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# GRAPHS OF FUNCTIONS, REGULAR SETS AND S-STRAIGHT SETS

**Abstract**

A subset $E$ of $\mathbb{R}^p$ is s-straight if $E$ has finite Hausdorff s-dimensional outer measure which equals its Method I s-outer measure. The graph of a continuously differentiable function is shown to be the countable union of closed 1-straight sets together with a set of Hausdorff 1-measure zero. This result is extended to the graphs of absolutely continuous functions and to regular sets.

## 1 Introduction

In [6], Foran introduced the notion of s-straight and proposed a subset $E$ of the unit circle such that $E$ is 1-straight and has positive measure. A detailed analysis of this proposed set is given in [2] together with other examples and results for 1-straight sets.

Given a nonempty bounded subset $B$ of $\mathbb{R}^p$, define $\operatorname{diam}(B) = \sup\{d(x, y) : x, y \in B\}$ where $d(x, y)$ denotes the usual distance function in $\mathbb{R}^p$. Define $\operatorname{diam}(\emptyset) = 0$. We write $\operatorname{diam}^s(E)$ in place of $[\operatorname{diam}(E)]^s$.

**Definition 1.1.** *Let $E$ be a subset of $\mathbb{R}^p$ and $s > 0$. Given $\infty \geq \delta > 0$, define*

$$\overline{m}^s_\delta(E) = \inf\left\{\sum_{i=1}^{\infty} \operatorname{diam}^s(E_i) : E = \bigcup_{i=1}^{\infty} E_i,\ \operatorname{diam}(E_i) < \delta \text{ for } i = 1, 2, \dots\right\}.$$

*Set $\overline{m}^s(E) = \sup_{\delta>0} \overline{m}^s_\delta(E)$ and set $\overline{m}^s_I(E) = \overline{m}^s_\infty(E)$.*

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The outer measure $\overline{m}_I^s$ is known as a Method I outer measure. The outer measure $\overline{m}^s$ is a metric outer measure on $\mathbb{R}^p$. Hence, every closed subset of $\mathbb{R}^p$ is $\overline{m}^s$-measurable. See [3, pp. 132–144] for details. If $E$ is an $\overline{m}^s$-measurable subset of $\mathbb{R}^p$, then we write $\mathcal{H}^s(E)$ in place of $\overline{m}^s(E)$. If $(A_n)_{n=1}^{\infty}$ is a decreasing sequence of compact subsets of $\mathbb{R}^p$ and if $\delta > 0$, then $\overline{m}_\delta^s(\bigcap_{n=1}^{\infty} A_n) = \lim_{n\to\infty} \overline{m}_\delta^s(A_n)$. In [6], Foran defines s-straight and proves Theorem 1.3, providing a useful equivalent formulation of s-straight.

**Definition 1.2.** *Given a subset $E$ of $\mathbb{R}^p$ and $s > 0$, we say $E$ is s-straight if $\overline{m}^s(E) < \infty$ and $\overline{m}^s(E) = \overline{m}_I^s(E)$.*

**Theorem 1.3.** *Let $E$ be a subset of $\mathbb{R}^p$ with $\overline{m}^s(E) < \infty$. Then $E$ is s-straight if and only if $\overline{m}^s(E \cap K) \leq \operatorname{diam}^s(E \cap K)$ for each compact subset $K$ of $\mathbb{R}^p$.*

## 2 Graphs of Functions and S-Straight Sets

**Definition 2.1.** *Let $A_1, \ldots, A_n$ be subsets of $\mathbb{R}^p$. We say that $A_1, \ldots, A_n$ are s-aligned if $\operatorname{diam}^s(B) \geq \sum_{i=1}^n \operatorname{diam}^s(B \cap A_i)$ for each bounded subset $B$ of $A_1 \cup \cdots \cup A_n$.*

**Example.** If $\operatorname{diam}(A \cup B) \geq \operatorname{diam}(A) + \operatorname{diam}(B)$, then $A$ and $B$ may not be 1-aligned. Let $A = \{(x, \sin x) : 0 \leq x \leq \pi\}$ and $B = \{(x, \sin x) : -\pi \leq x \leq 0\}$. Then $\operatorname{diam}(A \cup B) = \operatorname{diam}(A) + \operatorname{diam}(B)$. If $A_1 = \{(x, \sin x) : 0 \leq x \leq \pi/2\}$, then $\operatorname{diam}(A_1 \cup B) < \operatorname{diam}(A_1) + \operatorname{diam}(B)$. Hence, $A$ and $B$ are not 1-aligned.

The motivation for the definition of s-aligned arises from the following result.

**Proposition 2.2.** *Let $A_1, \ldots, A_n$ be s-aligned subsets of $\mathbb{R}^p$. If $A_1, \ldots, A_n$ are s-straight, then $A_1 \cup \cdots \cup A_n$ is s-straight.*

**Proof.** Let $K$ be a compact set in $\mathbb{R}^p$. Then $\overline{m}^s[K \cap (A_1 \cup \cdots \cup A_n)] \leq \sum_{i=1}^n \overline{m}^s(K \cap A_i) \leq \sum_{i=1}^n \operatorname{diam}^s(K \cap A_i) \leq \operatorname{diam}^s[K \cap (A_1 \cup \cdots \cup A_n)]$.

Suppose $A$ and $B$ are s-aligned subsets of $\mathbb{R}^p$. Let $\overline{A}$ and $\overline{B}$ denote the closure of $A$ and $B$, respectively. Then $\overline{A} \cap \overline{B}$ contains at most one point. Hence, $\overline{m}^s(A_1 \cup B_1) = \overline{m}^s(A_1) + \overline{m}^s(B_1)$ for each $A_1 \subseteq \overline{A}$ and $B_1 \subseteq \overline{B}$.

**Theorem 2.3.** *Let $(q_n)_{n=1}^{\infty}$ be a sequence of positive integers. Set $Q^n = \{(i_1, \ldots, i_n) : 1 \leq i_1 \leq q_1, \ldots, 1 \leq i_n \leq q_n\}$ for $n = 1, 2, \ldots$. Let $\{A_\beta : \beta \in Q^n$ for some $n \geq 1\}$ be a family of compact subsets of $\mathbb{R}^p$ such that $A_1, \ldots, A_{q_1}$ are s-aligned and such that*

*(a)* $A_{(\beta,1)}, \ldots, A_{(\beta,q_n)}$ *are s-aligned if* $\beta \in Q^{n-1}$,
*(b)* $A_\beta \supseteq A_{(\beta,1)} \cup \cdots \cup A_{(\beta,q_n)}$ *if* $\beta \in Q^{n-1}$ *and*

*(c)* $\lim_{n\to\infty} \max\{\operatorname{diam}^s(A_\beta) : \beta \in Q^n\} = 0$.
*Let* $P_n = \bigcup\{A_\beta : \beta \in Q^n\}$. *Then* $\bigcap_{n=1}^\infty P_n$ *is s-straight.*

**Proof**. If $E \subseteq P_n$, then $\operatorname{diam}^s(E) \geq \sum_{\beta\in Q^n} \operatorname{diam}^s(E \cap A_\beta)$. Let $P = \bigcap_{n=1}^\infty P_n$. The set $P$ is closed and $P_n \supseteq P_{n+1}$ for $n \geq 1$. Let $K$ be a compact subset of $\mathbb{R}^p$. It suffices to show that $\operatorname{diam}^s(P \cap K) \geq \overline{m}_\delta^s(P \cap K)$ for each $\delta > 0$. Let $\delta > 0$. Choose $\ell$ such that $\operatorname{diam}^s(A_\beta) < \delta$ for each $\beta \in Q^\ell$. Then for each $n \geq \ell$

$$\operatorname{diam}^s(P_n \cap K) \geq \sum_{\beta\in Q^n} \operatorname{diam}^s(A_\beta \cap K) \geq \overline{m}_\delta^s(P_n \cap K).$$

Hence, $\lim_{n\to\infty} \operatorname{diam}^s(P_n \cap K) \geq \lim_{n\to\infty} \overline{m}_\delta^s(P_n \cap K) \geq \overline{m}_\delta^s(P \cap K)$. Since $P$ is closed, $\operatorname{diam}^s(P \cap K) = \lim_{n\to\infty} \operatorname{diam}^s(P_n \cap K)$. Thus, $\operatorname{diam}^s(P \cap K) \geq \overline{m}_\delta^s(P \cap K)$. It follows that $P$ is s-straight.

**Lemma 2.4.** *Let* $f : [a, b] \to \mathbb{R}$ *be continuously differentiable with* $0 \leq m \leq f'(t) \leq M$. *If* $I$ *is a subinterval of* $[a, b]$, *let* $F(I) = \{(x, f(x)) : x \in I\}$. *Suppose* $0 < w < 1$ *satisfies* $(1 + w)\sqrt{1 + m^2} \geq 2w\sqrt{1 + M^2}$. *Then for each positive integer* $n$, *there exist disjoint subintervals* $I_1, \ldots, I_n$ *of* $[a, b]$ *each of length* $w(b - a)/n$ *such that* $\mathcal{H}^1[\bigcup_{k=1}^n F(I_k)] \geq w^2\mathcal{H}^1[F([a, b])]$ *and* $F(I_1), \ldots, F(I_n)$ *are 1-aligned.*

**Proof**. It suffices to consider the case where $[a, b] = [0, 1]$. Let $F(s, t) = \{(x, f(x)) : s \leq x \leq t\}$ if $0 \leq s \leq t \leq 1$. Assume $(1 + w)\sqrt{1 + m^2} \geq 2w\sqrt{1 + M^2}$ where $0 < w < 1$. Set $\alpha = \arctan m$ and $\beta = \arctan M$. Then $\tan\alpha \leq f' \leq \tan\beta$ and $(1 + w)\sec\alpha \geq 2w \sec\beta$. If $0 \leq s \leq t \leq 1$, then $(t - s)\sec\alpha \leq \mathcal{H}^1[F(s, t)] \leq (t - s)\sec\beta$. Let $n$ be a positive integer. Set $\delta = (1 - w)/(2n)$. Define $a_k = (k - 1)/n + \delta$, $b_k = k/n - \delta$ and $I_k = [a_k, b_k]$ for $k = 1, \ldots, n$. Then $\mathcal{H}^1[\bigcup_{k=1}^n F(I_k)] = \sum_{k=1}^n \mathcal{H}^1[F(I_k)] \geq \sum_{k=1}^n |I_k| \sec\alpha = w \sec\alpha$ where $|I_k|$ denotes the length of $I_k$. Hence

$$\mathcal{H}^1[\bigcup_{k=1}^n F(I_k)] \geq w\frac{\sec\alpha}{\sec\beta}\mathcal{H}^1[F(0, 1)] \geq \frac{2w^2}{1 + w}\mathcal{H}^1[F(0, 1)] \geq w^2\mathcal{H}^1[F(0, 1)].$$

Since $f$ is increasing, to show that $F(I_1), \ldots, F(I_n)$ are 1-aligned, it suffices to show that $\operatorname{diam}[F(s, t)] \geq \sum_{k=1}^n \operatorname{diam}[F([s, t] \cap I_k)]$ if $0 \leq s < t \leq 1$. If $s, t$ belong to a single $I_k$, this inequality is clear. Let $a_j \leq s < b_j$ and let $a_k < t \leq b_k$ where $j < k$. Set

$$\Phi(s, t) = \operatorname{diam}[F(s, t)] - \operatorname{diam}[F(s, b_j)] - \operatorname{diam}[F(a_k, t)] - \sum_{i=j+1}^{k-1} \operatorname{diam}[F(a_i, b_i)].$$

To show that $F(I_1), \ldots, F(I_n)$ are 1-aligned, it suffices to show that $\Phi(s,t) \geq 0$. Set $m = k+1-j \geq 2$. We have

$$\begin{aligned}
\Phi(s,t) &\geq (t-s)\sec\alpha - \left[(b_j - s) + (t - a_k) + \textstyle\sum_{i=j+1}^{k-1}(b_i - a_i)\right]\sec\beta \\
&\geq (b_k - a_j)\sec\alpha - \left[(b_j - a_j) + (b_k - a_k) + \textstyle\sum_{i=j+1}^{k-1}(b_i - a_i)\right]\sec\beta \\
&= \frac{m-(1-w)}{n}\sec\alpha - \frac{mw}{n}\sec\beta \\
&= \frac{m}{n}\left[\frac{m-1+w}{m}\sec\alpha - w\sec\beta\right] \\
&\geq \frac{m}{n}\left[\frac{1+w}{2}\sec\alpha - w\sec\beta\right].
\end{aligned}$$

Hence, $\Phi(s,t) \geq 0$ and so $F(I_1), \ldots, F(I_n)$ are 1-aligned.

**Theorem 2.5.** *Let $f : [a,b] \to \mathbb{R}$ be continuously differentiable with $0 \leq f' \leq M$. Let $\Gamma = \{(x, f(x)) : a \leq x \leq b\}$. Then there exists closed 1-straight subset $P$ of $\Gamma$ such that $\mathcal{H}^1(P) \geq \mathcal{H}^1(\Gamma)/[4(1+M^2)]$.*

**Proof**. If $I$ is a subinterval of $[a,b]$, set $F(I) = \{(x, f(x)) : x \in I\}$, $m(f', I) = \min\{f'(x) : x \in I\}$ and $M(f', I) = \max\{f'(x) : x \in I\}$. Set $\beta = \arctan(M)$ and let $w_1$ be the positive number such that $1 + w_1 = 2w_1 \sec\beta$, that is, $w_1 = 1/(2\sec\beta - 1)$. Then $w_1^2 > 1/[4(1+M^2)]$. Choose $w_2, w_3, \ldots$ in $(0,1)$ such that $\prod_{k=1}^{\infty} w_k^2 = 1/[4(1+M^2)]$. By Lemma 2.4 and by uniform continuity of $f'$ on $[a,b]$, we may choose an integer $q_1 \geq 2$ and closed subintervals $I_1, \ldots, I_{q_1}$ of $[a,b]$ each of length $w_1(b-a)/q_1$ such that

$$\text{(step 1)} \quad \begin{cases} F(I_1), \ldots, F(I_{q_1}) \text{ are 1-aligned,} \\ \mathcal{H}^1[\bigcup_{j=1}^{q_1} F(I_j)] \geq w_1^2 \mathcal{H}^1(\Gamma) \text{ and} \\ \dfrac{1+w_2}{2w_2}\sqrt{\dfrac{1+m^2(f', I_j)}{1+M^2(f', I_j)}} \geq 1 \text{ for } j = 1, \ldots, q_1 \end{cases}$$

Let $P_1 = F(I_1) \cup \cdots \cup F(I_{q_1})$. Then $\mathcal{H}^1(P_1) \geq w_1^2 \mathcal{H}^1(\Gamma)$. By Lemma 2.4 and by uniform continuity of $f'$ on $[a,b]$, we may choose an integer $q_2 \geq 2$ and closed subintervals $I_{i,1}, \ldots, I_{i,q_2}$ of $I_i$ each of length $w_1 w_2(b-a)/(q_1 q_2)$ for $i = 1, \ldots, q_1$ such that

$$\text{(step 2)} \quad \begin{cases} F(I_{i,1}), \ldots, F(I_{i,q_2}) \text{ are 1-aligned for } i = 1, \ldots, q_1 \\ \mathcal{H}^1[\bigcup_{j=1}^{q_2} F(I_{i,j})] \geq w_2^2 \mathcal{H}^1[F(I_i)] \text{ for } i = 1, \ldots, q_1 \text{ and} \\ \dfrac{1+w_3}{2w_3}\sqrt{\dfrac{1+m^2(f', I_{i,j})}{1+M^2(f', I_{i,j})}} \geq 1 \text{ for } i = 1, \ldots, q_1 \text{ and } j = 1, \ldots, q_2 \end{cases}$$

Let $P_2 = \bigcup\{F(I_{i,j}) : 1 \leq i \leq q_1 \text{ and } 1 \leq j \leq q_2\}$. Then $\mathcal{H}^1(P_2) \geq w_1^2 w_2^2 \mathcal{H}^1(\Gamma)$. Continuing this process, we obtain, in accordance with Theorem 2.3, a decreasing sequence $(P_n)_{n=1}^{\infty}$ of compact subsets of $\Gamma$ such that $\mathcal{H}^1(P_n) \geq (\prod_{k=1}^{n} w_k^2)\mathcal{H}^1(\Gamma)$ for each $n \geq 1$ and $\bigcap_{n=1}^{\infty} P_n$ is 1-straight. Let $P = \bigcap_{n=1}^{\infty} P_n$. Then $P$ is a closed 1-straight subset of $\Gamma$ with $\mathcal{H}^1(P) \geq \mathcal{H}^1(\Gamma)/[4(1+M^2)]$.

Let $f : [a,b] \to \mathbb{R}$ be continuous and let $\Gamma$ be the graph of $f$. Assume that $\mathcal{H}^1(\Gamma)$ is finite. If each closed subset of $\Gamma$ with positive $\mathcal{H}^1$ measure contains a closed 1-straight subset of positive $\mathcal{H}^1$ measure, then $\Gamma$ is the countable union of closed 1-straight subsets together with a set of $\mathcal{H}^1$ measure zero since the measure $\mathcal{H}^1$ is regular. See [2] for a similar result.

**Corollary 2.6.** *Let $f : [a,b] \to \mathbb{R}$ be continuously differentiable. Then the graph of $f$ is the countable union of closed 1-straight sets together with a set of $\mathcal{H}^1$ measure zero.*

**Proof**. If $I$ is a subinterval of $[a,b]$, let $F(I) = \{(x, f(x)) : x \in I\}$. Choose closed subintervals $I_1, \ldots, I_n$ of $[a,b]$ that cover $[a,b]$ such that $f' \geq 0$, $f' \leq 0$ or $|f'| \leq \sqrt{3}/3$ on $I_k$ for each $k = 1, \ldots, n$. If $|f'| \leq \sqrt{3}/3$ on $I_j$, then $F(I_j)$ rotated by $30^\circ$ coincides with the graph of a continuously differentiable function $g$ with $g' \geq 0$. It follows from Theorem 2.5 that each $F(I_k)$ is the countable union of closed 1-straight subsets together with a subset of $\mathcal{H}^1$ measure zero and so likewise for the graph of $f$.

**Theorem 2.7.** *Let $f : [a,b] \to \mathbb{R}$ be absolutely continuous. Then the graph of $f$ is the countable union of closed 1-straight sets together with a set of $\mathcal{H}^1$ measure zero.*

**Proof**. Let $\Gamma_f = \{(x, f(x)) : a \leq x \leq b\}$. Let $\mu$ denote Lebesgue measure on $[a,b]$. Let $E$ be a closed subset of $\Gamma_f$ with positive $\mathcal{H}^1$ measure. Set $B = \{x \in [a,b] : (x, f(x)) \in E\}$. Since $f$ is absolutely continuous, $B$ is a compact subset of $[a,b]$ with $\mu(B) > 0$. But $f$ is differentiable almost everywhere and $f'$ is $\mu$-measurable. By a theorem due to Federer, see [5, Theorem 3.1.15] and [1, p. 1160], there exists a compact subset $K$ of $B$ and a continuously differentiable function $g$ on $[a,b]$ such that $g = f$ on $K$ and $\mu(K) > 0$. Let $\Gamma_g(K) = \{(x, g(x)) : x \in K\}$. Then $\Gamma_g(K)$ is a compact subset of the graph of $g$ with positive $\mathcal{H}^1$ measure. By Corollary 2.6, there exists a closed 1-straight subset $P$ of the graph of $g$ with $\mathcal{H}^1(P \cap \Gamma_g(K)) > 0$. Since $P$ is 1-straight, $P \cap \Gamma_g(K)$ is a closed 1-straight subset of $E$. Hence, the graph of $f$ is the countable union of closed 1-straight subsets together with a set of $\mathcal{H}^1$ measure zero.

**Corollary 2.8.** *Let $f : [a, b] \to \mathbb{R}$ be continuous and increasing. Then the graph of $f$ is the countable union of closed 1-straight sets together with a set of $\mathcal{H}^1$ measure zero.*

**Proof**. The graph of $f$ rotated clockwise by 45° coincides with the graph of a continuous Lipschitz function, which is absolutely continuous.

## 3 Regular Sets and S-Straight Sets

If $E \subset \mathbb{R}^p$, we say $E$ is an s-set if $E$ is $\overline{m}^s$-measurable and $0 < \mathcal{H}^s(E) < \infty$. If $x \in \mathbb{R}^p$, let $B_r(x)$ denote the closed ball of radius $r > 0$ centered at $x$. We next recall some definitions concerning density, regular sets and contingents.

**Definition 3.1.** *[4, pp. 20-21]. We say $x$ is a regular point of $E$ if the $\mathcal{H}^s$-density of $x$ with respect to $E$, defined as $\lim_{r\to 0^+}[\mathcal{H}^s(E \cap B_r(x))/(2r)^s]$, exists and equals 1. A set $E$ is called a regular set if almost every point of $E$ is regular. Here we consider only the case $s = 1$ and sets $E \subseteq \mathbb{R}^2$.*

**Definition 3.2.** *[4, pp. 26-29]. A curve (or, Jordan curve) $\Gamma$ is the image of a continuous one-to-one function $\psi : [a, b] \to \mathbb{R}^p$, where $[a, b] \subseteq \mathbb{R}$ is a closed interval. In particular, a curve is not self-intersecting. If the length $\mathcal{L}(\Gamma)$ of a curve $\Gamma$ is defined is the usual way and if $\mathcal{L}(\Gamma) < \infty$, we say that $\Gamma$ is a rectifiable curve. A 1-set contained in a countable union of rectifiable curves is called a $Y$-set [4, p. 33]. Let $S(x, \theta, \varphi)$ be the closed one-way infinite cone with vertex $x$ and axis in direction of angle $\theta$ consisting of those points $y$ such that the line segment between $x$ and $y$ makes an angle of at most $\varphi$ with that axis.*

**Definition 3.3.** *[7, p. 262]. Let $E \subseteq \mathbb{R}^2$. For any point $x \in E$, the direction $\theta$ of a half-line originating at $x$ is the angle made by that half-line with a fixed direction, usually that of the horizontal axis. Such a half-line is denoted by $l(x, \theta)$. A half-line containing a point $y \neq x$ is denoted by $\overrightarrow{xy}$. A sequence $\{l_n(x, \theta_n)\}_{n=1}^{\infty}$ of half-lines is said to converge to the half-line $l(x, \theta)$ if $\lim_{n\to\infty} \theta_n = \theta$. A half-line $l(x, \theta)$ is called an intermediate half-tangent of $E$ at $x$ if there exists a sequence of points $\{x_n\}_{n=1}^{\infty} \subseteq E$, with $x_n \neq x$ for all $n$ such that both $\lim_{n\to\infty} x_n = x$, and the sequence of half-lines $\{\overrightarrow{xx_n}\}_{n=1}^{\infty}$ converges to $l(x, \theta)$. Finally, the contingent of $E$ at $x$, denoted by $\operatorname{contg}_E(x)$, is the set of all intermediate half-tangents of $E$ at $x$. (If $x_0$ is an isolated point, then $x_0$ has no intermediate half-tangents, and $\operatorname{contg}_E(x_0) = \emptyset$.)*

The following Lemma is part of the so-called Fundamental Theorem on Contingents of Plane Sets, found in Saks.

**Lemma 3.4.** *[7, p. 264]. Let $\theta$ be a fixed direction, and $E \subseteq \mathbb{R}^2$ be such that for each $x \in E$, the set $\operatorname{contg}_E(x)$ contains no half-line of direction $\theta$. If $\theta$ is the direction of the positive vertical axis, then $E = \cup_{i=1}^{\infty} E_i$ such that for each $i$ the set $E_i$ is the graph of a Lipschitz function $f_i : B_i \to \mathbb{R}$ where $B_i$ is a bounded subset of $\mathbb{R}$ and $\mathcal{H}^1(E_i) = \mathcal{L}(E_i) < \infty$.*

**Theorem 3.5.** *Let $E \subseteq \mathbb{R}^2$ be a regular 1-set. Then $E$ is a countable union of 1-straight sets together with a set of $\mathcal{H}^1$ measure zero.*

**Proof**. Besicovitch proved (see [4, p. 45]) that a regular 1-set $E \subseteq \mathbb{R}^2$ consists of a $Y$-set together with a set of $\mathcal{H}^1$-measure zero. In his proof (see [4, p. 32]) that a rectifiable curve $\Gamma$ in $\mathbb{R}^2$ has a tangent at almost all of its points, he proved in particular that for almost all $x \in \Gamma$ there exists a direction $\theta$ such that for suitable $\varphi, \rho > 0$ it follows that $\Gamma \cap [B_\rho(x) \setminus (S(x,\theta,\varphi) \cup S(x,\theta + \pi,\varphi))] = \emptyset$. So at almost all points $x$ of a rectifiable curve $\Gamma$, $\operatorname{contg}_\Gamma(x) \neq \mathbb{R}^2$. As in the proof of the Fundamental Theorem on Contingents of Plane Sets [7, p. 267], let $\{\theta_n\}$ be a countable everywhere dense set of directions in $\mathbb{R}^2$. Let $\Gamma_n$ be the set of points of $\Gamma$ at which $\operatorname{contg}_\Gamma(x)$ does not contain the half-line of direction $\theta_n$. Then since $\{\theta_n\}$ is dense, it is clear that $\Gamma = \cup_{n=1}^{\infty} \Gamma_n$, otherwise there exist points $x \in \Gamma$ such that $\operatorname{contg}_\Gamma(x) = \mathbb{R}^2$. By Lemma 3.4, with respect to a line of direction $\theta_n + \frac{\pi}{2}$, each $\Gamma_n$ is the countable union of finite length graphs of Lipschitz functions on bounded domain sets. Thus $\Gamma$ itself equals such a union. It is well-known, as in [7, p. 264], that any such Lipschitz function can be extended to be Lipschitz on the smallest closed interval containing its bounded domain. Since a Lipschitz function is absolutely continuous, by Theorem 2.7 it follows that the graph of each such extended Lipschitz function is a countable union of 1-straight sets. Since subsets of 1-straight sets are 1-straight and since translations and rotations of 1-straight sets are 1-straight (see [2]), the subset $\Gamma$ of the countable union of the graphs of these Lipschitz functions is also a countable union of 1-straight sets. Since this is true for each rectifiable curve $\Gamma$ in $\mathbb{R}^2$, it follows that $E$ is therefore a countable union of 1-straight sets together with a set of $\mathcal{H}^1$-measure zero.

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