The equivalence of the KHS, $\mathcal{W}$ and $\mathcal{V}$ integrals is known. This was proved for instance by Henstock in [10] (see Theorems 2.5.4 and 7.2.1). For the case of KHS and $\mathcal{W}$ integrals this was proved as early as 1957 by Kurzweil in [12] (see Theorem 1.2.1). The equivalence of the KHS and $\mathcal{D}^*S$ integrals follows from Theorem 6.1.

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