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Joanna Czarnowska,* Institute of Mathematics, University of Gdańsk, Wita Stwosza 57, Poland. e-mail: `jczarn@math.univ.gda.pl`

$\mathcal{F}$-CONNECTIVITY AND STRONG $\mathcal{F}$-CONNECTIVITY OF MULTIVALUED MAPS

Abstract

In the paper the general connectivity property is given for multivalued maps and the Darboux property, the intermediate value property, functional connectivity property, connectivity property etc. are considered as subcases of this property.

This general property is characterized locally, so as corollaries we obtain local characterization of the Darboux property, the intermediate value property etc. for multivalued maps and for real functions those classical results given by Bruckner, Ceder [2] and Garret, Nelms and Kellum [5].

Characterization of the sets of Darboux points, the intermediate value property points etc. for multivalued maps and for real functions are straightforward corollaries from one general theorem (Theorem 11).

1 Preliminaries

Let $\mathbb{R}$ denote the set of real numbers, I any interval contained in $\mathbb{R}$. If $A \subset I$, let $\overline{A}$ denote the closure of the set A in I and $A^c = I \setminus A$. For a non-empty set $A \subset \mathbb{R}^2$ and a number $\epsilon > 0$ we denote

$$K_\epsilon(A) = \{x \in \mathbb{R}^2 : \text{ there exists } y \in A \text{ such that } |x - y| < \epsilon\}.$$

For any sets $A, B \subset \mathbb{R}$ and any number $a \in \mathbb{R}$ we define

$$a < A \ (a > A) \iff a < y \ (a > y) \text{ for any } y \in A,$$

$$A < B \ (A > B) \iff x < y \ (x > y) \text{ for any } x \in A,\ y \in B.$$

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For $M \subset X \times Y$, where $X, Y \subset \mathbb{R}$, we put

$$\pi(M) = \{x \in X : \text{ there exists } y \in Y \text{ such that } (x, y) \in M\},$$

$$M_x = \{y \in Y : (x, y) \in M\}.$$

By $\operatorname*{Li}_{n\to\infty} A_n$ ($\operatorname*{Ls}_{n\to\infty} A_n$) we denote a lower (upper) limit of a sequence of sets $A_n \subset R$ (Kuratowski [7]).

In this paper $F : I \to \mathbb{R}$ denote a multivalued map which to each point $x \in I$ assigns a non-empty subset $F(x) \subset \mathbb{R}$. By the graph of F we mean the following set $\bigcup\{(x, y) : \; y \in F(x)\}$. We make no distinction between a map and its graph. For a set $A \subset I$ let $F(A) = \bigcup\{F(x) : \; x \in A\}$. F has the Darboux property if the image $F(E)$ is connected for any connected set $E \subset I$.

We say that $g \in \mathbb{R}$ is a left (right) limit number of a multivalued map F at a left (right) accumulation point x of the set I, if for any open set $V \subset \mathbb{R}$ such that $g \in V$ and for any $\varepsilon > 0$

$$F^{-}(V) \cap (x - \varepsilon, x) \neq \emptyset \qquad (F^{-}(V) \cap (x, x + \varepsilon) \neq \emptyset)$$

or equivalently, if there exist sequences $(x_n)_{n=1}^{\infty} \subset I$ and $(y_n)_{n=1}^{\infty}$ such that $x_n < x$ ($x_n > x$) and $y_n \in F(x_n)$ for $n \in N$, $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = g$. The set of all left (right) limit numbers of F at a point x is denoted by $L^{-}(F, x)$ ($L^{+}(F, x)$) and

$$L(F, x) = L^{-}(F, x) \cup L^{+}(F, x).$$

Remark 1. *(Bruckner, [1]) Let $I = A \cup B$ where A, B are non–empty, disjoint, bilaterally dense–in–itself sets. Then the frame $K = Fr_I(A) = Fr_I(B)$ is a perfect set in I and the sets $K \cap A$ and $K \cap B$ are dense in K.*

Lemma 1. *Let $M \subset I \times \mathbb{R}$ be a continuum. Then for any two different points $a, b \in \pi(M)$ there exists a continuum $C \subset M$ such that $\pi(C) = [a, b]$.*

PROOF. Assume that for some $a, b \in \pi(M)$, $a < b$ the assertion of the lemma does not hold and denote

$$\widetilde{M} = M \cap ([a, b] \times R),$$
$$M_1 = M \cap ((-\infty, a] \times \mathbb{R}),$$
$$M_2 = M \cap ([b, +\infty) \times \mathbb{R}).$$

The set $\widetilde{M}$ is a continuum so any component C of $\widetilde{M}$ is a continuum too and

$$C \cap M_1 \neq \emptyset \quad \text{or} \quad C \cap M_2 \neq \emptyset.$$

Let us define the following disjoint families of sets

$$\mathcal{C}_1 = \{C :\ C \text{ is a component of the set } \widetilde{M} \text{ such that } C \cap M_1 \neq \emptyset\},$$
$$\mathcal{C}_2 = \{C :\ C \text{ is a component of the set } \widetilde{M} \text{ such that } C \cap M_2 \neq \emptyset\}.$$

Let us put

$$X_1 = \bigcup \{C :\ C \in \mathcal{C}_1\},$$
$$X_2 = \bigcup \{C :\ C \in \mathcal{C}_2\}.$$

Let us see that

$$(X_1 \cap \overline{X_2}) \cup (\overline{X_1} \cap X_2) \neq \emptyset.$$

In the opposite case, since

$$\overline{X_1} \cap M_2 \subset \overline{X_1} \cap X_2,$$
$$\overline{X_2} \cap M_1 \subset \overline{X_2} \cap X_1,$$

then the sets $M_1 \cup X_1$, $M_2 \cup X_2$ will be the decomposition of the set M.

Assume that $X_1 \cap \overline{X_2} \neq \emptyset$ and select any $z_0 \in X_1$ such that $z_0 \in \overline{X_2}$. The point z_0 does not belong to any component of the family $\mathcal{C}_2$, so there exists a sequence $(z_n)_{n=1}^{\infty}$, $z_n \in C_n$, where $C_n \in \mathcal{C}_2$ such that

$$\lim_{n\to\infty} z_n = z_0.$$

We may assume that $C_k \neq C_l$ for $k \neq l$.

Since $z_0 \in \operatorname*{Li}_{n\to\infty}(C_n)$ and for any $n \in N$ the sets C_n are compact and connected, then the upper limit

$$K = \operatorname*{Ls}_{n\to\infty}(C_n) = \bigcap_{n=1}^{\infty} \overline{\bigcup_{k=n}^{\infty} C_k}$$

is a compact and connected set (Kuratowski, [7] p.180). Denote $z_0 = (x_0, y_0)$ and let us see that for any $x \in (x_0, b]$ there exists $n_x \in \mathbb{N}$ such that for any $n > n_x$ the sections $(C_n)_x$ are non–empty, compact sets contained in the compact set M_x. Then

$$\bigcap_{n=1}^{\infty} \overline{\bigcup_{k=n}^{\infty} (C_k)_x} \neq \emptyset \qquad \text{and} \qquad \bigcap_{n=1}^{\infty} \overline{\bigcup_{k=n}^{\infty} (C_k)_x} \subset K_x,$$

so $K_x \neq \emptyset$ for any $x \in (x_0, b]$. Since $z_0 \in K$, then $[x_0, b] \subset \pi(K)$. The set $K \cup C_0$, where $C_0 \in \mathcal{C}_1$ and $z_0 \in C_0$ is a continuum contained in M with projection $\pi(K \cup C_0) = [a, b]$. This is a contradiction and the lemma is proved.

Lemma 2. *Let $O_1, O_2 \subset I \times \mathbb{R}$ be disjoint, open subsets of $I \times \mathbb{R}$. Then for any right (left) accumulation point $x_0 \in I$ of the set I and for any points $a, b \in \mathbb{R}$ such that $(x_0, a) \in O_1$ and $(x_0, b) \in O_2$, there exist a continuum $C \subset (I \times \mathbb{R}) \setminus (O_1 \cup O_2)$ and a number $\delta > 0$ such that*

$$\pi(C) = [x_0, x_0 + \delta] \quad (\pi(C) = [x_0 - \delta, x_0]) \qquad \text{and} \qquad C_{x_0} \subset (a, b).$$

PROOF. Without loss of generality we may assume that $x_0 \in \mathrm{Int}(I)$. Let $(x_0, a) \in O_1$, $(x_0, b) \in O_2$ and assume that $a < b$. There exist some positive numbers δ and ε such that

$$P_1 = [x_0, x_0 + \delta] \times [a - \varepsilon, a + \varepsilon] \subset O_1,$$
$$P_2 = [x_0, x_0 + \delta] \times [b - \varepsilon, b + \varepsilon] \subset O_2.$$

Let us denote

$$X = [x_0, x_0 + \delta] \times [a - \varepsilon, b + \varepsilon].$$

Select a component O of the set $O_1 \cap X$ such that $P_1 \subset O$. If the set $X \setminus O$ is connected then we put

$$X_1 = O \text{ and } X_2 = X \setminus O.$$

In the opposite case let X_2 be a component of $X \setminus O$ such that $P_2 \subset X_2$ and

$$X_1 = X \setminus X_2.$$

The set X_1 is connected (Kuratowski, [7] p.149). Then

$$X = \overline{X_1} \cup \overline{X_2},$$

the sets $\overline{X_1}$ $\overline{X_2}$ are compact and connected so the set

$$C = \overline{X_1} \cap \overline{X_2}$$

is a continuum (Kuratowski, [7] p.171, 435).

Since $P_1 \subset X_1$, $P_2 \subset X_2$ and $P_1 \cap P_2 = \emptyset$, then

$$\pi(C) = [x_0, x_0 + \delta],$$
$$C \subset [x_0, x_0 + \delta] \times [a + \varepsilon, b - \varepsilon],$$

and

$$C_{x_0} \subset (a, b).$$

It is easy to show that $C \cap (O_1 \cup O_2) = \emptyset$. In the same way we may show that there exists a continuum C with projection $\pi(C) = [x_0 - \delta, x_0]$ for some positive number δ.

Lemma 3. *Let $F : I \to \mathbb{R}$ and let $B \subset \mathbb{R}$ be a bilaterally dense–in–itself set. Then for any $x \in \overline{B} \setminus B$ and for any $n \in \mathbb{N}$, there exists a closed interval J such that $x \in \operatorname{Int}(J)$ and*

$$F|_{B \cap J} \subset K_{\frac{1}{n}}(\{x\} \times L(F|_B, x)) \cup (I \times ((-\infty, -n) \cup (n, +\infty))).$$

Proof. Let us assume that for some $x \in \overline{B} \setminus B$ and $n_0 \in \mathbb{N}$, the assertion of the lemma is false. Then, there exist sequences $(z_n)_{n=1}^{\infty}$ and $(y_n)_{n=1}^{\infty}$ such that $z_n \in B$,

$$\lim_{n \to \infty} z_n = x, \; y_n \in F(z_n)$$

and

$$(z_n, y_n) \notin K_{\frac{1}{n_0}}(\{x\} \times L(F|_B, x)) \cup (I \times ((-\infty, -n_0) \cup (n_0, +\infty))).$$

Without loss of generality we may assume that $z_n > x$ for any $n \in \mathbb{N}$. The sequence $(y_n)_{n=1}^{\infty}$ is bounded, so it contains convergent subsequence $(y_{n_k})_{k=1}^{\infty}$. We obtain a contradiction that $y = \lim\limits_{k \to \infty} y_{n_k}$ is right limit number of $F|_B$ at the point x and $(x, y) \notin K_{\frac{1}{n_0}}(\{x\} \times L(F|_B, x))$.

Theorem 1. *Let $F : I \to \mathbb{R}$ has connected values and let the following conditions hold.*

(i) *$F(x) \cap L^-(F, x) \neq \emptyset$ and $F(x) \cap L^+(F, x) \neq \emptyset$ for any $x \in I$.*

(ii) *There exist disjoint, open sets $O_1, O_2 \subset I \times \mathbb{R}$ such that $F \subset O_1 \cup O_2$, $F \cap O_1 \neq \emptyset$ and $F \cap O_2 \neq \emptyset$.*

Then for some $x_0 \in I$ there exist limit numbers $g_1, g_2 \in L^-(F, x_0)$ or $g_1, g_2 \in L^+(F, x_0)$ such that $(x_0, g_1) \in O_1$ and $(x_0, g_2) \in O_2$.

Proof. At the beginning we show that for some point $x_0 \in I$ there exist limit numbers $g_1, g_2 \in L(F, x_0)$ such that $(x_0, g_1) \in O_1$ and $(x_0, g_2) \in O_2$. Let us define the sets A, B as

$$A = \{x \in I : \; \{x\} \times F(x) \subset O_1\},$$
$$B = \{x \in I : \; \{x\} \times F(x) \subset O_2\}.$$

They are non-empty, disjoint and $A \cup B = I$. Since the sets O_1, O_2 are open in $I \times \mathbb{R}$, then from (i) we get that A and B are bilaterally dense–in-it-self. By Remark (1), the frame $K = Fr_I(A) = Fr_I(B)$ is a perfect set in I and the sets $K \cap A$, $K \cap B$ are dense in K. Let us define the set

$$M = (I \times \mathbb{R}) \setminus (O_1 \cup O_2)$$

and assume contrary that for any $x \in I$

$$L(F,x) \subset \mathbb{R} \setminus (O_1)_x \quad \text{or} \quad L(F,x) \subset \mathbb{R} \setminus (O_2)_x. \tag{1}$$

For $a \in K \cap A$, since $\{a\} \times F(a) \subset O_1$ and $F(a) \cap L(F,a) \neq \emptyset$ then by (1) $L(F,a) \subset \mathbb{R} \setminus (O_2)_a$. Let us note that $(\{a\} \times L(F|_B, a)) \cap O_1 = \emptyset$. From this

$$L(F|_B, a) \subset M_a. \tag{2}$$

In the same way we can show that if $b \in K \cap B$ then $L(F|_A, b) \subset M_b$.

We now construct a sequence of closed intervals $\{I_n\}_{n=1}^{\infty}$ such that for any $n \in N$,

$$K \cap I_{n+1} \subset K \cap I_n \quad \text{and} \quad F|_{K \cap I_n} \subset D_n,$$

where

$$D_n = K_{\frac{1}{n}}(M) \cup (I \times ((-\infty, -n) \cup (n, +\infty))).$$

Let us take $x_1 \in K \cap A \cap \mathrm{Int}(I)$. Then by (2), $L(F|_B, x_1) \subset M_{x_1}$ and by Lemma (3), there exists a closed interval $J_1 \subset I$ such that $x_1 \in \mathrm{Int}(J_1)$ and $F|_{K \cap B \cap J_1} \subset D_1$. Let $y_1 \in K \cap B \cap \mathrm{Int}(J_1)$. In this case $L(F|_A, y_1) \subset M_{y_1}$ and by Lemma (3), there exists a closed interval $I_1 \subset J_1$ such that $y_1 \in \mathrm{Int}(I_1)$ and $F|_{K \cap A \cap I_1} \subset D_1$. Consequently $F|_{K \cap I_1} \subset D_1$.

Let us assume that we have the closed intervals $I_1, I_2, ..., I_{n-1}$ such that

$$K \cap I_{i+1} \subset K \cap I_i \quad \text{and} \quad K \cap I_i \subset D_i$$

for $i = 1, 2, ..., n-1$.

Select $x_n \in K \cap A \cap \mathrm{Int}(I_{n-1})$. Similarly, there exists a closed interval $J_n \subset I_{n-1}$ such that $x_n \in \mathrm{Int}(J_n)$ and $F|_{K \cap B \cap J_n} \subset D_n$. Let us put $y_n \in K \cap B \cap \mathrm{Int}(J_n)$. There exists interval $I_n \subset J_n$, $y_n \in \mathrm{Int}(I_n)$ and $F|_{K \cap A \cap I_n} \subset D_n$. Then $F|_{K \cap I_n} \subset D_n$ and the sequence $\{I_n\}_{n=1}^{\infty}$ is defined.

Then the set

$$C = \bigcap_{n=1}^{\infty} (K \cap I_n)$$

is non-empty. If $x \in C$, then $\{x\} \times F(x) \subset D_n$ for any $n \in N$, and from this

$$\{x\} \times F(x) \subset \bigcap_{n=1}^{\infty} D_n = M.$$

This contradicts that $M \cap F = \emptyset$. It was shown then, that for some point $x_0 \in I$, there exist limit numbers $g_1, g_2 \in L(F, x_0)$ such that $(x_0, g_1) \in O_1$ and $(x_0, g_2) \in O_2$.

Since F has connected values, then two cases are possible

$$\{x_0\} \times F(x_0) \subset O_1 \quad \text{or} \quad \{x_0\} \times F(x_0) \subset O_2.$$

Let us assume that the first of the cases holds. Then by (i), we can choose numbers $y' \in F(x_0) \cap L^-(F, x_0)$ and $y'' \in F(x_0) \cap L^+(F, x_0)$. Then if g_2 is left or right limit number the y', g_2 or y'', g_2 are required limit numbers .

2 $\mathcal{F}$-connectivity Property

We introduce the following denotations

$\mathcal{M} = \big\{ M \subset I \times \mathbb{R} :\ M$ is a continuum with non degenerate projection $\pi(M) \big\}$,

$\mathcal{P} = \big\{ P \in \mathcal{M} :\ P$ is a horizontal interval contained in $I \times \mathbb{R} \big\}$,

$\mathcal{G} = \big\{ M \subset I \times \mathbb{R} :\ M$ is the graph of the continuous function $f : [a, b] \to \mathbb{R}$, where $[a, b] \subset I,\ a < b \big\}$.

Point 1. *Let $\mathcal{F}$ be any family of subsets of the family $\mathcal{M}$ for which the following conditions hold*

(1) $\mathcal{P} \subset \mathcal{F}$

(2) *If $M \in \mathcal{F}$, $C \subset M$ and $C \in \mathcal{M}$, then $C \in \mathcal{F}$.*

In the paper by $\mathcal{F}$ we mean the subfamily of $\mathcal{M}$ for which the conditions (1) and (2) hold.

Let us introduce for multivalued map the following definition of $\mathcal{F}$- connectivity property.

Definition 1. A multivalued map $F : I \to \mathbb{R}$ with connected values is $\mathcal{F}$-connected, if for any distinct points $x_1, x_2 \in I$ and for any subset $M \in \mathcal{F}$ such that $\pi(M) = [x_1, x_2]$, if

$$\begin{aligned} &F(x_1) < M_{x_1} \quad \text{and} \quad F(x_2) > M_{x_2} \\ \text{or} \quad &F(x_1) > M_{x_1} \quad \text{and} \quad F(x_2) < M_{x_2} \end{aligned}$$

then $M \cap F|_{(x_1,x_2)} \neq \emptyset$.

If $F : I \to \mathbb{R}$ is a real function, then taking in Definition (1) as $\mathcal{F}$ the families $\mathcal{P}$, $\mathcal{M}$ or $\mathcal{G}$ we obtain a Darboux function, function with connected graph (Garrett, Nelms, Kellum, [5]) or respectively functionally connected function (Jastrzębski, Jędrzejewski [6]).

It is easy to see that for multivalued maps, if $\mathcal{F} = \mathcal{P}$ then $\mathcal{F}$-connectivity is equivalent to the Darboux property. If $\mathcal{F} = \mathcal{M}$ or $\mathcal{F} = \mathcal{G}$ it will be said that F is connected or is functionally connected.

Definition 2. A multivalued map $F : I \to \mathbb{R}$, with connected values is right $\mathcal{F}$-connected at a point x_0 if

(i) $F(x_0) \cap L^+(F, x_0) \neq \emptyset$

(ii) for any two numbers $g_1, g_2 \in L^+(F, x_0)$ and any set $M \in \mathcal{F}$, if $\pi(M) = [x_0, x_0 + \varepsilon]$, for some $\varepsilon > 0$ and $M_{x_0} \subset (g_1, g_2)$, then $M \cap F|_{(x_0, x_0+\varepsilon)} \neq \emptyset$.

We define left $\mathcal{F}$-connectivity at a point in a similar way. A multivalued map which is both left and right $\mathcal{F}$-connected at a certain point is called $\mathcal{F}$-connected at this point.

By $C^-_{\mathcal{F}}(F)$ and $C^+_{\mathcal{F}}(F)$ we denote the sets of left and respectively right $\mathcal{F}$-connectivity points and by $C_{\mathcal{F}}(F)$ the set of $\mathcal{F}$-connectivity points.

Notice that if $F : I \to \mathbb{R}$ is a real function then taking in Definition (2) as $\mathcal{F}$ the families $\mathcal{P}$, $\mathcal{M}$ or $\mathcal{G}$, we obtain respectively the Darboux property at a point (Bruckner, Ceder, [2]), connectivity at a point (Garrett, Nelms, Kellum, [5]) or functional connectivity at a point (Jastrzębski, Jędrzejewski, [6]).

If $F : I \to \mathbb{R}$ is a multivalued map then in this three cases it will be said, that F has the Darboux property at a point, is connected at a point or is functionally connected at a point.

Theorem 2. *If a multivalued map $F : I \to \mathbb{R}$ with connected values is $\mathcal{F}$-connected at each point, then it is $\mathcal{F}$-connected.*

Let us assume that F has connected and compact values. If F is $\mathcal{F}$- connected then it is $\mathcal{F}$-connected at each point.

PROOF. Assume contrary that F is $\mathcal{F}$-connected at each point and it is not $\mathcal{F}$-connected. Then there exists a set $M \in \mathcal{F}$ such that $\pi(M) = [x_1, x_2]$

$$F(x_1) < M_{x_1},\ F(x_2) > M_{x_2} \quad \text{and} \quad M \cap F|_{(x_1,x_2)} = \emptyset.$$

Since M is a continuum, then there exist disjoint, open sets O_1, O_2 such that

$$F \subset O_1 \cup O_2,$$

$$O_1 \cap F|_{[x_1,x_2]} \neq \emptyset \quad \text{and} \quad O_2 \cap F|_{[x_1,x_2]} \neq \emptyset.$$

F is $\mathcal{F}$-connected at each point, so

$$F(x) \cap L^-(F,x) \neq \emptyset,$$
$$F(x) \cap L^+(F,x) \neq \emptyset$$

for each $x \in [x_1, x_2]$. By Theorem (1), there exists a point $x_0 \in [x_1, x_2]$ and two limit numbers $g_1,\ g_2 \in L(F, x_0)$ – let assume that $g_1,\ g_2 \in L^+(F, x_0)$ – such that

$$(x_0, g_1) \in O_1 \quad \text{and} \quad (x_0, g_2) \in O_2.$$

By Lemma (2), there exists a continuum $C \subset (I * R)/(O_1 \cup O_2)$ such that

$$C_{x_0} \subset (g_1, g_2) \quad \text{and} \quad \pi(C) = [x_0, x_3],$$

where $x_0 < x_3 \leq x_2$. Since $C \in \mathcal{F}$ and $C \cap F|_{(x_0,x_3)} = \emptyset$ then F is not right $\mathcal{F}$-connected at the point x_0, a contradiction. This finishes the proof of the first part of the theorem.

Now, assume that F is $\mathcal{F}$-connected and for some $x_0 \in I$

$$F(x_0) \cap L^+(F, x_0) = \emptyset.$$

Denote $F(x_0) = [a, b]$, and assume that $a \leq b$. There exist positive numbers δ, ε such that

$$((x_0, x_0 + \delta) \times (a - \varepsilon, b + \varepsilon)) \cap F = \emptyset.$$

Select $x_1 \in (x_0, x_0 + \delta)$ and assume that $F(x_1) < a - \varepsilon$. In the case, when $F(x_1) > b + \varepsilon$ the proof is similar. Let M be a horizontal interval such that $\pi(M) = [x_0, x_1]$ and $M \subset [x_0, x_1] \times (a - \varepsilon, a)$. Then

$$F(x_0) > M_{x_0} \quad \text{and} \quad F(x_1) < M_{x_1}$$

and $M \cap F|_{(x_0,x_1)} = \emptyset$. We get a contradiction that F is not $\mathcal{F}$-connected. In the same way we can show that $F(x_0) \cap L^-(F, x_0) \neq \emptyset$.

Let us assume now that for some $x_0 \in I$ there exist two limit numbers $g_1, g_2 \in L^+(F, x_0)$ and there exists a set $M \in \mathcal{F}$ such that $\pi(M) = [x_0, x_0 + \delta]$; for some $\delta > 0$, $M_{x_0} \subset (g_1, g_2)$ and $M \cap F|_{(x_0,x_0+\delta)} = \emptyset$. Without loss of generality we may assume that $g_1 < g_2$. Since M is a compact set and F has connected values, then there exist two different points $a, b \in (x_0, x_0 + \delta)$ such that $a < b$,

$$F(a) < M_a \quad \text{and} \quad F(b) > M_b.$$

By Lemma (1), there exists a continuum $C \subset M$ such that $\pi(C) = [a,b]$. Then $C \in \mathcal{F}$,

$$F(a) < C_a \quad \text{and} \quad F(b) > C_b$$

and $C \cap F|_{(a,b)} = \emptyset$. It means that F is not $\mathcal{F}$-connected, a contradiction.

The following theorem is straightforward corollary from Theorem 2.

Theorem 3. *Let a multivalued map $F : I \to \mathbb{R}$ has connected values. If F has the Darboux property or respectively is functionally connected at each point then it has the Darboux property or respectively is functionally connected.*

If we assume that the values are compact then the inverse implication is true.

Corollary 1. *(Bruckner, Ceder, [2]) The function $f : I \to \mathbb{R}$ has the Darboux property if and only if it has the Darboux property at each point.*

Corollary 2. *(Jastrzębski, Jędrzejewski, [6]) The function $f : I \to \mathbb{R}$ is functionally connected if and only if f is functionally connected at each point.*

Theorem 4. *Let $F : I \to \mathbb{R}$ has connected values and let $\mathcal{F} = \mathcal{M}$. The following conditions are equivalent*

(i) *F is $\mathcal{F}$-connected at each point,*

(ii) *F has connected graph.*

PROOF. (i) $\Rightarrow$ (ii) Assume contrary that F is $\mathcal{F}$-connected at each point and the graph of F is not connected. Then

$$F(x) \cap L^-(F,x) \neq \emptyset \quad \text{and} \quad F(x) \cap L^+(F,x) \neq \emptyset$$

for any $x \in I$. There exist disjoint, open sets $O_1, O_2 \subset I \times \mathbb{R}$ such that

$$F \subset O_1 \cup O_2,$$
$$F \cap O_1 \neq \emptyset \quad \text{and} \quad F \cap O_2 \neq \emptyset.$$

By Theorem (1), for some point $x_0 \in I$ there exist limit numbers $g_1, g_2 \in L(F, x_0)$ such that

$$(x_0, g_1) \in O_1 \quad \text{and} \quad (x_0, g_2) \in O_2.$$

Let us assume that $g_1, g_2 \in L^+(F, x_0)$ and $g_1 < g_2$. By Lemma (2), there exists a continuum $C \subset (I \times \mathbb{R}) \setminus (O_1 \cup O_2)$ such that

$$C_{x_0} \subset (g_1, g_2)$$

and $\pi(C) = [x_0, x_0+\delta]$, where δ is positive number. Since $C \cap F|_{(x_0,x_0+\delta)} = \emptyset$, we get a contradiction that F is not connected at the point x_0 from the right side.

(ii) $\Rightarrow$ (i) Let us assume contrary that F has connected graph and $F(x_0) \cap L^+(F, x_0) = \emptyset$ for some $x_0 \in I$. Then for any $y \in F(x_0)$, there exist positive numbers δ_y, ε_y such, that $U_y \cap F = \emptyset$, where

$$U_y = [x_0, x_0 + \delta_y) \times (y - \varepsilon_y, y + \varepsilon_y).$$

Let us define open in $I \times \mathbb{R}$ sets O_1, O_2 as follows

$$O_1 = ((x_0, +\infty) \cap I) \times \mathbb{R},$$
$$O_2 = (((-\infty, x_0) \cap I) \times \mathbb{R}) \cup \bigcup \{U_y : \ y \in F(x_0)\}.$$

Then

$$F = (F \cap O_1) \cup (F \cap O_2),$$
$$F \cap O_1 \neq \emptyset \quad \text{and} \quad F \cap O_2 \neq \emptyset$$

and

$$(F \cap O_1) \cap (F \cap O_2) = \emptyset.$$

This contradicts the connectivity of the graph of F.

In the same way we can show that the set $F(x) \cap L^-(F, x) \neq \emptyset$ for any $x \in I$.

Let us assume now that there exists a point $x_0 \in I$ and two limit numbers $g_1, g_2 \in L^+(F, x_0)$, $g_1 < g_2$ and there exists a continuum $M \in \mathcal{F}$ such that $\pi(M) = [x_0, x_0 + \delta]$; for some $\delta > 0$, $M_{x_0} \subset (g_1, g_2)$ and $M \cap F|_{(x_0,x_0+\delta)} = \emptyset$. Since M is a compact set and F has connected values, there exist two different points $a,\ b \in (x, x_0 + \delta)$ such that $a < b$, $F(a) < M_a$ and $F(b) > M_b$. By Lemma (1), there exists a continuum $C \subset M$ such that $\pi(C) = [a, b]$, $F(a) < C_a$ and $F(b) > C_b$ and $C \cap F|_{(a,b)} = \emptyset$. Then we get a contradiction that the graph $F|_{[a,b]}$ is not connected.

Corollary 3. *(Garrett, Nelms, Kellum [5]) A function $f : I \to \mathbb{R}$ has connected graph if and only if f is connected at each point.*

Theorem 5. *Let $F : I \to \mathbb{R}$ has connected values. If F is $\mathcal{F}$-connected, where $\mathcal{F} = \mathcal{M}$, and*

$$F(x) \cap L^-(F, x) \neq \emptyset \quad \text{and} \quad F(x) \cap L^+(F, x) \neq \emptyset$$

for any $x \in I$, then F has connected graph.

PROOF. We show that F is $\mathcal{F}$-connected at each point $x \in I$. Assume contrary that for some x_0 there exist two limits numbers $g_1, g_2 \in L^+(F, x_0)$ and there exists a set $M \in \mathcal{F}$ such that $\pi(M) = [x_0, x_0 + \delta]$; for some $\delta > 0$, $M_{x_0} \subset (g_1, g_2)$ and $M \cap F|_{(x_0, x_0+\delta)} = \emptyset$. As in proof of Theorem (4), we get a contradiction that there exists a continuum $C \subset M$, $C \in \mathcal{F}$, such that $\pi(C) = [a, b]$,

$$F(a) < C_a \quad \text{and} \quad F(b) > C_b$$

and $C \cap F|_{(a,b)} = \emptyset$. Thus F is $\mathcal{F}$-connected at each point $x \in I$ and by Theorem (4), it has connected graph.

3 Strong $\mathcal{F}$-connectivity Property

Let us introduce for multivalued maps the following definition of strong $\mathcal{F}$-connectivity property.

Definition 3. A multivalued map $F : I \to \mathbb{R}$ is strongly $\mathcal{F}$-connected if for any two different points $x_1, x_2 \in I$ and for any $y_1 \in F(x_1)$, there exists $y_2 \in F(x_2)$ such that for any set $M \in \mathcal{F}$, if $\pi(M) = [x_1, x_2]$ and

$$y_1 < M_{x_1} \text{ and } y_2 > M_{x_2}$$
$$\text{or } y_1 > M_{x_1} \text{ and } y_2 < M_{x_2},$$

then $M \cap F|_{(x_1,x_2)} \neq \emptyset$.

For real functions $\mathcal{F}$-connectivity and strong $\mathcal{F}$-connectivity are equivalent.

Lemma 4. *If $F : I \to \mathbb{R}$ is strongly $\mathcal{F}$-connected then for any $x_0 \in I$*

$$F(x_0) \subset L^-(F, x_0) \cap L^+(F, x_0).$$

PROOF. Assume that there exist $x_0 \in I$ and $y_0 \in F(x_0)$ such that y_0 is not a right limit number – we consider one of the two possible cases. There exist then positive numbers δ, ε such that

$$F \cap ((x_0, x_0 + \delta) \times (y_0 - \varepsilon, y_0 + \varepsilon)) = \emptyset.$$

Select any $x' \in (x_0, x_0+\delta)$. Then for any $y \in F(x')$ we may choose a horizontal interval $M \in \mathcal{F}$ such that $\pi(M) = [x_0, x']$ and

$$M \subset [x_0, x'] \times (y_0, y_0 + \varepsilon) \text{ if } y > y_0 + \varepsilon$$

or

$$M \subset [x_0, x'] \times (y_0 - \varepsilon, y_0) \text{ if } y < y_0 - \varepsilon.$$

In this two cases

$$M \cap F|_{(x_0,x')} = \emptyset.$$

It means that contrary to the assumption, F is not strongly $\mathcal{F}$-connected.

Remark 2. *A map $F : I \to R$, with connected values, is strongly $\mathcal{F}$-connected iff and only iff for any two different points $x_1, x_2 \in I$ and for any set $M \in \mathcal{F}$, with projection $\pi(M) = [x_1, x_2]$ if there exist $y_1 \in F(x_1)$, $y_2 \in F(x_2)$ such that*

$$\begin{aligned} & y_1 < M_{x_1} \text{ and } y_2 > M_{x_2} \\ \text{or} \quad & y_1 > M_{x_1} \text{ and } y_2 < M_{x_2}, \end{aligned}$$

then $M \cap F|_{(x_1,x_2)} \neq \emptyset$.

Proof. Assume that F is strongly $\mathcal{F}$-connected and there exist two different points $x_1, x_2 \in I$, $x_1 < x_2$ and $y_1 \in F(x_1)$, $y_2 \in F(x_2)$ such that for some set $M \in \mathcal{F}$ with projection $\pi(M) = [x_1, x_2]$ we have

$$y_1 < M_{x_1} \quad \text{and} \quad y_2 > M_{x_2}$$

— we consider one of the two possible cases — and $M \cap F|_{(x_1,x_2)} = \emptyset$. By Lemma (4), $y_2 \in L^-(F, x_2)$, so there exists $x' \in (x_1, x_2)$ such that

$$F(x') > M_{x'}.$$

By Lemma (1), there exists a continuum $C \subset M$ such that $\pi(C) = [x_1, x']$. Then $C \in \mathcal{F}$ and for any $y \in F(x')$ we have

$$y_1 < C_{x_1} \quad \text{and} \quad y > C_{x'}$$

and

$$C \cap F|_{(x_1,x')} = \emptyset.$$

Then contrary to the assumption, F is not strongly $\mathcal{F}$-connected. The inverse implication is obvious.

It follows from Remark (2), that strongly $\mathcal{F}$-connected multivalued map with connected values is $\mathcal{F}$-connected. If we put as $\mathcal{F}$ the family $\mathcal{P}$ then we obtain the intermediate value property of multivalued maps which is equivalent to those given by Ceder ([3]). Taking by $\mathcal{F}$ the families $\mathcal{G}$ or $\mathcal{M}$ we obtain strong functional connectivity or respectively strong connectivity.

Theorem 6. *A map $F : I \to \mathbb{R}$, with connected values, is strongly $\mathcal{F}$-connected iff and only iff for any two different points $a, b \in I$ a map $\widetilde{F}$:*

$[a,b] \to \mathbb{R}$ *defined as follows*

$$\widetilde{F}(x) = \begin{cases} F(x); & x \in (a,b) \\ y_x; & x \in \{a,b\}, \ \textit{where } y_x \textit{ is any element of } F(x), \end{cases}$$

is $\widetilde{\mathcal{F}}_{a,b}$*-connected, where*

$$\widetilde{\mathcal{F}}_{a,b} = \{M \subset [a,b] \times \mathbb{R} : \ M \in \mathcal{F}\}.$$

PROOF. Let us assume that F is strongly $\mathcal{F}$-connected. Select two different points $a, b \in I$ and let $y_a \in F(a)$, $y_b \in F(b)$. Let us put

$$\widetilde{F}(x) = \begin{cases} F(x); & x \in (a,b) \\ y_a; & x = a \\ y_b; & x = b. \end{cases}$$

Select the set $M \in \widetilde{\mathcal{F}}_{a,b}$ such that $\pi(M) = [x_1, x_2]$,

$$\widetilde{F}(x_1) < M_{x_1} \quad \text{and} \quad \widetilde{F}(x_2) > M_{x_2}.$$

We consider one of the cases, in the second one the proof is similar. Since $\widetilde{F}(x) \subset F(x)$ for any $x \in [a,b]$, $M \in \mathcal{F}$ and F is strongly $\mathcal{F}$-connected, then

$$M \cap F|_{(x_1,x_2)} \neq \emptyset.$$

Since $F|_{(x_1,x_2)} = \widetilde{F}|_{(x_1,x_2)}$, then

$$M \cap \widetilde{F}|_{(x_1,x_2)} \neq \emptyset.$$

It means that $\widetilde{F}$ is $\widetilde{\mathcal{F}}_{a,b}$-connected.

Let us take now the set $M \in \mathcal{F}$, such that $\pi(M) = [x_1, x_2]$,

$$y_1 < M_{x_1} \text{ and } y_2 > M_{x_2}$$

for some $y_1 \in F(x_1)$ and $y_2 \in F(x_2)$. Let $\widetilde{F} : [x_1, x_2] \to \mathbb{R}$ be a map defined as follows

$$\widetilde{F}(x) = \begin{cases} F(x); & x \in (x_1, x_2) \\ y_1; & x = x_1 \\ y_2; & x = x_2. \end{cases}$$

The map $\widetilde{F}$ is $\widetilde{\mathcal{F}}_{x_1,x_2}$-connected, $M \in \widetilde{\mathcal{F}}_{x_1,x_2}$,

$$\widetilde{F}(x_1) < M_{x_1} \quad \text{and} \quad \widetilde{F}(x_2) > M_{x_2},$$

so

$$M \cap \widetilde{F}|_{(x_1,x_2)} \neq \emptyset.$$

Then

$$M \cap F|_{(x_1,x_2)} \neq \emptyset$$

which means, that F is stro ngly $\mathcal{F}$-connected.

From Theorem (6) we get the following theorems as corollaries.

Theorem 7. *A map $F : I \to \mathbb{R}$ with connected values, has the intermediate value property iff and only iff for any two different points $a, b \in I$ and for any $y_a \in F(a)$, $y_b \in F(b)$ the set*

$$F((a, b)) \cup \{y_a,\ y_b\}$$

is connected.

Theorem 8. *A map $F : I \to \mathbb{R}$, with connected and compact values, is strongly connected iff and only iff for any two different points $a, b \in I$ and for any $y_a \in F(a)$, $y_b \in F(b)$ a map $\widetilde{F} : [a, b] \to \mathbb{R}$ defined as follows*

$$\widetilde{F}(x) = \begin{cases} F(x); & x \in (a, b) \\ y_a; & x = a \\ y_b; & x = b \end{cases}$$

has connected graph.

Let us introduce for multivalued maps the following definition of strongly $\mathcal{F}$-connectivity at a point.

Definition 4. A map $F : I \to \mathbb{R}$, with connected values, is strongly $\mathcal{F}$-connected from the right side at a point $x_0 \in I$ if

(i) $F(x_0) \subset L^+(F, x_0)$,

(ii) for any two different points $g_1, g_2 \in L^+(F, x_0)$ and for any set $M \in \mathcal{F}$, if $\pi(M) = [x_0, x_0 + \varepsilon]$, for some positive number $\varepsilon > 0$, and $M_{x_0} \subset (g_1, g_2)$, then $M \cap F|_{(x_0,x_0+\varepsilon)} \neq \emptyset$.

In the same way we define left strong $\mathcal{F}$-connectivity at the point x_0 and we say that F is strongly $\mathcal{F}$-connected at x_0 if it is strongly $\mathcal{F}$-connected both from the left and right side at this point.

By $S^-_{\mathcal{F}}(F)$ and $S^+_{\mathcal{F}}(F)$ we denote the sets of left and respectively right strong $\mathcal{F}$-connectivity points and by $S_{\mathcal{F}}(F)$ the set of all strong $\mathcal{F}$-connectivity points. Notice that

$$S_{\mathcal{F}}(F) \subset C_{\mathcal{F}}(F).$$

If we put in Definition (4) as $\mathcal{F}$ the families $\mathcal{P}$, $\mathcal{G}$ or $\mathcal{M}$, then we obtain the intermediate value property, strong functional connectivity or respectively strong connectivity at a point.

Theorem 9. *A map $F : I \to \mathbb{R}$, with connected values, is strongly $\mathcal{F}$- connected iff and only iff it is strongly $\mathcal{F}$-connected at each point.*

PROOF. Assume that F is strongly $\mathcal{F}$-connected. By Lemma (4), $F(x) \subset L^-(F,x) \cap L^+(F,x)$ for any $x \in I$. Let us assume now that for some $x_0 \in I$ there exist two limit numbers $g_1, g_2 \in L^+(F, x_0)$ and there exists a set $M \in \mathcal{F}$ such that $\pi(M) = [x_0, x_0 + \delta]$; for some $\delta > 0$, $M_{x_0} \subset (g_1, g_2)$ and $M \cap F|_{(x_0,x_0+\delta)} = \emptyset$. Without loss of generality we may assume that $g_1 < g_2$. Since M is a compact set and F has connected values, then there exist two different points $a, b \in (x_0, x_0 + \delta)$ such that $a < b$ and

$$F(a) < M_a \text{ and } F(b) > M_b.$$

By Lemma (1), there exists a continuum $C \subset M$ such that $\pi(C) = [a, b]$. Then $C \in \mathcal{F}$,

$$F(a) < C_a \text{ and } F(b) > C_b$$

and $C \cap F|_{(a,b)} = \emptyset$. It means that F is not strongly $\mathcal{F}$-connected, a contradiction.

Let us assume now that F is strongly $\mathcal{F}$-connected at each point. Then F is $\mathcal{F}$-connected at each point and by Theorem (2), F is $\mathcal{F}$-connected. Assume contrary that F is not strongly $\mathcal{F}$-connected. By Remark (2), there exist two different points $a, b \in I$, $a < b$ and the set $M \in \mathcal{F}$ with projection $\pi(M) = [a, b]$, such that for some two points $y_1 \in F(a)$, $y_2 \in F(b)$,

$$y_1 < M_a \quad \text{and} \quad y_2 > M_b$$

— we consider one of the two possible cases —

$$M \cap F|_{(a,b)} = \emptyset.$$

Since F has connected values and $y_1 \in L^+(F, a)$, $y_2 \in L^-(F, b)$, then there exist points $x_1, x_2 \in (a, b), x_1 < x_2$ such that

$$F(x_1) < M_{x_1} \quad \text{and} \quad F(x_2) > M_{x_2}.$$

By Lemma (1), there exists a continuum $C \subset M$ such that $\pi(C) = [x_1, x_2]$. Then $C \in \mathcal{F}$,

$$F(x_1) < C_{x_1} \quad \text{and} \quad F(x_2) > C_{x_2}$$

and $C \cap F|_{(x_1,x_2)} = \emptyset$, which contradicts that F is $\mathcal{F}$-connected.

From Theorem (9) we get the following corollaries.

Corollary 4. *Czarnowska [4]) A map $F : I \to \mathbb{R}$ has the intermediate value property iff and only iff it has the intermediate value property at each point.*

Corollary 5. *(Czarnowska [4]) A map $F : I \to \mathbb{R}$ is strongly functionally connected iff and only iff it is strongly functionally connected at each point.*

4 Characterization of the Sets of $\mathcal{F}$- connectivity and Strong $\mathcal{F}$-connectivity Points

Lemma 5. *(Czarnowska [4]) For any multivalued map $F : I \to \mathbb{R}$ the set*

$$\{x \in I : \ L^-(F,x) \div L^+(F,x) \neq \emptyset\}$$

is countable.

Lemma 6. *For any $F : I \to \mathbb{R}$ the set*

$$\{x \in I : \ F(x) \not\subset L^-(F,x) \cap L^+(F,x)\}$$

is countable.

Proof. Let us denote

$$\begin{aligned} E &= \{x \in I : \ F(x) \not\subset L^-(F,x) \cap L^+(F,x)\}, \\ B_1 &= \{x \in I : \ L^-(F,x) \div L^+(F,x) \neq \emptyset\}, \\ B_2 &= \{x \in I : \ L^-(F,x) = L^+(F,x)\}. \end{aligned}$$

Notice that $E = (E \cap B_1) \cup (E \cap B_2)$. By Lemma (5), the set B_1 is countable. Now we show that the set $E \cap B_2$ is countable. To do this, select $x_0 \in E \cap B_2$. There exists element $y_0 \in F(x_0)$ such that $y_0 \notin L^-(F,x_0)$ and $y_0 \notin L^+(F,x_0)$. Let us take rational numbers q_0, q_1, q_2, q_3 such that $q_0 < x_0 < q_1$, $q_2 < y_0 < q_3$ and

$$((q_0,q_1) \times (q_2,q_3)) \cap F \subset \{x_0\} \times F(x_0).$$

Thus four rational numbers are assigned to each point in $E \cap B_2$. It is not difficult to see that this is an injective mapping so the set $E \cap B_2$ is countable and the set E is countable too.

Lemma 7. *For any multivalued map $F : I \to \mathbb{R}$ the set*

$$C_{\mathcal{F}}(F) \div S_{\mathcal{F}}(F)$$

is countable.

PROOF. The assertion of the lemma follows from the Lemma (6) and the following inclusions.

$$S_{\mathcal{F}}(F) \subset C_{\mathcal{F}}(F),$$
$$C_{\mathcal{F}}(F) \setminus S_{\mathcal{F}}(F) \subset \{x \in I :\ F(x) \not\subset L^-(F,x) \cap L^+(F,x)\}.$$

Theorem 10. *Let $F : I \to \mathbb{R}$ be a map with connected values. Then the sets*

$$C^-_{\mathcal{F}}(F) \div C^+_{\mathcal{F}}(F),$$
$$S^-_{\mathcal{F}}(F) \div S^+_{\mathcal{F}}(F)$$

are countable.

PROOF. Now we show that the set $A = C^-_{\mathcal{F}}(F) \setminus C^+_{\mathcal{F}}(F)$ is countable. Let us denote

$$B = \{x \in I :\ L^-(F,x) \div L^+(F,x) \neq \emptyset\}.$$

By Lemma (5) the set B is countable. Thus it is enough to show that the set $A \setminus B$ is countable. To do this, select $x_0 \in A \setminus B$. Let us see that

$$F(x_0) \cap L^+(F, x_0) \neq \emptyset.$$

Thus there exist limit numbers $g_1, g_2 \in L^+(F, x_0)$ and a set $M \in \mathcal{F}$ such that $\pi(M) = [x_0, x_0 + \varepsilon]$, for some $\varepsilon > 0$, $M_{x_0} \subset (g_1, g_2)$ and

$$M \cap F|_{(x_0, x_0+\varepsilon)} = \emptyset.$$

There exist rational numbers q_0, q_1, q_2 such that

$$g_1 < q_1 < M_{x_0} < q_2 < g_2,\ x_0 < q_0 < x_0 + \varepsilon,$$
$$M \cap ([x_0, q_0] \times \mathbb{R}) \subset [x_0, q_0] \times [q_1, q_2],$$
$$M \cap F|_{(x_0, q_0)} = \emptyset.$$

So three rational numbers are assigned to each point in $A \setminus B$. Let us show that this is an injective mapping. Suppose, on the contrary, that the same triple (q_0, q_1, q_2) is assigned to x_0 and x_1 from $A \setminus B$, and assume that $x_1 < x_0$. Then there exists a set $P \in \mathcal{F}$ such that

(1) $P \cap ([x_1, q_0] \times \mathbb{R}) \subset [x_1, q_0] \times [q_1, q_2],\ [x_1, q_0] \subset \pi(P)$,

(2) $P \cap F|_{(x_1,q_0)} = \emptyset$.

By Lemma (1), there exists a continuum $C \subset P$ such that $\pi(C) = [x_1, x_0]$. From (1), we get that $C_{x_0} \subset (g_1, g_2)$. Since $L^-(F, x_0) = L^+(F, x_0)$, then $g_1, g_2 \in L^-(F, x_0)$nd by (2), $C \cap F|_{(x_1,x_0)} = \emptyset$. Since $C \in \mathcal{F}$, then contrary to the assumption $x_0 \notin C^-_{\mathcal{F}}(F)$.

In the same way we can show that the set $C^+_{\mathcal{F}}(F) \setminus C^-_{\mathcal{F}}(F)$ is countable.

By Lemma (6) the set

$$E = \{x \in I :\ F(x) \not\subset L^-(F, x) \cap L^+(F, x)\}$$

is countable. Since

$$S^-_{\mathcal{F}}(F) \setminus E = C^-_{\mathcal{F}}(F) \setminus E,$$
$$S^+_{\mathcal{F}}(F) \setminus E = C^+_{\mathcal{F}}(F) \setminus E,$$

then

$$S^-_{\mathcal{F}}(F) \div S^+_{\mathcal{F}}(F) \subset (C^-_{\mathcal{F}}(F) \div C^+_{\mathcal{F}}(F)) \cup E.$$

Finally the set $S^-_{\mathcal{F}}(F) \div S^+_{\mathcal{F}}(F)$ is countable too.

Theorem 11. *For any map $F : I \to \mathbb{R}$, $C_{\mathcal{F}}(F)$ and $S_{\mathcal{F}}(F)$ are G_δ-sets.*

Proof. Without loss of generality we may assume that the set I is open. Let $x \in C_{\mathcal{F}}(F)$. For any $n \in N$, there exists an open interval U^x_n with diameter less than $\frac{1}{n}$, contained x such that

$$L(F, z) \cup F(z) \subset K_{\frac{1}{n}}(L(F, x)) \cup (-\infty, -n) \cup (n, +\infty)$$

for any $z \in U^x_n$. Let us define for any $n \in N$ the open sets

$$U_n = \bigcup \{U^x_n :\ x \in C_{\mathcal{F}}(F)\}.$$

We have $C_{\mathcal{F}}(F) \subset \bigcap\limits_{n=1}^{\infty} U_n$. It is enough to show that

(1) $\bigcap\limits_{n=1}^{\infty} U_n \subset C_{\mathcal{F}}(F) \cup (C^-_{\mathcal{F}}(F) \div C^+_{\mathcal{F}}(F)$,

since $C_{\mathcal{F}}(F)$ will be G_δ-set as a different $C_{\mathcal{F}}(F) = \bigcap\limits_{n=1}^{\infty} U_n \setminus B$ of G_δ-set and countable set $B \subset C^-_{\mathcal{F}}(F) \div C^+_{\mathcal{F}}(F)$.

Let us show the inclusion (1). Select $x_0 \in \bigcap_{n=1}^{\infty} U_n$ and assume that $x_0 \notin C_{\mathcal{F}}(F)$. It means that for any $n \in N$, there exists an element $x_n \in C_{\mathcal{F}}(F)$, $x_n \neq x_0$ such that $x_0 \in U_n^{x_n}$. Then for any $n \in N$ we have

(2) $|x_n - x_0| < \frac{1}{n}$ and $L(F, x_0) \cup F(x_0) \subset K_{\frac{1}{n}}(L(F, x_n)) \cup (-\infty, -n) \cup (n, +\infty)$.

There exists a subsequence of the sequence $(x_n)_{n=1}^{\infty}$ convergent to x from the left or right sight. Let us assume that the second of the cases holds. Without loss of generality we may assume that $x_n > x$ for any $n \in N$. Now we show that $x_0 \in C_{\mathcal{F}}^{+}(F)$. From (2) we get

(3) $F(x_0) \subset L^{+}(F, x_0)$.

Let $y_0 \in F(x_0)$. There exists $k \in N$ such that $y_0 \in K_{\frac{1}{n}}(L(F, x_n))$ for any $n > k$. Then for any $n > k$ there exists $g_n \in L(F, x_n)$ such that $|g_n - y_0| < \frac{1}{n}$. Then $\lim_{n\to\infty} g_n = y_0$ and $y_0 \in L^{+}(F, x_0)$.

Assume that there exist limit numbers $g_1, g_2 \in L^{+}(F, x_0)$, $g_1 < g_2$ and there exists a set $M \in \mathcal{F}$ such that $\pi(M) = [x_0, x_0 + \delta]$, for some $\delta > 0$, $M_{x_0} \subset (g_1, g_2)$ and $M \cap F|_{(x_0, x_0+\delta)} = \emptyset$. There exist then positive numbers δ_1, ε such that $\delta_1 < \delta$ and

$$M \cap ([x_0, x_0 + \delta_1] \times \mathbb{R}) \subset [x_0, x_0 + \delta_1] \times [g_1 + \varepsilon, g_2 - \varepsilon].$$

From (2), we get that $g_1, g_2 \in K_{\frac{1}{n}}(L(F, x_n)) \cup (-\infty, -n) \cup (n, +\infty)$ for any $n \in N$. There exists $l \in N$ such that $g_1, g_2 \in K_{\frac{1}{n}}(L(F, x_n))$ for any $n > l$. Let us take $n_0 \in N$ such that $n_0 > l$ and $\frac{1}{n_0} < \min(\delta_1, \varepsilon)$. Then $x_{n_0} \in (x_0, x_0 + \delta_1)$ and

$$g_1, g_2 \in K_{\frac{1}{n_0}}(L(F, x_{n_0})).$$

There exist limit numbers $y', y'' \in L(F, x_{n_0})$ such that $|g_1 - y'| < \frac{1}{n_0}$ and $|g_2 - y''| < \frac{1}{n_0}$. So

$$y' < M_{x_{n_0}} \quad \text{and} \quad y'' > M_{x_{n_0}}.$$

The set $F(x_{n_0})$ is connected, then

$$F(x_{n_0}) < M_{x_{n_0}} \quad \text{or} \quad F(x_{n_0}) > M_{x_{n_0}}.$$

Let us assume that the first of the cases holds — in the second one the proof is similar. Suppose that $y'' \in L^{+}(F, x_{n_0})$. Since $x_{n_0} \in C_{\mathcal{F}}(F)$, then there exists $y \in L^{+}(F, x_{n_0}) \cap F(x_{n_0})$. By Lemma (1), there exists a continuum $C \subset M$ such that $\pi(C) = [x_{n_0}, x_0 + \delta_1]$. Then $C \in \mathcal{F}$, $C_{x_{n_0}} \subset (y, y'')$, $y, y'' \in L^+(F, x_{n_0})$ and

$$C \cap F|_{(x_{n_0}, x_0+\delta_1)} = \emptyset,$$

which contradicts that $x_{n_0} \in C^+_{\mathcal{F}}(F)$.

If $y'' \in L^-(F, x_{n_0})$, then we can take $y \in L^-(F, x_{n_0}) \cap F(x_{n_0})$ and a continuum $C \subset M$ such that $\pi(C) = [x_0, x_{n_0}]$, and we obtain that $C \in \mathcal{F}$, $C_{x_{n_0}} \subset (y, y'')$, $y, y'' \in L^-(F, x_{n_0})$ and

$$C \cap F|_{(x_0, x_{n_0})} = \emptyset.$$

This contradicts that $x_{n_0} \in C^-_{\mathcal{F}}(F)$. From (3), we get that $x_0 \in C^+_{\mathcal{F}}(F)$.

If there exist a subsequence of the sequence $(x_n)_{n=1}^{\infty}$ convergent to x from the left side, then in a similar way we may show that $x_0 \in C^-_{\mathcal{F}}(F)$. Thus the proof that $C_{\mathcal{F}}(F)$ is G_δ-set is finished. By Lemma (6), the set $E = \{x \in I : F(x) \not\subset L^-(F, x) \cap L^+(F, x)\}$ is countable. Since $S_{\mathcal{F}}(F) = C_{\mathcal{F}}(F) \setminus E$, then $S_{\mathcal{F}}(F)$ is G_δ-set.

From Theorem (11), we get the following corollaries.

Corollary 6. *The sets of Darboux points, connectivity points, functional connectivity points, the intermediate value property points (Czarnowska [4]) or strongly functional connectivity points (Czarnowska [4]) are* G_δ.

Theorem 12. *(Rosen [9]) The sets of Darboux points or connectivity points of real function are* G_δ.

Theorem 13. *(Jastrzębski, Jędrzejewski [6]) The set of functional connectivity points of real function is* G_δ.

J. S. Lipiński ([8]) has shown that for any G_δ-sets G and H such that $G \subset H$, there exists function $f : \mathbb{R} \to \mathbb{R}$ such that H is the set of Darboux points and G is the set of connectivity points of f. From Theorem (11) we get the following corollary.

Theorem 14. *The set* $G \subset \mathbb{R}$ *is the set of* $\mathcal{F}$*-connectivity points or strong* $\mathcal{F}$*-connectivity points for some multivalued map iff and only iff it is* $G_\delta - set$.

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