Real Analysis Exchange
Vol. (), , pp. 827–830

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NOTE ON THE OUTER MEASURES OF IMAGES OF SETS

Abstract

Let f be a real function on $\mathbb{R}$, let $\{I_v\}$ be a family of intervals covering a set E such that $m(E \cap I_v) \geq m\big(f(E \cap I_v)\big)$ for each I_v. We prove that $m\big(f(E)\big) \leq 2 \cdot m(E)$. No coefficient smaller than 2 will suffice here in general.

This note concerns the following well-known [HS] result.

Proposition. *Let f be a real valued function on the line $\mathbb{R}$, differentiable at each point of a set $E \subset \mathbb{R}$, such that $\big|f'(x)\big| \leq 1$ for each $x \in E$. Then $m\big(f(E)\big) \leq m(E)$, where m denotes Lebesgue outer measure.*

The usual proofs [HS] employ appropriate Vitali coverings of E or of $f(E)$. We wonder if the conclusion holds when we dispense with the derivative and just let E be covered by a family of intervals $\{I\}$ such that $m(E \cap I) \geq m\big(f(E \cap I)\big)$ for each I in the family. That $m(E) \geq m\big(f(E)\big)$ need not hold is shown in Proposition 1. Indeed $m\big(f(E)\big)$ could be almost twice $m(E)$.

Theorem 1. *Let f be a real valued function on $\mathbb{R}$, let E be a subset of $\mathbb{R}$ and let $\{I_v\}$ be a family of intervals covering E. Let $m(I_v) \geq m\big(f(E \cap I_v)\big)$ for each I_v. Then $m\big(f(E)\big) \leq 2m\big(\cup_v I_v\big)$. If moreover $m(E \cap I_v) \geq m\big(f(E \cap I_v)\big)$ for each I_v, then $m\big(f(E)\big) \leq 2 \cdot m(E)$.*

Proof. Let $U = \cup_v(\text{interior} I_v)$. Every point in $E \setminus U$ must be an endpoint of a component of U. Thus $E \setminus U$ is at most a countable set. Without loss of generality we can (and do) assume that $E \subset U$ and each I_v is an open interval. By the Lindelöf Theorem [C], there are countably many intervals $I_1, I_2, I_3, \ldots$ such that $U = \cup_{j=1}^{\infty} I_j$.

Let k be a real number such that $k < m\big(f(E)\big)$. Choose an index N such that

$$m\big(f(E \cap \cup_{j=1}^{N} I_j)\big) \geq k\,. \tag{1}$$

Key Words: coverings, Lebesgue outer measure
Mathematical Reviews subject classification: 28A12
Received by the editors October 31, 2000

We consider each subfamily of $X = \{I_1, I_2, \dots, I_N\}$ for which the union of its intervals equals $I_1 \cup I_2 \cup \dots \cup I_N$. Let $\{K_1, K_2, \dots, K_t\}$ be such a subfamily with a minimum number of intervals. Then no K_i can be a subset of $\cup_{j \neq i} K_j$; otherwise we could delete K_i. Thus no two K_i can have the same left endpoint. Say the K_i are $(a_1, b_1),\ (a_2, b_2), \dots, (a_t, b_t)$, where $a_1 < a_2 < \dots < a_t$. Now (a_1, b_1) cannot meet (a_j, b_j) for any $j > 2$; for if it did, either $(a_j, b_j) \subset (a_2, b_2)$ where $b_j \le b_2$, or $(a_2, b_2) \subset (a_1, b_1) \cup (a_j, b_j)$ where $b_j > b_2$. It follows likewise that the intervals $(a_1, b_1),\ (a_3, b_3),\ (a_5, b_5),\ \dots$ are mutually disjoint, and the intervals $(a_2, b_2),\ (a_4, b_4),\ (a_6, b_6),\ \dots$ are mutually disjoint. But

$$\begin{aligned} m\big(\cup_v I_v\big) &\ge m\big((a_1, b_1)\big) + m\big((a_3, b_3)\big) + m\big((a_5, b_5)\big) + \dots \\ &\ge m\Big(f\big(E \cap (a_1, b_1)\big)\Big) + m\Big(f\big(E \cap (a_3, b_3)\big)\Big) \\ &\quad + m\Big(f\big(E \cap (a_5, b_5)\big)\Big) + \dots \end{aligned} \tag{2}$$

and

$$\begin{aligned} m\big(\cup_v I_v\big) &\ge m\big((a_2, b_2)\big) + m\big((a_4, b_4)\big) + m\big((a_6, b_6)\big) + \dots \\ &\ge m\Big(f\big(E \cap (a_2, b_2)\big)\Big) + m\Big(f\big(E \cap (a_4, b_4)\big)\Big) \\ &\quad + m\Big(f\big(E \cap (a_6, b_6)\big)\Big) + \dots . \end{aligned} \tag{3}$$

We add (2) and (3) and obtain

$$\begin{aligned} 2 \cdot m\big(\cup_v I_v\big) &\ge m\Big(f\big(E \cap (a_1, b_1)\big)\Big) + m\Big(f\big(E \cap (a_2, b_2)\big)\Big) \\ &\quad + \dots + m\Big(f\big(E \cap (a_t, b_t)\big)\Big) \\ &\ge m\Big(f\big(E \cap \cup_{j=1}^t (a_j, b_j)\big)\Big) . \end{aligned} \tag{4}$$

But because $\cup_{j=1}^t (a_j, b_j) = \cup_{i=1}^N I_i$

$$m\Big(f\big(E \cap \cup_{j=1}^t (a_j, b_j)\big)\Big) = m\Big(f\big(E \cap \cup_{i=1}^N I_i\big)\Big). \tag{5}$$

From (1), (4) and (5) we obtain $2 \cdot m\big(\cup_v I_v\big) \ge k$. Because k was arbitrary, we have $2 \cdot m\big(\cup_v I_v\big) \ge m\big(f(E)\big)$.

For the second conclusion in Theorem 1, replace (a_i, b_i) with $E \cap (a_i, b_i)$ in (2), (3) and (4). $\square$

We now show that 2 is the smallest coefficient we can use in Theorem 1.

Proposition 1. *For each number $d < 2$, there is a continuous piecewise linear function F_d on $\mathbb{R}$ and intervals J_{d1}, J_{d2} in $\mathbb{R}$ for which*

$$m(J_{d1}) = m(J_{d2}) = m\big(F_d(J_{d1})\big) = m\big(F_d(J_{d2})\big)\,,$$

and

$$m\big(F_d(J_{d1} \cup J_{d2})\big) > d \cdot m\big(J_{d1} \cup J_{d2}\big)\,.$$

PROOF. Fix a real number p for which $0 < p < \frac{1}{4}$. In the plane $\mathbb{R}^2$, draw the segments from $(-\infty, 0)$ to $(0,0)$, from $(0,0)$ to $(p,1)$, from $(p,1)$ to $(1-p,1)$, from $(1-p,1)$ to $(1,2)$, and from $(1,2)$ to $(\infty,2)$. Let f_p be the real function on $\mathbb{R}$ whose graph is the broken line just constructed. From this graph we infer that $f_p\Big(\frac{p^2}{1-p}\Big) = \frac{p}{1-p}$, $f_p\Big(\frac{1-p-p^2}{1-p}\Big) = \frac{2-3p}{1-p}$ and $f_p(p) = f_p(1-p) = 1$. Let I_1 be the interval on the x-axis with endpoints $x = \frac{p^2}{1-p}$ and $x = 1-p$. Let I_2 be the interval on the x-axis with endpoints $x = p$ and $\frac{1-p-p^2}{1-p}$. It follows that the interval $f_p(I_1)$ on the y-axis has endpoints $y = \frac{p}{1-p}$ and $y = 1$; the interval $f_p(I_2)$ on the y-axis has endpoints $y = 1$ and $y = \frac{2-3p}{1-p}$. Direct computations give

$$m(I_1) = m(I_2) = m\big(f_p(I_1)\big) = m\big(f_p(I_2)\big) = \frac{1-2p}{1-p}\,,$$

$$m(I_1 \cup I_2) = \frac{1-p-2p^2}{1-p} \quad \text{and} \quad m\big(f_p(I_1 \cup I_2)\big) = \frac{2-4p}{1-p}\,.$$

It follows that $\frac{m\big(f_p(I_1 \cup I_2)\big)}{m(I_1 \cup I_2)} = \frac{2-4p}{1-p-2p^2}$. Clearly, for any positive number $d < 2$, there is a positive number $p < \frac{1}{4}$ (depending on d), for which

$$m\big(f_p(I_1 \cup I_2)\big) > d \cdot m(I_1 \cup I_2).$$

We infer from this the conclusion of Proposition 1. $\square$

We now list some consequences of our Theorem 1.

Corollary 1. *Let $f \in \mathbb{R}^{\mathbb{R}}$ and let $\{I_v\}$ be a family of intervals such that* $\big(\text{diameter } f(I_v)\big) \leq (\text{diameter} I_v)$ *for each I_v. Then* $m\big(f(\cup_v I_v)\big) \leq 2 \cdot m\big(\cup_v I_v\big)$.

PROOF. Note that $m\big(f(I_v)\big) \leq \big(\text{diameter } f(I_v)\big)$. Put $E = \cup_v I_v$ in Theorem 1. $\square$

Corollary 2. *Let f be absolutely continuous on $\mathbb{R}$ and let $\{I_v\}$ be a family of intervals such that $\cup_v I_v \subset [0,1]$, $m\big(\cup_v I_v\big) = 1$ and $m\big(f(I_v)\big) \leq m(I_v)$ for each I_v. Then* $\max f[0,1] \;-\; \min f[0,1] \;\leq\; 2$.

Proof. Now $\cup_v I_v$ is evidently measurable and $m\Big([0,1]\backslash(\cup_v I_v)\Big) = 0$. Because f is absolutely continuous we have $m\Big(f\big([0,1]\setminus(\cup_v I_v)\big)\Big) = 0$. Apply Theorem 1 with $E = \cup_v I_v$. $\square$

Corollary 3. *Let f be continuous on $\mathbb{R}$ and let $\{I_v\}$ be a family of intervals such that $\cup_v I_v \subset [0,1]$, the set $[0,1]\backslash(\cup_v I_v)$ is a countable set, and $m\big(f(I_v)\big) \leq m(I_v)$ for each I_v. Then* $\max f[0,1] \; - \; \min f[0,1] \; \leq \; 2$.

Proof. Use a scheme much like the proof of Corollary 2. $\square$

Corollary 4. *Let f be nondecreasing and let $\big\{(a_v, b_v)\big\}$ be a family of intervals such that $f(b_v) - f(a_v) \leq b_v - a_v$ for each interval (a_v, b_v). Then*

$$m\Big(f\big(\cup_v(a_v, b_v)\big)\Big) \;\leq\; 2 \cdot m\Big(\cup_v(a_v.b_v)\Big).$$

Proof. Corollary 1. $\square$

Corollary 5. *Let f be a continuous strictly increasing function on $\mathbb{R}$ and let $\big\{(a_v, b_v)\big\}$ be a family of intervals such that $f(b_v) - f(a_v) \geq b_v - a_v$ for each (a_v, b_v). Then $m\Big(f\big(\cup_v(a_v, b_v)\big)\Big) \geq \frac{m\big(\cup_v(a_v,b_v)\big)}{2}$.*

Proof. Let g be the inverse function of f on $f(\mathbb{R})$. Apply Corollary 4 to g and the family of intervals $\big\{f(a_v), f(b_v)\big\}$. $\square$

In the proof of Theorem 1 we should not expect that there necessarily exists a subfamily of mutually nonoverlapping intervals $I_1, I_2, I_3, \ldots$ with $\sum_j m(I_j) \;\geq\; \frac{m(\cup_v I_v)}{2}$. Let, for example, $J_n = \Big(-5^{-n}, \; \sum_{i=1}^{n} 2^{-i}\Big)$ for each positive integer n, and $E = \Big(\cup_{n=1}^{\infty} J_n\Big) \cup \Big(\cup_{n=1}^{\infty}(-J_n)\Big)$. Then each subfamily of mutually nonoverlapping intervals is a singleton family, but the length of no interval here is as large as $\frac{m(E)}{2}$.

References

[C] H. Cullen, *Introduction to General Topology*, D. C. Heath, Boston, 1968 (Theorem 18.15).

[HS] E. Hewitt and K, Stromberg, *Real and Abstract Analysis*, Springer- Verlag, New York, 1965 (Exercises (17.25), (17.26), (17.27)).