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PACKING MEASURE IN GENERAL METRIC SPACE

Abstract

Packing measures are counterparts to Hausdorff measures, used in measuring fractal dimension of sets. C. Tricot defined them for subsets of finite-dimensional Euclidean space. We consider here the proper way to phrase the definitions for use in general metric spaces, and for Hausdorff functions other than the simple powers, in particular non-blanketed Hausdorff functions. The question of the Vitali property arises in this context. An example of a metric space due to R. O. Davies illustrates the concepts.

1 Packing Measure

We begin with the definition of the packing measure (see [24], [22], [21]). Let d be a positive integer, and write $\mathbb{R}^d$ for d-dimensional Euclidean space. For $x \in \mathbb{R}^d$ and $r > 0$, the **open** and **closed balls** are

$$B_r(x) = \left\{ y \in \mathbb{R}^d : \|y - x\| < r \right\}, \qquad \overline{B}_r(x) = \left\{ y \in \mathbb{R}^d : \|y - x\| \le r \right\}.$$

Let $A \subseteq \mathbb{R}^d$ be a set. A **centered-ball packing** of A is a countable disjoint collection of open balls with center in A:

$$\{B_{r_1}(x_1), B_{r_2}(x_2), \cdots\},$$

where $x_i \in A$ and $B_{r_i}(x_i) \cap B_{r_j}(x_j) = \varnothing$ for $i \neq j$.

Definition 1.1. Let s be a positive number. For $\varepsilon > 0$, define

$$\mathcal{P}^s_\varepsilon(A) = \sup \sum_i \left(\operatorname{diam} B_{r_i}(x_i)\right)^s,$$

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where the supremum is over all packings of A by centered balls with diameter $\leq \varepsilon$. The **s-dimensional packing pre-measure** of A is

$$\widetilde{\mathcal{P}}^s(A) = \lim_{\varepsilon \to 0} \mathcal{P}^s_\varepsilon(A).$$

The **s-dimensional packing outer measure** is the outer measure $\mathcal{P}^s$ defined from the set function $\widetilde{\mathcal{P}}^s$ by method I. That is,

$$\mathcal{P}^s(A) = \inf \left\{ \sum_{D \in \mathcal{D}} \widetilde{\mathcal{P}}^s(D) : \mathcal{D} \text{ is a countable cover of } A \right\}.$$

Then $\mathcal{P}^s$ is a metric outer measure on $\mathbb{R}^d$.

1.1 Generalizations

Now we will consider possible generalizations of the definition. First, we will consider a general metric space (S, ρ) in place of $\mathbb{R}^d$. As has been observed by C. Cutler [3] and H. Haase [9],[1] it is preferable to use the "radius" definition $\sum \bigl(2r_i\bigr)^s$ instead of the "diameter" definition $\sum \bigl(\operatorname{diam} B_{r_i}(x_i)\bigr)^s$. In order to handle this difference conveniently, we introduce the next definition.

Definition 1.2. A **constituent** in the metric space (S, ρ) is an ordered pair (x, r), where $x \in S$ and $r > 0$.

We think of the constituent (x, r) as representing the ball $B_r(x)$. In metric spaces other than Euclidean spaces it is possible that two balls $B_r(x), B_{r'}(x')$ may be equal as point-sets even if $x \neq x'$ and/or $r \neq r'$.

Definition 1.3. Let $\varepsilon > 0$. A collection π of constituents is said to be **ε-fine** iff $r \leq \varepsilon$ for all $(x, r) \in \pi$.

Secondly, we wish to use a Hausdorff function φ not necessarily of the form

$$\varphi(r) = (2r)^s. \tag{1}$$

Let us say that $\varphi : [0, \infty) \to [0, \infty)$ is a **Hausdorff function** iff φ is continuous,[2] nondecreasing, and $\varphi(r) = 0$ if and only if $r = 0$. In place of sums of the form $\sum \bigl(2r_i\bigr)^s$, we will use sums of the form $\sum \varphi\bigl(r_i\bigr)$.

[1] Additional reasons to prefer the "radius" definition over the "diameter" definition are pointed out by McClure [18] §3.5 and Mattila & Mauldin [17] §5.

[2] Continuity will not be an important consideration here. It seems to me that left-continuity is enough when we work with packing measures, but right-continuity is what we want when working with Hausdorff (or covering) measures.

One use of this generality is the possibility of making delicate adjustments in the Hausdorff function by including logarithmic factors ([11], § 8):

$$\varphi(r) = r^{s_0}(\log(1/r))^{-s_1}(\log\log(1/r))^{-s_2}. \tag{2}$$

Another use is to measure the dimension of infinite-dimensional sets, where Hausdorff functions that vanish more rapidly than any power may be useful (see [15], [2], [8], [19]). For example:

$$\varphi(r) = 2^{-M/r^\alpha}. \tag{3}$$

Following Larman [16] we will say that a Hausdorff function φ is **blanketed** iff there is a constant $C < \infty$ such that

$$\varphi(2r) \le C\varphi(r) \tag{4}$$

for all $r \le 1$. (This is also known as the **doubling condition**, the **Orlicz condition**, or the Δ_2 **condition**.) For example, Hausdorff functions of the form (1) or (2) are blanketed, but those of the form (3) are not.

The final variants we wish to consider are in the definition of **packing**.

Definition 1.4. Let (S, ρ) be a metric space, and let π be a collection of constituents.

(a) π is an **(a)-packing** iff $\rho(x, x') \ge r \vee r'$ for all $(x, r) \ne (x', r')$ in π;

(b) π is a **(b)-packing** iff $B_r(x) \cap B_{r'}(x') = \varnothing$ for all $(x, r) \ne (x', r')$ in π;

(c) π is a **(c)-packing** iff $\rho(x, x') \ge r + r'$ for all $(x, r) \ne (x', r')$ in π.

Note that $\rho(x, x') \ge r \vee r'$ in (a) may be interpreted as: $x' \notin B_r(x)$ and $x \notin B_{r'}(x')$. This sort of packing was used in [14]. Although (b) and (c) are the same thing in Euclidean space, for a general metric space we have: every (c)-packing is a (b)-packing, and every (b)-packing is an (a)-packing.

If we were using closed balls rather than open balls, the inequalities would perhaps be changed to (a) $\rho(x, x') > r \vee r'$, and (c) $\rho(x, x') > r + r'$.

Definition 1.5. An **(a)-packing of a set** A is an (a)-packing π such that $x \in A$ for all $(x, r) \in \pi$. Similarly for (b)- and (c)-packings.

¿From these three types of packing, we may define three packing measures. I will write out case (a).

Definition 1.6. Let (S,ρ) be a metric space, let φ be a Hausdorff function, and let $A \subseteq S$. For $\varepsilon > 0$, define

$$^{(\mathrm{a})}\mathcal{P}^{\varphi}_{\varepsilon}(A) = \sup \sum_{(x,r)\in\pi} \varphi(r),$$

where the supremum is over all ε-fine (a)-packings π of A. Let

$$^{(\mathrm{a})}\widetilde{\mathcal{P}}^{\varphi}(A) = \lim_{\varepsilon\to 0} {}^{(\mathrm{a})}\mathcal{P}^{\varphi}_{\varepsilon}(A)$$

and

$$^{(\mathrm{a})}\mathcal{P}^{\varphi}(A) = \inf \left\{ \sum_{D\in\mathcal{D}} {}^{(\mathrm{a})}\widetilde{\mathcal{P}}^{\varphi}(D) : \mathcal{D} \text{ is a countable cover of } A \right\}.$$

Then $^{(\mathrm{a})}\mathcal{P}^{\varphi}$ is a metric outer measure on S.

Similar definitions may be given for the (b)-packing measure $^{(\mathrm{b})}\mathcal{P}^{\varphi}$ and the (c)-packing measure $^{(\mathrm{c})}\mathcal{P}^{\varphi}$. Because of the relations between the three types of packings, we have

$$^{(\mathrm{c})}\mathcal{P}^{\varphi}(A) \le {}^{(\mathrm{b})}\mathcal{P}^{\varphi}(A) \le {}^{(\mathrm{a})}\mathcal{P}^{\varphi}(A).$$

Note that if π is an (a)-packing, then the "halved" set of constituents

$$\pi_{\mathrm{h}} = \{\, (x, r/2) : (x,r) \in \pi \,\}$$

is a (c)-packing. This shows that if φ is blanketed, with constant C as in (4), then

$$^{(\mathrm{a})}\mathcal{P}^{\varphi}(A) \le C \cdot {}^{(\mathrm{c})}\mathcal{P}^{\varphi}(A).$$

So for blanketed φ, all three definitions agree within a constant factor. But for Hausdorff functions that are not blanketed (such as (3)), they need not agree within a constant factor.

1.2 Gauge Variation

It was shown in [20] (in the real line and in [6] for metric space) that the packing measure is a gauge variation (in the sense of Henstock [13] and Thomson [23]). But that was proved only for Hausdorff functions of the form (1) and for (b)-packings. We will consider here how this works in the present setting. Primarily we will simply follow [6].

Definition 1.7. Let (S,ρ) be a metric space and let $A \subseteq S$. A **gauge** for A is a function $\Delta : A \to (0,\infty)$.

Definition 1.8. If Δ is a gauge and π is a collection of constituents, then we say π is **Δ-fine** iff $r \le \Delta(x)$ for all $(x,r) \in \pi$.

Definition 1.9. Let φ be a Hausdorff function. For a gauge Δ, define

$$^{(a)}\mathcal{P}^{\varphi}_{\Delta}(A) = \sup \sum_{(x,r)\in\pi} \varphi(r),$$

where the supremum is over all Δ-fine (a)-packings π of A. As Δ decreases pointwise, the value $^{(a)}\mathcal{P}^{\varphi}_{\Delta}(A)$ decreases. For the limit, write

$$^{(a)}\mathcal{P}^{\varphi}_{\bullet}(A) = \inf_{\Delta} {}^{(a)}\mathcal{P}^{\varphi}_{\Delta}(A),$$

where the infimum is over all gauges Δ for A.

The set-function $^{(a)}\mathcal{P}^{\varphi}_{\bullet}$ is a metric outer measure. The proof is the same as in [23], [6].

If the gauge Δ is the constant ε, then the notations agree:

$$^{(a)}\mathcal{P}^{\varphi}_{\varepsilon}(A) = {}^{(a)}\mathcal{P}^{\varphi}_{\Delta}(A).$$

We will see below that $^{(a)}\mathcal{P}^{\varphi}_{\bullet}(A) = {}^{(a)}\mathcal{P}^{\varphi}(A)$.

In the definition of $^{(a)}\mathcal{P}^{\varphi}_{\Delta}(A)$, it suffices to use *finite* packings, because of the sup in the definition. Also, since we are using $\varphi(r)$ and not $\varphi(\operatorname{diam} B_r(x))$, it does not matter if we use closed balls instead of open balls for the definitions: Indeed, if $\{\overline{B}_{r_i}(x_i)\}$ is a packing by closed balls, then $\{B_{r_i}(x_i)\}$ will be a packing by open balls with the same radii r_i; and if $\{B_{r_i}(x_i)\}$ is a packing by open balls, then $\{\overline{B}_{r_i-\eta_i}(x_i)\}$ is a packing by closed balls for any positive values of η_i, and by (left-)continuity $\sum \varphi(r_i - \eta_i)$ may be made as close as we like to $\sum \varphi(r_i)$.

Proposition 1.10. *Let $\overline{A}$ denote the closure of A. Then*

$$^{(a)}\widetilde{\mathcal{P}}^{\varphi}(A) = {}^{(a)}\widetilde{\mathcal{P}}^{\varphi}\left(\overline{A}\right).$$

PROOF. Any ε-fine (a)-packing of A is also an ε-fine (a)-packing of $\overline{A}$, so $^{(a)}\mathcal{P}^{\varphi}_{\varepsilon}(A) \le {}^{(a)}\mathcal{P}^{\varphi}_{\varepsilon}(\overline{A})$, and thus $^{(a)}\widetilde{\mathcal{P}}^{\varphi}(A) \le {}^{(a)}\widetilde{\mathcal{P}}^{\varphi}(\overline{A})$. On the other hand, let $\pi = \{(x_i, r_i)\}_{i=1}^n$ be a finite (a)-packing of $\overline{A}$. Given $\eta > 0$, choose $y_i \in A$ with $\rho(y_i, x_i) < \eta$. Then $\{(y_i, r_i - 2\eta)\}_{i=1}^n$ is an (a)-packing of A and choosing η small enough makes $\sum_{i=1}^n \varphi(r_i - 2\eta)$ as close as we like to $\sum_{i=1}^n \varphi(r_i)$, because

φ is (left-)continuous. This shows that, for any $\varepsilon > 0$, ${}^{(a)}\mathcal{P}^{\varphi}_{\varepsilon}(A) \geq {}^{(a)}\mathcal{P}^{\varphi}_{\varepsilon}(\overline{A})$. [Note: this does not work for ${}^{(a)}\mathcal{P}^{\varphi}_{\Delta}$, since $\Delta(y_i)$ could be much smaller than $\Delta(x_i)$.] So we have ${}^{(a)}\widetilde{\mathcal{P}}^{\varphi}(A) = {}^{(a)}\widetilde{\mathcal{P}}^{\varphi}(\overline{A})$. $\square$

¿From the preceding result we see that in the definition of ${}^{(a)}\mathcal{P}^{\varphi}(E)$, we may use covers by closed sets:

$$\mathcal{P}^s(A) = \inf\left\{ \sum_{D\in\mathcal{D}} \widetilde{\mathcal{P}}^s(D) : \mathcal{D} \text{ is a countable cover of } A \text{ by closed sets} \right\}.$$

The next consequence is the **regularity** of ${}^{(a)}\mathcal{P}^{\varphi}$: If A is any set, then there is an $F_{\sigma\delta}$-set E with $A \subseteq E$ and ${}^{(a)}\mathcal{P}^{\varphi}(A) = {}^{(a)}\mathcal{P}^{\varphi}(E)$. ¿From regularity we conclude: if $E_n \nearrow E$, then ${}^{(a)}\mathcal{P}^{\varphi}(E_n) \to {}^{(a)}\mathcal{P}^{\varphi}(E)$, even for non-measurable E_n.

Proposition 1.11. *For any set E, we have ${}^{(a)}\mathcal{P}^{\varphi}_{\bullet}(E) = {}^{(a)}\mathcal{P}^{\varphi}(E)$.*

Proof. Constants ε are among the gauges Δ, so ${}^{(a)}\widetilde{\mathcal{P}}^{\varphi}(E) \geq {}^{(a)}\mathcal{P}^{\varphi}_{\bullet}(E)$. If $E \subseteq \bigcup E_n$, then ${}^{(a)}\mathcal{P}^{\varphi}_{\bullet}(E) \leq \sum_n {}^{(a)}\mathcal{P}^{\varphi}_{\bullet}(E_n) \leq \sum_n {}^{(a)}\widetilde{\mathcal{P}}^{\varphi}(E_n)$. Take the infimum over all covers to obtain ${}^{(a)}\mathcal{P}^{\varphi}_{\bullet}(E) \leq {}^{(a)}\mathcal{P}^{\varphi}(E)$.

On the other hand, suppose Δ is a gauge on a set E. For each positive integer n, let

$$E_n = \left\{ x \in E : \Delta(x) \geq \frac{1}{n} \right\}.$$

Then $E_n \nearrow E$. For each n,

$${}^{(a)}\mathcal{P}^{\varphi}_{\Delta}(E) \geq {}^{(a)}\mathcal{P}^{\varphi}_{\Delta}(E_n) \geq {}^{(a)}\mathcal{P}^{\varphi}_{1/n}(E_n) \geq {}^{(a)}\widetilde{\mathcal{P}}^{\varphi}(E_n) \geq {}^{(a)}\mathcal{P}^{\varphi}(E_n).$$

Take the limit as $n \to \infty$ to get ${}^{(a)}\mathcal{P}^{\varphi}_{\Delta}(E) \geq {}^{(a)}\mathcal{P}^{\varphi}(E)$. This is true for all gauges Δ, so ${}^{(a)}\mathcal{P}^{\varphi}_{\bullet}(E) \geq {}^{(a)}\mathcal{P}^{\varphi}(E)$. $\square$

1.3 (b)- and (c)-Packings

The interested reader may check that everything in subsection 1.2 works also for the (b)- and (c)-packing measures.

2 Strong Vitali Property

To what extent do the known results about packing measures $\mathcal{P}^s$ in Euclidean space generalize to other metric spaces, other Hausdorff functions, and/or other types of packings? As we have seen above, many such results remain true with the same proofs (at least when we are careful to use (twice the) radius and not the actual diameter of a ball).

But in my discussion [7] of the packing measures $\mathcal{P}^s$, there are a few places where the "strong Vitali property" is used. Since Euclidean spaces $\mathbb{R}^d$ have this property it is not a restrictive assumption. Does this mean, however, that the results do not generalize beyond Euclidean space?

Here are the usual definitions.

Definition 2.1. Let (S, ρ) be a metric space, and let $A \subseteq S$. A **fine cover** of A is a collection β of constituents (x, r) such that for every $x \in A$ and every $\varepsilon > 0$, there is $(x, r) \in \beta$ with $r < \varepsilon$.

Definition 2.2. Let (S, ρ) be a metric space, and let μ be a Borel measure on S. We say that μ has the **strong Vitali property** iff for any Borel set $E \subseteq S$ with $\mu(E) < \infty$ and any fine cover β of E, there exists a countable $\pi \subseteq \beta$ such that the balls $B_r(x)$ with $(x, r) \in \pi$ are disjoint and

$$\mu\left(E \setminus \bigcup_{(x,r)\in\pi} B_r(x)\right) = 0.$$

According to Besicovitch [1] every Borel measure in Euclidean space $\mathbb{R}^d$ has the strong Vitali property (see [7], Theorem 1.3.13). More generally, a metric space is said to be "finite-dimensional in the sense of Larman" iff there is a fixed constant K so that every ball $B_r(x)$ can be covered by K balls of radius $r/2$. Larman [16] showed that every Borel measure in such a metric space has the strong Vitali property.

What can be said about other metric spaces? Certainly the strong Vitali property fails for some measures in some metric spaces, but is it perhaps always true for packing measures ${}^{(\mathrm{a})}\mathcal{P}^{\varphi}$? A counterexample to this is given below. Or perhaps the strong Vitali property is true for packing measures $\mathcal{P}^s$ for finite s? Even that is not clear, since a metric space that is finite-dimensional in the sense of Hausdorff measure or packing measure need not be finite-dimensional in the sense of Larman. This question is (1.8.5) in [7]. Note that Haase ([10], Theorem 2) proved that certain packing-type measures necessarily have a Vitali property. But (as Cutler noted in [3], Remark 3.19) Haase's theorem does not include the centered-ball radius-packing measures we are considering here.

We may formulate variants of the strong Vitali property corresponding to our three types of packings. Many other variant Vitali properties may be found in [12].

Definition 2.3. Let (S, ρ) be a metric space, and let μ be a Borel measure on S. We say that μ has the **(a)-Vitali property** (respectively, (b)-Vitali, (c)-Vitali) iff for any Borel set $E \subseteq S$ with $\mu(E) < \infty$ and any fine cover β of E, there exists a countable (a)-packing $\pi \subseteq \beta$ of E (respectively, (b)-packing, (c)-packing) such that

$$\mu\left(E \setminus \bigcup_{(x,r)\in\pi} B_r(x)\right) = 0.$$

Of course, the (b)-Vitali property is just another name for the strong Vitali property defined above. If a measure μ has the (c)-Vitali property, then μ has the (b)-Vitali property. If μ has the (b)-Vitali property, then μ has the (a)-Vitali property.

2.1 Density Inequality

One result that uses the strong Vitali property in its proof (at least in [7], Theorem 1.5.11) is the density inequality, useful in estimating a packing measure or packing dimension.

Definition 2.4. Let (S, ρ) be a metric space, let $x \in S$, let φ be a Hausdorff function, and let μ be a finite Borel measure. The **lower φ-density** of μ at x is:

$$\underline{D}^{\varphi}_{\mu}(x) = \liminf_{r \searrow 0} \frac{\mu\big(B_r(x)\big)}{\varphi(r)}.$$

The upper φ-density is defined with lim sup instead of lim inf.

Here is the theorem using the strong Vitali property. Since it is normally stated only in Euclidean space, or only for packing measures $\mathcal{P}^s$, I will repeat the proof here.

Theorem 2.5. *Let S be a metric space, and let μ be a finite Borel measure Let $E \subseteq S$ be a Borel set, and let φ be a Hausdorff function. Then*

$$^{(\mathrm{b})}\mathcal{P}^{\varphi}(E) \inf_{x\in E} \underline{D}^{\varphi}_{\mu}(x) \le \mu(E). \tag{5}$$

It follows that

$$^{(\mathrm{c})}\mathcal{P}^{\varphi}(E) \inf_{x\in E} \underline{D}^{\varphi}_{\mu}(x) \le \mu(E).$$

If μ has the (a)-Vitali property, then

$$\mu(E) \le {}^{(\mathrm{a})}\mathcal{P}^{\varphi}(E) \sup_{x\in E} \underline{D}^{\varphi}_{\mu}(x), \tag{6}$$

provided this product is not 0 *times* ∞. *If* μ *has the (b)-Vitali property, then*

$$\mu(E) \leq {}^{(\mathrm{b})}\mathcal{P}^{\varphi}(E) \sup_{x\in E} \underline{D}_{\mu}^{\varphi}(x), \tag{7}$$

provided this product is not 0 *times* ∞. *If* μ *has the (c)-Vitali property, then*

$$\mu(E) \leq {}^{(\mathrm{c})}\mathcal{P}^{\varphi}(E) \sup_{x\in E} \underline{D}_{\mu}^{\varphi}(x), \tag{8}$$

provided this product is not 0 *times* ∞. *Finally, if the Hausdorff function* φ *satisfies (4), then even if* μ *satisfies no Vitali property, we have*

$$\mu(E) \leq C^2 \cdot {}^{(\mathrm{b})}\mathcal{P}^{\varphi}(E) \sup_{x\in E} \underline{D}_{\mu}^{\varphi}(x), \tag{9}$$

provided this product is not 0 *times* ∞.

PROOF. We begin with the proof of ${}^{(\mathrm{b})}\mathcal{P}^{\varphi}(E) \inf_{x\in E} \underline{D}_{\mu}^{\varphi}(x) \leq \mu(E)$. We may assume that $\inf_{x\in E} \underline{D}_{\mu}^{\varphi}(x) > 0$. Let $h > 0$ be a constant such that $\underline{D}_{\mu}^{\varphi}(x) > h$ for all $x \in E$. I must show that $h \cdot {}^{(\mathrm{b})}\mathcal{P}^{\varphi}(E) \leq \mu(E)$. Let $\varepsilon > 0$ be given. Then there is an open set $V \supseteq E$ such that $\mu(V) < \mu(E) + \varepsilon$. For $x \in E$, let $\Delta(x) > 0$ be so small that

$$\begin{gathered} \tfrac{\mu\left(B_r(x)\right)}{\varphi(r)} > h \quad \text{for all } r < \Delta(x) \\ \Delta(x) < \operatorname{dist}(x, S \setminus V). \end{gathered}$$

Then Δ is a gauge for E. Let π be a Δ-fine (b)-packing of E. Then $\bigcup_{\pi} B_r(x)$ is contained in V, and

$$\sum_{(x,r)\in\pi} \varphi(r) < \frac{1}{h} \sum_{\pi} \mu\big(B_r(x)\big) \leq \frac{1}{h}\mu(V).$$

This shows that

$${}^{(\mathrm{b})}\mathcal{P}^{\varphi}(E) \leq {}^{(\mathrm{b})}\mathcal{P}_{\Delta}^{\varphi}(E) \leq \frac{1}{h}\mu(V) \leq \frac{1}{h}\big(\mu(E) + \varepsilon\big).$$

Let $\varepsilon \to 0$ to obtain ${}^{(\mathrm{b})}\mathcal{P}^{\varphi}(E) \leq (1/h)\mu(E)$ as required.

Next, suppose μ has the (a)-Vitali property. I must show that

$$\mu(E) \leq {}^{(\mathrm{a})}\mathcal{P}^{\varphi}(E) \sup_{x\in E} \underline{D}_{\mu}^{\varphi}(x).$$

We may assume that $\sup_{x\in E} \underline{D}^{\varphi}_{\mu}(x) < \infty$. Let $h < \infty$ satisfy $\underline{D}^{\varphi}_{\mu}(x) < h$ for all $x \in E$. I must show that $\mu(E) \le h \cdot {}^{(a)}\mathcal{P}^{\varphi}(E)$. Let Δ be a gauge on E. Then

$$\beta = \left\{ (x,r) : x \in E, r < \Delta(x), \frac{\mu\big(B_r(x)\big)}{\varphi(r)} \le h \right\}$$

is a fine cover of E. By the (a)-Vitali property, there is an (a)-packing $\pi \subseteq \beta$ of E with $\mu(E) = \mu\big(E \cap \bigcup_\pi B_r(x)\big)$. Thus

$$\mu(E) = \mu\left(E \cap \bigcup_{\pi} B_r(x) \right) \le \sum_{\pi} \mu\big(B_r(x)\big) \le h \sum_{\pi} \varphi(r).$$

So $\mu(E) \le h \cdot {}^{(a)}\mathcal{P}^{\varphi}_{\Delta}(E)$. But Δ was arbitrary, so $\mu(E) \le h \cdot {}^{(a)}\mathcal{P}^{\varphi}(E)$ as required.

The proofs for the (b)- and (c)-Vitali properties are the same.

Now if φ satisfies (4), then we have $\varphi(4r) \le C^2\varphi(r)$. I must show that

$$\mu(E) \le C^2 \cdot {}^{(b)}\mathcal{P}^{\varphi}(E) \sup_{x\in E} \underline{D}^{\varphi}_{\mu}(x).$$

We may assume that $\sup_{x\in E} \underline{D}^{\varphi}_{\mu}(x) < \infty$. Let $h < \infty$ satisfy $\underline{D}^{\varphi}_{\mu}(x) < h$ for all $x \in E$. I must show that $\mu(E) \le hC^2 \cdot {}^{(a)}\mathcal{P}^{\varphi}(E)$. Let Δ be a gauge on E. Then

$$\beta = \left\{ (x,r) : x \in E, r < \Delta(x), \frac{\mu\big(B_{4r}(x)\big)}{\varphi(4r)} \le h \right\}$$

is a fine cover of E. Now use for example [7], Theorem 1.3.1, to conclude that there is a (b)-packing $\{ (x_i, r_i) : i = 1, 2, \cdots \} \subseteq \beta$ such that

$$E \subseteq \bigcup_{i=1}^{\infty} \overline{B}_{3r_i}(x_i) \subseteq \bigcup_{i=1}^{\infty} B_{4r_i}(x_i).$$

Thus

$$\mu(E) \le \sum_{i=1}^{\infty} \mu\big(B_{4r_i}(x_i)\big) \le h \sum_{i=1}^{\infty} \varphi(4r_i) \le hC^2 \sum_{i=1}^{\infty} \varphi(r_i).$$

So $\mu(E) \le hC^2 \cdot {}^{(b)}\mathcal{P}^{\varphi}_{\Delta}(E)$. But Δ was arbitrary, so $\mu(E) \le hC^2 \cdot {}^{(b)}\mathcal{P}^{\varphi}(E)$ as required. $\square$

3 Davies' Space

R. O. Davies [4] constructed an interesting example of a metric space in which the strong Vitali property fails. This space (or a variant of it) will be described here.

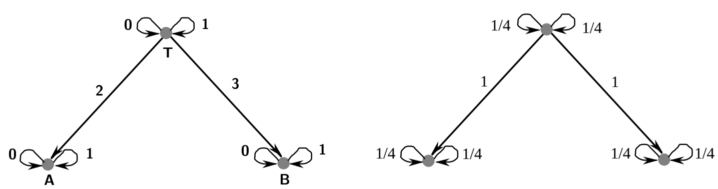

Figure 1: Graph $\mathbf{G}(5)$...

3.1 The Metric Space

For a given positive integer N, let $\mathbf{G}(N)$ be a (finite) graph defined as follows. The vertices are pairs (i, j) of integers with $1 \leq i \leq N$, $0 \leq j \leq N$. The total number of vertices is $N(N+1)$. Vertices $(i, 0)$ are called **central** vertices, and vertices $(i, j), j \neq 0$, are called **peripheral** vertices. The edges in the graph $\mathbf{G}(N)$ are as follows: A peripheral vertex (i, j) is joined only to $(i, 0)$, called its **central neighbor**. A central vertex $(i, 0)$ is joined to all other central vertices $(i', 0)$, as well as to the vertices $(i, j), j \neq 0$, called the **peripheral neighbors** of $(i, 0)$. Given two vertices v, u, we will write $v \sim u$ if $v = u$ or v is joined to u by an edge. We write $v \nsim u$ if not $v \sim u$. Note that each vertex has an odd number of neighbors: a peripheral vertex has only one neighbor; while a central vertex has $2N - 1$ neighbors.

Now we choose a sequence $N_1, N_2, \cdots$ of integers ≥ 2 such that

$$\prod_{n=1}^{\infty} \frac{N_n - 1}{N_n + 1} \geq \frac{1}{3}.$$

For any sequence N_n with $\sum 1/N_n < \infty$, this infinite product converges, and omitting the first few factors if necessary will give us a sequence so that this product is $\geq 1/3$. Now if we define recursively

$$\gamma_0 = 1, \qquad \gamma_n = \frac{1}{N_n(N_n+1)} \gamma_{n-1},$$

then the sequence γ_n strictly decreases to 0.

We will define our metric space. The set of points is the Cartesian product of countably many graphs:

$$\Omega = \prod_{n=1}^{\infty} \mathbf{G}(N_n).$$

If $u \in \Omega$, we will write $u = (u_1, u_2, u_3, \cdots)$, where $u_i \in \mathbf{G}(N_i)$ for all i. Next we define the metric ρ on Ω. If $u \in \Omega$, then of course $\rho(u,u) = 0$. If $u, v \in \Omega$ and $u \neq v$, let n be the least integer such that $u_n \neq v_n$; if $u_n \sim v_n$ in $\mathbf{G}(N_n)$, then $\rho(u,v) = (1/2)^n$; if $u_n \nsim v_n$, then $\rho(u,v) = (1/2)^{n-1}$. It may be checked that this defines a metric on Ω. Note that Ω is compact, separable, and totally disconnected.

Given a finite sequence $w_1 \in \mathbf{G}(N_1), w_2 \in \mathbf{G}(N_2), \cdots, w_n \in \mathbf{G}(N_n)$, define a **cylinder**:

$$\Omega(w_1, w_2, \cdots, w_n) = \{\, u \in \Omega : u_1 = w_1, u_2 = w_2, \cdots, u_n = w_n \,\}.$$

The diameter of cylinder $\Omega(w_1, w_2, \cdots, w_n)$ is $(1/2)^n$. We say that a cylinder $\Omega(w_1, w_2, \cdots, w_n)$ is **central** or **peripheral** according as the last coordinate w_n is central or peripheral. Every cylinder is closed and open (clopen).

We may now describe the open balls in the metric space Ω. Let $u \in \Omega$ and r be given, $0 < r < 1$. Then define n such that $(1/2)^n < r \leq (1/2)^{n-1}$. Then

$$B_r(u) = \{\, v : u_1 = v_1, \cdots, u_{n-1} = v_{n-1}, u_n \sim v_n \,\}.$$

If $u_n = (i, 0)$ is central, then $B_r(u) =$

$$\bigcup \{\, \Omega(u_1, u_2, \cdots, u_{n-1}, w) : w = (i', 0) \text{ for some } i' \text{ or } w = (i,j) \text{ for some } j \,\}$$

is a union of $2N_n$ cylinders, half central and half peripheral. Such a ball will be called a **central ball**. If $u_n = (i,j)$ is peripheral, then

$$B_r(u) = \bigcup \{\, \Omega(u_1, u_2, \cdots, u_{n-1}, w) : w = (i,j) \text{ or } w = (i,0) \,\}$$

is a union of two cylinders, one central and one peripheral. Such a ball will be called a **peripheral ball**.

For future use, we will identify a particular subset of each ball. If u, r, n are as before, then the ball $B_r(u)$ has just been described. If u_n is peripheral in $\mathbf{G}(N_n)$, so that $B_r(u)$ is a peripheral ball, let

$$B_r^{\mathrm{p}}(u) = \Omega(u_1, u_2, \cdots, u_{n-1}, u_n).$$

This cylinder, "half" of the ball, will be called the **principal cylinder** of $B_r(u)$. On the other hand, if $u_n = (i,0)$ is central in $\mathbf{G}(N_n)$, so that $B_r(u)$ is a central ball, then we set (arbitrarily)

$$B_r^{\mathrm{p}}(u) = \Omega(u_1, u_2, \cdots, u_{n-1}, (i,1)).$$

This cylinder will also be called the **principal cylinder** of the ball $B_r(u)$, although there is no reason for the choice of $(i,1)$ over any other peripheral

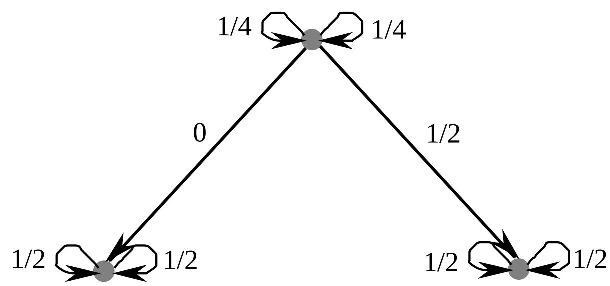

Figure 2: Relation between two balls (p and c mean peripheral and central)

(i, j). So with these definitions we have a principal cylinder associated to each ball. Of course $B_r^{\mathrm{p}}(u) \subseteq B_r(u)$. When n and r are related as usual, the diameter of the principal cylinder $B_r^{\mathrm{p}}(u)$ is $(1/2)^n$.

Now suppose we are given two balls, $B_r(u)$, $B_s(v)$. Let us discuss the criteria for

(a) $B_r(u) \cap B_s(v) = \varnothing$; and for

(b) $v \notin B_r(u)$ and $u \notin B_s(v)$.

These criteria will be useful information when we describe packings.

First consider the case where the balls of of the same size in the sense $(1/2)^n < r, s \leq (1/2)^{n-1}$ for some n. (i) If $u_i \neq v_i$ for some $i < n$, then the balls are disjoint. (ii) If $u_i = v_i$ for all $i < n$ and $u_n \sim v_n$, then each ball contains the center of the other (this includes the case $u_n = v_n$, when the balls coincide). (iii) If $u_i = v_i$ for all $i < n$ and $u_n \nsim v_n$, then we must consider further cases: u_n and v_n are not both central; (iiia) if u_n and v_n are both peripheral with different central neighbors, then the balls are disjoint; (iiib) if u_n and v_n are both peripheral with the same central neighbor, then the balls are not disjoint, but neither ball contains the center of the other; (iiic) if one of u_n, v_n is central and the other is peripheral, then again the balls are not disjoint, but neither ball contains the center of the other. So in all of these cases, either each ball contains the center of the other, or neither ball contains the center of the other.

Now consider the case where the balls are of different sizes: say $(1/2)^n < r \le (1/2)^{n-1} \le (1/2)^m < s \le (1/2)^{m-1}$. (iv) If $u_i \ne v_i$ for some $i < m$, then the balls are disjoint. (v) If $u_i = v_i$ for all $i < m$ and $u_m \not\sim v_m$, then again the balls are disjoint. (vi) If If $u_i = v_i$ for all $i < m$ and $u_m \sim v_m$, then $B_r(u) \subseteq B_s(v)$ (this includes the case where $u_m = v_m$). So for these balls, either one contains the other, or else they are disjoint.

An observation we can make by examining these case is that for two balls $B_r(u)$, $B_s(v)$, if $u \notin B_s(v)$ and $v \notin B_r(u)$, then their principal cylinders $B_r^{\mathrm{p}}(u), B_s^{\mathrm{p}}(v)$ are disjoint.

3.2 Uniform Measure

The **uniform measure** on Ω is defined as follows. Let $\mu\big(\Omega(w_1, w_2, \cdots, w_n)\big) = \gamma_n$. With $\mu(\Omega) = \gamma_0 = 1$, we get a set-function defined on the semi-ring of cylinders. Since $\mathbf{G}(N_n)$ has $N_n(N_n+1)$ points, the recursion for γ_n shows that this set function is additive. It is trivially σ-additive in the sense that a disjoint union of cylinders is only a cylinder when the union is actually finite. So the usual extension theorem yields an extension to the Borel sets, which will also be called μ, and which will be called the uniform measure on Ω.

¿From the description of the balls, above, we may compute the measure of a ball. Let $u \in \Omega$ and r be given, with $0 < r < 1$. Choose n so that $(1/2)^n < r \le (1/2)^{n-1}$. If u_n is a central vertex of $\mathbf{G}(N_n)$, then $B_r(u)$ is a central ball, and $\mu(B_r(u)) = 2N_n\gamma_n$. If u_n is a peripheral vertex of $\mathbf{G}(N_n)$, then $B_r(u)$ is a peripheral ball, and $\mu(B_r(u)) = 2\gamma_n$.

3.3 (a)-Packing Measure

Note that the sequence γ_n decreases to 0, so there is a Hausdorff function φ with

$$\varphi\big((1/2)^{n-1}\big) = \gamma_n. \tag{10}$$

This is the type of Hausdorff function that we will consider in our metric space Ω. We will show that the uniform measure μ is the (a)-packing measure ${}^{(\mathrm{a})}\mathcal{P}^\varphi$.

Proposition 3.1. *Let φ be a Hausdorff function satisfying (10) and let $E \subseteq \Omega$. Then ${}^{(\mathrm{a})}\mathcal{P}^\varphi(E) \le \mu(E)$.*

Proof. If $B_r(u)$ is a ball, let n be such that $(1/2)^n < r \le (1/2)^{n-1}$. The principal cylinder $B_r^{\mathrm{p}}(u)$ has measure γ_n, so

$$\mu\big(B_r^{\mathrm{p}}(u)\big) = \gamma_n = \varphi\big((1/2)^{n-1}\big) \ge \varphi(r).$$

Recall that if π is an (a)-packing, then the principal cylinders of the balls in π are disjoint.

Now let $E \subseteq \Omega$ be given. Let U be an open set with $E \subseteq U$. Then $\Delta(u) = \mathrm{dist}(u, \Omega \setminus U)$ is a gauge on E. Let π be a Δ-fine (a)-packing of E. Then by disjointness,

$$\sum_{(u,r)\in\pi} \varphi(r) \leq \sum_{(u,r)\in\pi} \mu\big(B_r^{\mathrm{p}}(u)\big) \leq \mu(U).$$

Take the supremum on π to get ${}^{(\mathrm{a})}\mathcal{P}^{\varphi}_{\Delta}(E) \leq \mu(U)$. Thus ${}^{(\mathrm{a})}\mathcal{P}^{\varphi}(E) \leq \mu(U)$. Take the infimum on U to conclude ${}^{(\mathrm{a})}\mathcal{P}^{\varphi}(E) \leq \mu(E)$. $\square$

Lemma 3.2. *Let φ be a Hausdorff function satisfying (10). Then ${}^{(\mathrm{a})}\mathcal{P}^{\varphi}(\Omega) = 1$.*

PROOF. Wrote $\overline{\mu}$ for the outer measure generated by μ. Let $\varepsilon > 0$ be given and let Δ be a gauge on Ω. Now $\Delta(u) > 0$ for all u, so (by regularity of the finite Borel measure μ) we may choose $m \in \mathbb{N}$ so that

$$\overline{\mu}\left\{\, u \in \Omega : \Delta(u) > (1/2)^{m-1} \,\right\} > 1 - \frac{\varepsilon}{2}.$$

(We have used the outer measure $\overline{\mu}$ because we do not know that the set is measurable.)

Choose $n \geq m$ so that $1/(N_n+1) < \varepsilon/2$. The graph $\mathbf{G}(N_n)$ has $N_n(N_n+1)$ elements, of which N_n^2 are peripheral, so $\mu\{\, u : u_n \text{ is peripheral} \,\} > 1 - \varepsilon/2$. Write

$$A = \left\{\, u : u_n \text{ is peripheral and } \Delta(u) > (1/2)^{n-1} \,\right\}.$$

So $\overline{\mu}(A) > 1 - \varepsilon$.

Now that n has been chosen, we will write $r = (1/2)^{n-1}$ and define a packing π_n. Let $R_n = \{\, (w_1, \cdots, w_n) : \Omega(w_1, \cdots, w_n) \cap A \neq \varnothing \,\}$, and let M_n be the number of elements of R_n. To estimate M_n, note that the measure of each cylinder $\Omega(w_1, \cdots, w_n)$ is γ_n and

$$\bigcup_{(w_1,\cdots,w_n)\in R_n} \Omega(w_1, \cdots, w_n) \supseteq A,$$

so $M_n \gamma_n \geq \overline{\mu}(A) \geq 1 - \varepsilon$. Let the packing π_n consist of constituents (u, r), where one $u \in \Omega(w_1, \cdots, w_n) \cap A$ is chosen for each $(w_1, \cdots, w_n) \in R_n$. Because each w_n is peripheral, this is a Δ-fine (a)-packing for Ω. Therefore

$${}^{(\mathrm{a})}\mathcal{P}^{\varphi}_{\Delta}(\Omega) \geq \sum_{\pi_n} \varphi(r) = M_n \gamma_n \geq 1 - \varepsilon.$$

This is true for all Δ, so ${}^{(\mathrm{a})}\mathcal{P}^{\varphi}(\Omega) \geq 1 - \varepsilon$. This is true for all $\varepsilon > 0$, so ${}^{(\mathrm{a})}\mathcal{P}^{\varphi}(\Omega) \geq 1$.

The opposite inequality is from 3.1. $\square$

Theorem 3.3. *Let $E \subseteq \Omega$ be any Borel set and let φ be a Hausdorff function that satisfies (10). Then* ${}^{(a)}\mathcal{P}^{\varphi}(E) = \mu(E)$.

PROOF. First, by Proposition 3.1, $\mu(E) \geq {}^{(a)}\mathcal{P}^{\varphi}(E)$. Of course this is also true for the complement: $\mu(\Omega \setminus E) \geq {}^{(a)}\mathcal{P}^{\varphi}(\Omega \setminus E)$. Now E is a measurable set, so we get $\mu(E) = 1 - \mu(\Omega \setminus E) \leq 1 - {}^{(a)}\mathcal{P}^{\varphi}(\Omega \setminus E) = {}^{(a)}\mathcal{P}^{\varphi}(E)$. Therefore $\mu(E) = {}^{(a)}\mathcal{P}^{\varphi}(E)$. $\square$

3.4 (b)-Packing Measure

Next let us consider the (b)-packing measure on Davies' space Ω. Now the situation is different. For any Hausdorff function φ, either ${}^{(b)}\mathcal{P}^{\varphi}(\Omega) = 0$ or ${}^{(b)}\mathcal{P}^{\varphi}(\Omega) = \infty$. We will characterize the Hausdorff functions for which ${}^{(b)}\mathcal{P}^{\varphi}(\Omega)$ is 0 or ∞ in terms of convergence of the series

$$\sum_{n=1}^{\infty} \frac{\varphi\left((1/2)^{n-1}\right)}{(N_n+1)\gamma_n}. \tag{11}$$

Proposition 3.4. *Let φ be a Hausdorff function such that (11) converges. Then* ${}^{(b)}\mathcal{P}^{\varphi}(\Omega) = 0$.

PROOF. Fix $m \in \mathbb{N}$, and write $\varepsilon = (1/2)^{m-1}$. Let π be an ε-fine (b)-packing for Ω. Any $(u,r) \in \pi$ has $(1/2)^n < r \leq (1/2)^{n-1}$ for some $n \geq m$. Inside a given cylinder $\Omega(w_1, w_2, \cdots, w_{n-1})$, among the balls $B_r(u)$ with $(1/2)^n < r \leq (1/2)^{n-1}$, the packing π contains at most one central ball $B_r(u)$ with $(1/2)^n < r \leq (1/2)^{n-1}$ or at most N_n peripheral balls $B_r(u)$ with $(1/2)^n < r \leq (1/2)^{n-1}$. Thus

$$\begin{aligned}
&\sum_{(u,r)\in\pi} \varphi(r) \\
&\leq \sum_{n=m}^{\infty} N_1(N_1+1)N_2(N_2+1)\cdots N_{n-1}(N_{n-1}+1)N_n\varphi((1/2)^{n-1}) \\
&= \sum_{n=m}^{\infty} \frac{\varphi\left((1/2)^{n-1}\right)}{(N_n+1)\gamma_n}.
\end{aligned}$$

Thus ${}^{(b)}\mathcal{P}^{\varphi}_{\varepsilon}(\Omega) \leq$ the same sum, a tail of a convergent series. So as $m \to \infty$ we see ${}^{(b)}\widetilde{\mathcal{P}}^{\varphi}(\Omega) = 0$ and therefore ${}^{(b)}\mathcal{P}^{\varphi}(\Omega) = 0$. $\square$

Lemma 3.5. *Let φ be a Hausdorff function such that (11) diverges. Then, for any nonempty open set $U \subseteq \Omega$, we have* ${}^{(b)}\widetilde{\mathcal{P}}^{\varphi}(U) = \infty$.

Proof. Any nonempty open set contains a cylinder $\Omega(w_1, \cdots, w_m)$. Let $\varepsilon > 0$ be given. By decreasing the value of ε and/or increasing the value of m, we may assume $\varepsilon = (1/2)^{m-1}$. We will define an ε-fine (b)-packing π for $\Omega(w_1, \cdots, w_m)$. Let us say a node (i, j) of the graph $\mathbf{G}(N_n)$ is **distinguished** if $j = 1$. Define the collection π to consist of all constituents (u, r) such that: (a) $r = (1/2)^{n-1}$ for some $n > m$; (b) $u_1 = w_1, \cdots, u_m = w_m$; (c) $u_{m+1}, \cdots, u_{n-1}$ are peripheral but not distinguished; and (d) u_n is distinguished. The balls of π are pairwise disjoint: any pair of them falls in case (iiia), (iv), or (v). So π is an ε-fine (b)-packing for $\Omega(w_1, \cdots, w_m)$. For each $n > m$, there are

$$N_{m+1}(N_{m+1} - 1) \cdots N_{n-1}(N_{n-1} - 1)N_n$$

constituents in π with radius $(1/2)^{n-1}$. Recall that the infinite product

$$\prod_{n=1}^{\infty} \frac{N_n - 1}{N_n + 1}$$

converges to a value $\geq 1/3$. Now compute

$$\begin{aligned}
&\sum_{(u,r)\in\pi} \varphi(r) \\
&= \sum_{n=m+1}^{\infty} N_{m+1}(N_{m+1} - 1) \cdots N_{n-1}(N_{n-1} - 1)N_n \varphi\left((1/2)^{n-1}\right) \\
&\geq \frac{1}{3} \sum_{n=m+1}^{\infty} N_{m+1}(N_{m+1} + 1) \cdots N_{n-1}(N_{n-1} + 1)N_n \varphi\left((1/2)^{n-1}\right) \\
&= \frac{\gamma_m}{3} \sum_{n=m+1}^{\infty} \frac{\varphi\left((1/2)^{n-1}\right)}{(N_n + 1)\gamma_n} = \infty.
\end{aligned}$$

Thus ${}^{(\mathrm{b})}\mathcal{P}^{\varphi}_{\varepsilon}(U) = \infty$; and since ε was arbitrary, ${}^{(\mathrm{b})}\widetilde{\mathcal{P}}^{\varphi}(U) = \infty$. $\square$

Proposition 3.6. *Let φ be a Hausdorff function such that (11) diverges. Then ${}^{(\mathrm{b})}\mathcal{P}^{\varphi}(\Omega) = \infty$.*

Proof. Cover $\Omega = \bigcup_{k=1}^{\infty} E_k$. By the Baire Category Theorem, at least one closure $\overline{E_k}$ has nonempty interior. But then ${}^{(\mathrm{b})}\widetilde{\mathcal{P}}^{\varphi}(E_k) = {}^{(\mathrm{b})}\widetilde{\mathcal{P}}^{\varphi}\left(\overline{E_k}\right) = \infty$. So $\sum_k {}^{(\mathrm{b})}\widetilde{\mathcal{P}}^{\varphi}(E_k) = \infty$. This is true for any countable cover of Ω, so ${}^{(\mathrm{b})}\mathcal{P}^{\varphi}(\Omega) = \infty$. $\square$

4 Vitali Property

4.1 Strong Vitali Property

The strong Vitali property fails for the packing measure ${}^{(a)}\mathcal{P}^\varphi$ in Davies' space Ω described above, where the Hausdorff function φ satisfies (10). For the construction that verifies this, we define

$$\theta_0 = 3, \qquad \theta_n = \frac{N_n - 1}{N_n + 1}\theta_{n-1}.$$

Then θ_n decreases to a limit ≥ 1.

Let $\delta_n = \gamma_n/\theta_n$, $\alpha_n = (\gamma_n + \delta_n)/2$, and $\beta_n = (\gamma_n - \delta_n)/2$. It follows that:

$$\alpha_0 = \tfrac{2}{3}, \qquad \beta_0 = \tfrac{1}{3}, \qquad \alpha_n \searrow 0, \qquad \beta_n \searrow 0, \qquad \alpha_n > \beta_n > 0$$
$$N_n^2\alpha_n + N_n\beta_n = \alpha_{n-1}, \qquad N_n^2\beta_n + N_n\alpha_n = \beta_{n-1}.$$

Now define two measures μ_1, μ_2, beginning with the cylinders

$$\Omega(w_1, w_2, \cdots, w_n).$$

We count how many of the vertices w_i are central. If w_i is central for an even number of i with $1 \leq i \leq n$, then

$$\mu_1\big(\Omega(w_1, w_2, \cdots, w_n)\big) = \alpha_n, \qquad \mu_2\big(\Omega(w_1, w_2, \cdots, w_n)\big) = \beta_n.$$

If w_i is central for an odd number of i with $1 \leq i \leq n$, then

$$\mu_1\big(\Omega(w_1, w_2, \cdots, w_n)\big) = \beta_n, \qquad \mu_2\big(\Omega(w_1, w_2, \cdots, w_n)\big) = \alpha_n.$$

(Since 0 is even, we have in particular that $\mu_1(\Omega) = \alpha_0 = 2/3$ and $\mu_2(\Omega) = \beta_0 = 1/3$.) These definitions produce set functions μ_1, μ_2 that are additive on cylinders, because of the relations $N_n^2\alpha_n + N_n\beta_n = \alpha_{n-1}, N_n^2\beta_n + N_n\alpha_n = \beta_{n-1}$. Thus, as before, these measures may be extended to all Borel sets. Note that $\alpha_n + \beta_n = \gamma_n$, so $\mu_1 + \mu_2 = \mu$ on cylinders, and therefore on all Borel sets.

Next we compute the measures of balls. If $u \in \Omega$ and r satisfies $(1/2)^n < r \leq (1/2)^{n-1}$, then the ball $B_r(u)$ is described above. If $u_n = (i, j)$ is peripheral, then $B_r(u)$ consists of two cylinders,

$$\Omega(u_1, u_2, \cdots, u_{n-1}, (i, j)) \quad \text{and} \quad \Omega(u_1, u_2, \cdots, u_{n-1}, (i, 0)).$$

One of these has an even number of central coordinates, and the other has an odd number. So the total measure is

$$\mu_1\big(B_r(u)\big) = \alpha_n + \beta_n = \gamma_n,$$

and similarly
$$\mu_2\big(B_r(u)\big) = \beta_n + \alpha_n = \gamma_n.$$

Similarly, if $u_n = (i, 0)$ is peripheral, then $B_r(u)$ consists of $2N_n$ cylinders, half central and half peripheral, and thus

$$\mu_1\big(B_r(u)\big) = N_n(\alpha_n + \beta_n) = N_n\gamma_n,$$

and similarly
$$\mu_2\big(B_r(u)\big) = N_n(\beta_n + \alpha_n) = N_n\gamma_n.$$

So for every ball $B_r(u)$ with $r < 1$ we have $\mu_1(B_r(u)) = \mu_2(B_r(u)) = (1/2)\mu(B_r(u))$.

Now consider the whole space Ω. Its packing measure is ${}^{(\mathrm{a})}\mathcal{P}^\varphi(\Omega) = \mu(\Omega) = 1$. The collection of all balls with radius < 1 is a fine cover of Ω. Suppose $\pi = \{B_{r_i}(u_i)\}$ is a collection of disjoint balls with $r_i < 1$. We claim that

$$\mu\left(\Omega \setminus \bigcup_{i=1}^{\infty} B_{r_i}(u_i)\right) \neq 0.$$

Indeed,

$$\begin{aligned}
\mu\left(\Omega \setminus \bigcup_{i=1}^{\infty} B_{r_i}(u_i)\right) &= 1 - \sum_{i=1}^{\infty} \mu\big(B_{r_i}(u_i)\big) \\
&= 1 - 2\sum_{i=1}^{\infty} \mu_2\big(B_{r_i}(u_i)\big) \\
&\geq 1 - 2\mu_2(\Omega) = 1 - 2\beta_0 = \frac{1}{3}.
\end{aligned}$$

So the strong Vitali property fails.

4.2 (a)-Vitali Property: Questions

The uniform measure μ (which is the packing measure ${}^{(\mathrm{a})}\mathcal{P}^\varphi$) fails the strong Vitali property. That is, μ fails the (b)-Vitali property. So μ also fails the (c)-Vitali property. What about the (a)-Vitali property? Is the (a)-Vitali property of any use, anyway?

4.3 Density Inequality

Consider the inequalities in Theorem 2.5. The lower bounds for $\mu(E)$ are stated only for ${}^{(\mathrm{b})}\mathcal{P}^\varphi$ and ${}^{(\mathrm{c})}\mathcal{P}^\varphi$, but not for ${}^{(\mathrm{a})}\mathcal{P}^\varphi$. Our example will show that the inequality is not correct for ${}^{(\mathrm{a})}\mathcal{P}^\varphi$.

Definition 4.1. A point $u = (u_1, u_2, \cdots)$ in the Davies metric space Ω is called **eventually peripheral** iff all but finitely many of the components u_k are peripheral.

Proposition 4.2. *The set P of eventually peripheral points satisfies $\mu(P) = 1$.*

PROOF. Write

$$P_m = \{\, u = (u_1, u_2, \cdots) \in \Omega : u_k \text{ is peripheral for all } k \geq m \,\}.$$

The graph $\mathbf{G}(N_k)$ has $N_k(N_k+1)$ vertices, and N_k^2 of them are peripheral. So

$$\mu(P_m) = \prod_{k=m}^{\infty} \frac{N_k^2}{N_k(N_k+1)} = \prod_{k=m}^{\infty} \frac{N_k}{N_k+1}.$$

By assumption, the infinite product $\prod (N_k - 1)/(N_k + 1)$ converges. But $N_k/(N_k + 1)$ is between $(N_k - 1)/(N_k + 1)$ and 1, so the infinite product $\prod N_k/(N_k+1)$ also converges. So the "tail" products must approach 1. That is, $\mu(P_m) \to 1$ as $m \to \infty$. But since P_m increases to P, we conclude that $\mu(P) = 1$. $\square$

Proposition 4.3. *Let Ω be the Davies metric space discussed above. Let μ be the uniform measure on Ω. Let φ be a Hausdorff function satisfying (10). Then*

$$^{(a)}\mathcal{P}^{\varphi}(E) \inf_{u \in E} \underline{D}_{\mu}^{\varphi}(u) \leq \mu(E). \tag{12}$$

is false for some Borel set E.

PROOF. In fact, (12) is false for the set P of eventually peripheral points. Let u be an eventually peripheral point. For r small enough, the ball $B_r(u)$ is a peripheral ball. Now if (as usual) n is chosen so that $(1/2)^n < r \leq (1/2)^{n-1}$, then

$$\frac{\mu\big(B_r(u)\big)}{\varphi(r)} = \frac{2\gamma_n}{\varphi(r)} \geq \frac{2\gamma_n}{\varphi\big((1/2)^{n-1}\big)} = \frac{2\gamma_n}{\gamma_n} = 2$$

with equality for $r = (1/2)^{n-1}$. Therefore $\underline{D}_{\mu}^{\varphi}(u) = 2$. So left-hand side of (12) is

$$^{(a)}\mathcal{P}^{\varphi}(P) \inf_{u \in P} \underline{D}_{\mu}^{\varphi}(u) = 2$$

but the right-hand side is $\mu(P) = 1$. $\square$

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