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AN UNPUBLISHED RESULT OF P. ŠLEICH: SETS OF TYPE $H^{(s)}$ ARE σ-BILATERALLY POROUS

Abstract

The sets of type $H^{(s)}$ form an important subclass of the class of sets of uniqueness (U-sets) for trigonometrical series. The main aim of this note is to present a proof of an unpublished result of P. Šleich which asserts that each set of type $H^{(s)}$ is σ-bilaterally porous.

1 Introduction

The classical notion of sets of uniqueness for trigonometrical series (U-sets) was investigated by many authors (cf. [1], [11], [2]). The sets of type H were introduced by A. Rajchman [4] in 1922. He proved that each H-set is a U-set and consequently there exists a non-empty perfect U-set (since e.g. the Cantor ternary set is of type H). Rajchman conjectured (cf. [1], p. 800) that each U-set can be covered by countably many of H-sets. This hypothesis was disproved by Piateckii-Shapiro [3] in 1952. He generalized Rajchman's notion of a H-set introducing the notion of a set of type $H^{(s)}$ (see Definition 2.3 below), $s \in \mathbf{N}$, in such a way that $H^{(1)}$-sets coincide with H-sets, each $H^{(s)}$-set is still a U-set and there exists a closed $H^{(2)}$-set which cannot be covered by countably many of H-sets. The book [1] (pp. 818–822) contains a relatively simple proof (which is also due to Piateckii-Shapiro but was not published by him) of the existence of such a $H^{(2)}$-set. (Piateckii-Shapiro in [3]

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even showed, by a more complicated proof, that, for each $s \in \mathbf{N}$, there exists a $H^{(s+1)}$-set which cannot be covered by countably many of $H^{(s)}$-sets).

As was noted already in [8], p. 344, the proof from [1] is based on the notion of porosity. This notion was used (under different nomenclature) by many authors (e.g. by Denjoy in 1920). The notion of σ-porous sets was introduced and used by Dolzhenko in 1967 (cf. [8] for more information).

The main steps of the mentioned proof from [1] (which is possibly the first application of a kind of σ-porosity) are the following:

(i) It is observed that the closure of each H-set is bilaterally porous.

(ii) A closed $H^{(2)}$-set F is constructed which contains a dense subset $D \subset F$ such that F is porous on the right at no point of D.

(iii) The Baire category theorem easily yields that F cannot be covered by countably many of H-sets.

(It was shown in [6] that the set F constructed in [1] has not the property from (ii) but an easy slight modification of the construction gives a set with this property.)

Thus there exists a $H^{(2)}$-set which cannot be covered by countably many of *closed* bilaterally porous sets. On the other hand, the set F from [1] is σ-porous and thus I asked the question, whether each $H^{(s)}$-set is σ-porous.This question was answered in positive by P. Šleich in [6] (cf. [9]). He does not work in mathematics now, will not publish the proof and agrees with publishing of this note. In the note a proof which is almost identical with this from [6] (and [7]) is presented. Only formal changes are made and some remarks and arguments are added.

Note that the natural question (Question 6.7. of [8]) whether each closed U-set is σ-porous was answered in negative in [10]. This result together with Šleich's result provides an alternative proof of the fact that there exists a closed U-set which cannot be covered by countably many of sets from $\bigcup_{s=1}^{\infty} H^{(s)}$. Moreover, the non-$\sigma$-porous U-set from [10] belongs even to the important subclass U' of U (for the definition of Piatecki-Shapiro's class U' see [2]). Thus, using Šleich's result, we obtain existence of a closed U'-set which cannot be covered by countably many of sets from $\bigcup_{s=1}^{\infty} H^{(s)}$. I do not know whether this result is new.

2 Definitions and lemmas

We will use the following notation.

The symbols $\mathbf{R}, \mathbf{Z}$ and $\mathbf{N}$ stand for the sets of all real numbers, integers, and positive integers, respectively.

If $K \subset \mathbf{R}$ and $x \in \mathbf{R}$, then we put $xK := \{xy : y \in K\}$ and $K + x := \{y + x : y \in K\}$.

For $x \in \mathbf{R}$ and $h > 0$ we put $B(x,h) = (x-h, x+h)$.

If $I \subset \mathbf{R}$ is an open bounded interval and $h > 0$ then we denote by $|I|$ the length of the interval I and by $h * I$ the open interval of length $h|I|$ which is concentric with I.

If $x \in \mathbf{R}$ then $\langle x \rangle := x - \max\{a \in \mathbf{Z} : a \leq x\}$ denotes the fractional part of x.

If $E \subset \mathbf{R}$ and I is an open interval, then $\pi(E, I)$ denotes the maximum of lenghts of open subintervals of I which do not intersect E (and $\pi(E, I) = 0$ if E is dense in I).

If $x \in \mathbf{R}$ and $E \subset \mathbf{R}$ then we put

$$p^+(E,x) := \limsup_{h \to 0+} \frac{\pi(E,(x,x+h))}{h}, \quad p^-(E,x) := \limsup_{h \to 0+} \frac{\pi(E,(x-h,x))}{h}.$$

We say E is bilaterally porous at x if both $p^+(E,x) > 0$ and $p^-(E,x) > 0$.

We say E is bilaterally porous if it is bilaterally porous at each of its points.

A set is said to be σ-bilaterally porous if it is a countable union of bilaterally porous sets.

Definition 2.1. ([1]) *Sequences $\{\lambda_k^{(1)}\}, \ldots, \{\lambda_k^{(s)}\}$ of real numbers are said to be independent if*

$$\lim_{k \to \infty} |\lambda_k^{(1)} n_1 + \ldots \lambda_k^{(s)} n_s| = \infty$$

for each non-zero vector $(n_1, \ldots, n_s)$ with integer components.

It is easy to see the following facts.

Lemma 2.2. *Let $\{\lambda_k^{(1)}\}, \ldots, \{\lambda_k^{(s)}\}$ be independent sequences of real numbers. Then the following assertions hold.*

(i) $\lim_{k \to \infty} |\lambda_k^{(i)}| = \infty$ *for* $i = 1, \ldots, s$.

(ii) *Let $L : \mathbf{R}^s \to \mathbf{R}^s$ be a linear mapping given by a regular matrix with rational components. Let*

$$(\mu_k^{(1)}, \ldots, \mu_k^{(s)}) := L(\lambda_k^{(1)}, \ldots, \lambda_k^{(s)}).$$

Then the sequences $\{\mu_k^{(1)}\}, \ldots, \{\mu_k^{(s)}\}$ are independent.

(iii) *If $\{k_j\}$ is an increasing sequence of natural numbers, then the sequences $\{\lambda_{k_j}^{(1)}\}, \dots, \{\lambda_{k_j}^{(s)}\}$ are independent.*

(iv) *If $s > 1$, then $\{\lambda_k^{(1)}\}, \dots, \{\lambda_k^{(s-1)}\}$ are independent.*

Definition 2.3. *Let $s \in \mathbf{N}$. We say $E \subset [0,1]$ is of type $H^{(s)}$ if there exist s independent sequences of integers $\{n_k^{(1)}\}, \dots, \{n_k^{(s)}\}$ and an open s-dimensional open interval $I = I_1 \times \dots \times I_s \subset [0,1]^s$ such that the condition $(\langle n_k^{(1)} x\rangle, \dots, \langle n_k^{(s)} x\rangle) \notin I$ holds for each $x \in E$ and $k \in \mathbf{N}$.*

Remark 1. *If we consider in the above definition independent sequences of real numbers, we obtain a definition of slightly more general sets of type $H^{(s)*}$. These sets are considered in [1], pp. 919–921; it is shown there that each set of type $H^{(s)*}$ is a disjoint finite union of sets of type $H^{(s)}$ (see also [11], vol. II, pp. 152–153, where the notation $H_*^{(s)}$ is used). Thus we know a priori that each $H^{(s)}$-set is σ-bilaterally porous iff each $H^{(s)*}$-set is of this type.*

Definition 2.4. *Let $K \subset \mathbf{R}$ and $\delta \neq 0$ be a real number. Then we put*

$$S(K,\delta) := \bigcup_{n \in \mathbf{Z}} \delta(K+n).$$

We omit the obvious proofs of the following easy facts.

Lemma 2.5. *(i)* *$S(K,\delta) = S(-K,-\delta)$ for every $K \subset \mathbf{R}$ and $\delta \neq 0$.*

(ii) *Let I be an open interval. Then there exists an open interval $J \subset (0,1)$ such that $S(J,\delta) \subset S(I,\delta)$ for each $\delta \neq 0$.*

(iii) *Let $\delta \neq 0$, $x \in \mathbf{R}$ and I, J be open intervals such that $|I| \leq 1$ and $|J| = 2|\delta|$. Then there exists $n \in \mathbf{Z}$ such that $\delta(I+n) \subset J$.*

(iv) *Let $\delta \neq 0$, I be an open interval, $r \leq (1/4)|\delta||I|$ and $x \in S(2^{-1} * I, \delta)$. Then $B(x,r) \subset S(I,\delta)$.*

(v) *Let $K \subset (0,1)$, $x \in \mathbf{R}$ and $\lambda \in \mathbf{R}$, $\lambda \neq 0$. Then $\langle \lambda x\rangle \in K$ iff $x \in S(K, 1/\lambda)$.*

(vi) *$S(K, p\delta) \subset S(pK, \delta)$ whenever $K \subset \mathbf{R}$, $\delta \neq 0$ and $0 \neq p \in \mathbf{Z}$.*

¿From Lemma 2.2. (i), (iii) and Lemma 2.5 (ii), (v) the following characterization of sets of type $H^{(s)*}$ easily follows. Note, however, that for our purposes we can consider the condition from Lemma 2.6 as the definition of $H^{(s)*}$-sets. For the proof of Theorem 3.1 we then need only the fact that each $H^{(s)}$-set satisfies the condition of Lemma 2.6.

Lemma 2.6. *A set $E \subset [0,1]$ is of type $H^{(s)*}$ iff there exist s independent sequences of non-zero real numbers $\{\lambda_k^{(1)}\}, \dots, \{\lambda_k^{(s)}\}$ and open intervals $I_1, \dots, I_s$ such that the condition*

$$x \notin \bigcap_{i=1}^{s} S(I_i, 1/\lambda_k^{(i)})$$

holds for each $x \in E$ and $k \in \mathbf{N}$.

We will need the following simple but nontrivial lemma.

Lemma 2.7. *Let $K \subset \mathbf{R}$ and $z, \delta_1, \delta_2 \in \mathbf{R}$, $\delta_1, \delta_2 \neq 0, \delta_1 \neq \delta_2$, be given. Then*

$$S(K - z, \delta_1\delta_2/(\delta_2 - \delta_1)) \cap S(\{z\}, \delta_2) \subset S(K, \delta_1).$$

Proof. Let $y \in S\left(K - z, \delta_1\delta_2/(\delta_2 - \delta_1)\right) \cap S(\{z\}, \delta_2)$. Then we can find $\xi \in K$ and integers m, n such that

$$y = (\xi - z + n)\frac{\delta_1\delta_2}{\delta_2 - \delta_1}, \quad y = (z + m)\delta_2, \quad \text{i.e.}$$

$$\left(\frac{1}{\delta_1} - \frac{1}{\delta_2}\right) y = \xi - z + n, \quad \frac{y}{\delta_2} = z + m.$$

Adding the last two equalities we obtain

$$\frac{y}{\delta_1} = \xi + (n + m),$$

which immediately implies $y \in S(K, \delta_1)$. $\square$

We will need also the following lemma which is an immediate consequence of Dirichlet's theorem on simultaneous Diophantine approximation (cf. [5] or [11], vol. I, p. 235).

Lemma 2.8. *Let $r_1 > 0, \dots, r_n > 0$ and $\varepsilon > 0$ be positive real numbers. Then there exist positive integers $q, p_1, \dots, p_n$ such that $|qr_i - p_i| < \varepsilon$ for $i = 1, \dots, n$.*

Lemma 2.9. *Let $E \subset [0,1]$ be of type $H^{(s)*}$, $s \geq 2$. Then there exist s independent sequences $\{\lambda_k^{(1)}\}, \dots, \{\lambda_k^{(s)}\}$ of non-zero real numbers, $s-1$ real numbers $r_1, \dots, r_{s-1} \in [0,1]$ and s open intervals $I_1, \dots, I_s$ such that:*

(i) $x \notin \bigcap_{i=1}^{s} S(I_i, 1/\lambda_k^{(i)})$ *for each $x \in E$ and $k \in \mathbf{N}$.*

(ii) $0 < \lambda_k^{(1)} \leq \lambda_k^{(2)} \leq \dots \leq \lambda_k^{(s)}$ *for each $k \in \mathbf{N}$.*

(iii) $\lim_{k\to\infty} \lambda_k^{(i)}/\lambda_k^{(s)} = r_i$ *for* $i = 1, \dots, s-1$.

(iv) $|I_1| \le 1, \dots, |I_s| \le 1$.

Proof. By Lemma 2.6 there exist s independent sequences $\{\tilde{\lambda}_p^{(1)}\}, \dots, \{\tilde{\lambda}_p^{(s)}\}$ of non-zero real numbers and open intervals $\tilde{I}_1, \dots, \tilde{I}_s$ such that $x \notin \bigcap_{i=1}^s S(\tilde{I}_i, 1/\tilde{\lambda}_p^{(i)})$ for each $x \in E$ and $p \in \mathbf{N}$. We can clearly suppose that $|\tilde{I}_1| \le 1, \dots, |\tilde{I}_s| \le 1$. It is easy to see that there exist a permutation $\pi : \{1, \dots, s\} \to \{1, \dots, s\}$, an increasing sequence of indices $\{p_j\}$ and numbers $\varepsilon_1, \dots, \varepsilon_s \in \{-1, 1\}$ such that

$$0 < \varepsilon_1 \tilde{\lambda}_{p_j}^{(\pi(1))} \le \varepsilon_2 \tilde{\lambda}_{p_j}^{(\pi(2))} \le \dots \le \varepsilon_s \tilde{\lambda}_{p_j}^{(\pi(s))} \quad \text{for each} \quad j \in \mathbf{N}.$$

Putting $\overline{\lambda}_j^{(i)} := \varepsilon_i \tilde{\lambda}_{p_j}^{(\pi(i))}$ and $I_i := \varepsilon_i \tilde{I}_{\pi(i)}$, we obtain by Lemma 2.2 (ii),(iii) and Lemma 2.5 (i) that the sequences $\{\overline{\lambda}_j^{(1)}\}, \dots, \{\overline{\lambda}_j^{(s)}\}$ are independent, $|I_i| \le 1$ and $x \notin \bigcap_{i=1}^s S(I_i, 1/\overline{\lambda}_j^{(i)})$ for each $x \in E$ and $j \in \mathbf{N}$. Since

$$\left(\frac{\overline{\lambda}_j^{(1)}}{\overline{\lambda}_j^{(s)}}, \dots, \frac{\overline{\lambda}_j^{(s-1)}}{\overline{\lambda}_j^{(s)}} \right) \in [0,1]^{s-1} \quad \text{for each} \quad j \in \mathbf{N},$$

we can find an increasing sequence $\{j_k\}$ of indices and $(r_1, \dots, r_{s-1}) \in [0,1]^{s-1}$ such that all conditions (i)-(iv) hold for $\lambda_k^{(i)} := \overline{\lambda}_{j_k}^{(i)}$. Since the sequences $\{\lambda_k^{(1)}\}, \dots, \{\lambda_k^{(s)}\}$ are independent by Lemma 2.2 (iii), we are done. □

We already noted in the Introduction that each H-set (i.e. $H^{(1)}$-set) is bilaterally porous. The generalization for $H^{(1)*}$-sets is straightforward; for the sake of completness we present the obvious proof.

Proposition 2.10. *Every set* $E \subset \mathbf{R}$ *of type* $H^{(1)*}$ *is bilaterally porous.*

Proof. By the definition and Lemma 2.2 (i) there exist a sequence λ_k of real non-zero numbers and an open interval I such that $\lim_{k\to\infty} |\lambda_k| = \infty$ and $E \cap S(I, 1/\lambda_k) = \emptyset$ for each $k \in \mathbf{N}$. We may and will suppose $|I| < 1$. Fix an arbitrary $x \in E$ and $k \in \mathbf{N}$. Using Lemma 2.5 (iii) (with $\delta := 1/\lambda_k$ and $J := (x, x + 2/|\lambda_k|)$) we obtain $n \in Z$ such that $(1/\lambda_k)(I + n) \subset (x, x + 2/|\lambda_k|)$. Since $(1/\lambda_k)(I+n) \cap E = \emptyset$ and $|(1/\lambda_k)(I+n)| = |I|/|\lambda_k|$, we obtain $\pi(E, (x, x + 2/|\lambda_k|)) \ge |I|/|\lambda_k|$. Because $2/|\lambda_k| \to 0$, we obtain $p^+(E, x) \ge |I|/2$. Quite analogously we can obtain $p^-(E, x) \ge |I|/2$. □

3 Proof of the theorem

Theorem 3.1. *Every set of type $H^{(s)}$ is σ-bilaterally porous.*

Proof. We will prove by induction that every set of type $H^{(s)*}$ is σ-bilaterally porous. (It is only a "formally stronger" result; cf. Remark 1.) We know that it is true for $s = 1$ (Proposition 2.10). Now suppose that $s \geq 2$ is given and that every set of type $H^{(s-1)*}$ is σ-bilaterally porous. We will prove that an arbitrary set E of type $H^{(s)*}$ is σ-bilaterally porous. By Lemma 2.9 we can find independent sequences of real numbers $\{\lambda_k^{(1)}\}, \dots, \{\lambda_k^{(s)}\}$, open intervals $I_1, \dots, I_s$ and real numbers $r_1, \dots, r_{s-1}$ from $[0,1]$ such that the conditions (i)-(iv) of Lemma 2.9 hold.

By (ii) and (iii) there exists $m \in \{1, \dots, s\}$ such that $r_i = 0$ for $1 \leq i < m$ and $r_i > 0$ for $m \leq i < s$. Put $\varepsilon := (1/16) \min\{|I_i| : i = 1, \dots, s-1\}$.

If $m < s$ then choose by Lemma 2.8 positive integers $q, p_m, \dots, p_{s-1}$ such that

$$|qr_i - p_i| < \varepsilon \quad \text{for} \quad i = m, \dots, s-1. \tag{1}$$

If $m = s$, put $q = 1$. Denote by c the center of the interval I_s and put

$$J_i := 2^{-1} * I_i \quad \text{and} \quad \{\mu_k^{(i)}\} := \{\lambda_k^{(i)}\} \ \text{ for } \ 1 \leq i < m;$$

$$J_i := \frac{1}{p_i}\left(2^{-1} * I_i\right) - \frac{c}{q} \quad \text{and} \quad \{\mu_k^{(i)}\} := \left\{\frac{1}{p_i q}\left(q\lambda_k^{(i)} - p_i\lambda_k^{(s)}\right)\right\} \ \text{ for } \ m \leq i < s.$$

By Lemma 2.2 (ii) (where we put $\{\mu_k^{(s)}\} := \{\lambda_k^{(s)}\}$) and Lemma 2.2 (iv) we easily obtain that the sequences $\{\mu_k^{(1)}\}, \dots, \{\mu_k^{(s-1)}\}$ are independent. By Lemma 2.2 (i), the condition (iii) of Lemma 2.9 and (1) we can choose $K \in \mathbf{N}$ such that $\mu_k^{(i)} \neq 0$ for $k \geq K,\ i = 1, \dots, s-1$,

$$\frac{\lambda_k^{(i)}}{\lambda_k^{(s)}} \leq \frac{\varepsilon}{q} \qquad \text{for} \qquad k \geq K,\ 1 \leq i < m \ \text{ and} \tag{2}$$

$$\left|\frac{q\lambda_k^{(i)}}{\lambda_k^{(s)}} - p_i\right| \leq \varepsilon \qquad \text{for} \qquad k \geq K,\ m \leq i < s. \tag{3}$$

Now define

$$M_n := E \setminus \bigcup_{k=n}^{\infty} \bigcap_{i=1}^{s-1} S\left(2^{-1} * J_i, 1/\mu_k^{(i)}\right) \qquad \text{for} \ \ n = K, K+1, \dots \ \text{ and}$$

$$Z := E \setminus \bigcup_{n=K}^{\infty} M_n = E \cap \bigcap_{n=K}^{\infty} \bigcup_{k=n}^{\infty} \bigcap_{i=1}^{s-1} S\left(2^{-1} * J_i, 1/\mu_k^{(i)}\right).$$

Considering the sequences $\{\mu_{n+k-1}^{(1)}\}, \ldots, \{\mu_{n+k-1}^{(s-1)}\}$ and the intervals $2^{-1} * J_1, \ldots, 2^{-1} * J_{s-1}$, we obtain immediately by Lemma 2.6 that each set $M_n,\ n \geq K$, is of type $H^{(s-1)*}$ and therefore it is σ-bilaterally porous.

Consequently it is sufficient to prove that the set Z is bilaterally porous. To this end fix an arbitrary $x \in Z$; we will prove $p^+(Z, x) > 0$. By definition of Z we can find an increasing sequence of indices k_j such that

$$k_j \geq K \quad \text{and} \quad x \in \bigcap_{i=1}^{s-1} S\left(2^{-1} * J_i, 1/\mu_{k_j}^{(i)}\right) \quad \text{for each} \quad j \in \mathbf{N}.$$

Now fix an arbitrary $j \in \mathbf{N}$ and denote $k := k_j$. By Lemma 2.5 (iv) we obtain

$$B(x, (1/4)\min\{|J_i|/|\mu_k^{(i)}| : i = 1, \ldots s-1\}) \subset \bigcap_{i=1}^{s-1} S(J_i, 1/\mu_k^{(i)}). \tag{4}$$

Using definitions of J_i, $\mu_k^{(i)}$ and ε, (2) and (3) we obtain that

$$|J_i|/|\mu_k^{(i)}| \geq 8q/\lambda_k^{(s)} \quad \text{for} \quad i = 1, \ldots, s-1.$$

Indeed, if $1 \leq i < m$, then we have

$$\frac{|J_i|}{|\mu_k^{(i)}|} = \frac{|I_i|}{2\lambda_k^{(i)}} \geq \frac{8\varepsilon}{\lambda_k^{(s)}} \frac{\lambda_k^{(s)}}{\lambda_k^{(i)}} \geq \frac{8q}{\lambda_k^{(s)}}$$

and if $m \leq i \leq s-1$, then

$$\frac{|J_i|}{|\mu_k^{(i)}|} = \frac{|I_i|}{2p_i} \frac{p_i q}{\lambda_k^{(s)}} \left| \frac{q\lambda_k^{(i)}}{\lambda_k^{(s)}} - p_i \right|^{-1} \geq \frac{8\varepsilon}{p_i} \frac{p_i q}{\lambda_k^{(s)}\varepsilon} = \frac{8q}{\lambda_k^{(s)}}.$$

Consequently (4) implies

$$B(x, 2q/\lambda_k^{(s)}) \subset \bigcap_{i=1}^{s-1} S(J_i, 1/\mu_k^{(i)}).$$

By (iv) of Lemma 2.9 and Lemma 2.5 (iii) (with $I := (1/q)I_s$ and $\delta := q/\lambda_k^{(s)}$) we can choose $m \in \mathbf{Z}$ such that

$$(1/\lambda_k^{(s)})I_s + mq/\lambda_k^{(s)} \subset (x, x + 2q/\lambda_k^{(s)}).$$

Consequently

$$d := c/\lambda_k^{(s)} + mq/\lambda_k^{(s)} \in B(x, 2q/\lambda_k^{(s)}) \subset \bigcap_{i=1}^{s-1} S(J_i, 1/\mu_k^{(i)}).$$

Thus we have $d \in S(2^{-1} * I_i, 1/\lambda_k^{(i)})$ for $1 \le i < m$. If $m \le i < s$ then we can apply Lemma 2.7 with $K := (1/p_i)\,(2^{-1} * I_i),\ z := c/q,\ \delta_1 := p_i/\lambda_k^{(i)}$ and $\delta_2 := q/\lambda_k^{(s)}$. Indeed, $\delta_1, \delta_2 \neq 0$ and $\delta_1 \neq \delta_2$ follows from $\mu_k^{(i)} \neq 0$. Since

$$\begin{aligned} d \in S(J_i, 1/\mu_k^{(i)}) &= S\left((1/p_i)(2^{-1} * I_i) - c/q,\ p_i q/(q\lambda_k^{(i)} - p_i\lambda_k^{(s)})\right) \\ &= S\left(K, \delta_1\delta_2/(\delta_2 - \delta_1)\right) \end{aligned}$$

and clearly $d \in S(\{c/q\}, q/\lambda_k^{(s)}) = S(\{z\}, \delta_2)$, Lemma 2.7 and Lemma 2.5 (vi) give

$$d \in S(K, \delta_1) = S\left((1/p_i)(2^{-1} * I_i),\ p_i/\lambda_k^{(i)}\right) \subset S(2^{-1} * I_i, 1/\lambda_k^{(i)}).$$

Since clearly $d \in S(2^{-1} * I_s, 1/\lambda_k^{(s)})$, we obtain

$$d \in \bigcap_{i=1}^{s} S(2^{-1} * I_i, 1/\lambda_k^{(i)}). \tag{5}$$

Denoting $\omega := \min\{|I_i| : i = 1, \ldots, s\}$, (5), Lemma 2.9 (i), (ii) and Lemma 2.5 (iv) imply

$$B(d, \omega/4\lambda_k^{(s)}) \subset \bigcap_{i=1}^{s} S(I_i, 1/\lambda_k^{(i)}) \subset \mathbf{R} \setminus E \subset \mathbf{R} \setminus Z.$$

Since clearly also $B(d, \omega/4\lambda_k^{(s)}) \subset (1/\lambda_k^{(s)})\, I_s + mq/\lambda_k^{(s)} \subset (x, x + 2q/\lambda_k^{(s)})$, we have

$$\frac{\pi(Z, (x, x + 2q/\lambda_k^{(s)}))}{2q/\lambda_k^{(s)}} \ge \frac{|B(d, \omega/4\lambda_k^{(s)})|}{2q/\lambda_k^{(s)}} = \frac{\omega}{4q}.$$

Since $\lim_{j\to\infty} \lambda_{k_j}^{(s)} = \infty$, we obtain $p^+(Z, x) \ge \omega/4q > 0$. By a quite symmetrical way we obtain $p^-(Z, x) \ge \omega/4q > 0$. Therefore Z is bilaterally porous at x which completes the proof. □

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