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# LEBESGUE INTEGRABILITY IMPLIES GENERALIZED RIEMANN INTEGRABILITY IN $\mathbb{R}^{]0,1]}$

**Abstract**

It is shown that in the space $\mathbb{R}^{]0,1]}$ any function which is Lebesgue integrable with respect to Wiener measure is also Henstock integrable.

## 1 Introduction

Generalized Riemann integration of Henstock type in infinite dimensional spaces, was examined in several studies (see [3, 7]). Here we consider a version of this integral as it was defined in [6], and investigate its relation to the Lebesgue integral.

In $\mathbb{R}^n$ it is known (see [1, 2]) that any Lebesgue integrable function is Henstock integrable with the same integral value and that these integrals are equivalent on the class of non-negative functions. We extend these results to the case of integration in $\mathbb{R}^{]0,1]}$ with respect to a measure coinciding with Wiener measure on cylindrical intervals.

The same problem was considered in [7] with a slightly different definition of the integral. But there was a gap in the proof of the corresponding results there related to the treatment of intervals attached to points with infinite coordinates. We are filling this gap in the proof presented here.

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## 2 Preliminaries

We review first the definition of the Henstock integral in one dimension and in $n$ dimensions. Let $I$ be a real interval of the form $]-\infty, v]$, $[u, v[$, or $[u, \infty[$. Let $\delta(x)$ be a positive function defined for $x \in \overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$. The function $\delta$ is called a *gauge*. We say that $I$ is *attached to* $x$ (or *associated with* $x$) if $x = -\infty$, $x = u$ or $v$, or $x = \infty$, respectively. If $I$ is attached to $x$, we say that $(x, I)$ is *$\delta$-fine* if

$$v < -\frac{1}{\delta(x)},\ v - u < \delta(x),\ u > \frac{1}{\delta(x)}$$

respectively. An *elementary set* $E$ is an interval or a finite union of intervals. $\mathbb{R}$ is an elementary set. A finite collection of attached point-interval pairs

$$\mathcal{E} = \{(x, I)\} = \{(x^{(1)}, I^{(1)}), \ldots, (x^{(m)}, I^{(m)})\}$$

is a *division* of $E$ if the $I^{(j)}$ are disjoint and have union $E$. The division $\mathcal{E}$ is *$\delta$-fine* if each $(x, I)$ of the division is $\delta$-fine. In such cases we sometimes write $\mathcal{E}$ as $\mathcal{E}_\delta$, and call it a *$\delta$-division.*

Suppose $h$ is a real- or complex-valued function of attached point-interval pairs $(x, I)$, with $h(x, I) := 0$ if $x$ is plus or minus infinity. For example, $h(x, I)$ could be a point function $f(x)$ multiplied by the interval length $|I| = v - u$, or by some other function of $I$. We say that $h$ is *generalized Riemann (or Henstock) integrable* on $E$, with integral $\alpha$, if for every $\varepsilon > 0$ there exists a gauge $\delta$ such that

$$\left|(\mathcal{E}_\delta) \sum h(x, I) - \alpha\right| < \varepsilon$$

for every $\delta$-division $\mathcal{E}_\delta$ of $E$. We write $\alpha = (\mathrm{H}) \int_E h(x, I)$.

For integration in $\mathbb{R}^n$, intervals $I$ are $I = I_1 \times \cdots \times I_n$, where each $I_j$ is a one-dimensional interval. $I$ is attached to $x = (x_1, \ldots, x_n)$ in $\mathbb{R}^n$ if each $I_j$ is attached to $x_j$ in $\mathbb{R}$. If a gauge $\delta$ is defined in the $n$-dimensional space, $\delta : \overline{\mathbb{R}}^n \mapsto \mathbb{R}_+$, an associated pair $(x, I)$ is $\delta$-fine provided each $(x_j, I_j)$ is $\delta$-fine in one-dimensional space (with $\delta(x_j) = \delta(x)$). For bounded intervals $I$ with $I_j = [u_j, v_j[$, $j = 1, \ldots, n$, this means that each $v_j - u_j < \delta(x)$. Provided we give the appropriate $n$-dimensional meaning to each of the parameters, the above definition of the integral serves also to define the integral in $n$-dimensional space of a function $h$ of attached point-interval pairs.

Turning to an infinite dimensional space $\mathbb{R}^B$, we can interpret it as the set of all real-valued functions defined on a set $B$. Respectively, $\overline{\mathbb{R}}^B$ is the set of all extended real functions on $B$.

We will define gauge and integral in $\mathbb{R}^B$. Although in the definitions which follow, $B$ can be taken to be any labelling set, in this paper $B$ will always stand for the interval $]0,1]$.

The integral in $\mathbb{R}^B$ will be defined by Riemann sums $\sum f(x)p(I)$ where the sets $I$ are *intervals* or *cylindrical intervals* in $\mathbb{R}^B$, and $p(I)$ is an interval function. If $N$ is a finite subset $\{t_1,\dots,t_n\}$ of $B$, a cylindrical interval $I$ or $I[N]$ is

$$\prod_{j=1}^{n} I_j \times \mathbb{R}^{B\setminus N}$$

where each $I_j$ is a one-dimensional interval. We write $\prod_{j=1}^{n} I_j$ as $I(N)$ and $(x(t_1),\dots,x(t_n))$ as $x(N)$. So

$$I = I[N] = \{x : x(N) \in I(N)\}.$$

We say that $x$, $N$ and $I[N]$ are *attached* (or *associated*) if $I_j$ is attached to $x(t_j)$ for $1 \leq j \leq n$. The Riemann sums for the function space integrals will be $\sum f(x)p(I)$ (or more generally, $\sum h(x,N,I)$) where the $x$, $N$, $I$ are attached. The integrand $h$ may depend explicitly on the dimension labels $t_j$ in $N$. Also, $h(x,N,I)$ will be zero by definition if $x(t_j)$ is $\pm\infty$ for at least one of the $t_j \in N$. The integral will exist if the Riemann sums converge, in some sense, to a single value. So we need some "shrinking rule" or gauge for the associated $I$, $N$, $x$, which appear in the terms of the Riemann sums.

We define a gauge on $\overline{\mathbb{R}}^B$ as follows.

- Choose a countable subset $A$ of $B$.
- Let $\mathbf{L}_A$ denote the family of all finite subsets of $A$. Let $L_A$ be a mapping from $\overline{\mathbb{R}}^B$ to $\mathbf{L}_A$,
$$L_A : \overline{\mathbb{R}}^B \mapsto \mathbf{L}_A,$$
so that $L_A(x)$ is a finite subset of $A$. Thus each $x \in \overline{\mathbb{R}}^B$ is allocated its own finite set of dimensions.
- Given $L_A(x)$, for each finite $N \supseteq L_A(x)$ let $\delta_N$ be a gauge on the finite-dimensional space $\mathbb{R}^N$, so
$$\delta_N : \overline{\mathbb{R}}^N \mapsto \mathbb{R}_+.$$
Now let $\overline{\delta}_{L_A}$ denote the family of all the finite-dimensional gauges selected. In effect $\overline{\delta}_{L_A}$ corresponds to the following collection of real numbers
$$\left\{\delta_N(x(N)) : N \supseteq L_A(x), x(N) \in \overline{\mathbb{R}}^N\right\}.$$

A gauge $\gamma$ on the infinite-dimensional space $\mathbb{R}^B$ is given by

$$(A, L_A, \overline{\delta}_{L_A}). \tag{1}$$

If $x$, $N$, and $I$ are attached in $\mathbb{R}^B$, we say that $(x, I[N])$ is $\gamma$-fine if

- $N \supseteq L_A(x)$, and
- $(x(N), I(N))$ is $\delta_N$-fine in $\mathbb{R}^N$.

Thus, the components of a gauge in $\mathbb{R}^B$ are, firstly, a minimal dimension set, denoted $L_A(x)$, for each $x$, and, secondly, a gauge $\delta_N$ for each finite dimension set $N$ containing $L_A(x)$.

We define a division $\mathcal{E} = \{(x, I[N])\}$ of $\mathbb{R}^B$ or of any elementary set $E$ in $\mathbb{R}^B$ in the same way as it was defined above in the one-dimensional case. We say that the division $\mathcal{E}$ is $\gamma$-fine, and denote it by $\mathcal{E}_\gamma$, if each member $(x, I[N])$ of $\mathcal{E}$ is $\gamma$-fine. We may call $\mathcal{E}_\gamma$ a $\gamma$-*division.*

**Definition 1.** Suppose $h$ is a function of attached integration elements $x$, $N$, $I$; with $h(x, N, I[N])$ defined to be zero if $x(t) = -\infty$ or $\infty$ for some $t \in N$. If $E$ is an elementary set, we say that $h$ is *integrable in the generalized Riemann* sense (or that it is *Henstock integrable* or H-*integrable*) on the elementary set $E$, with integral $\alpha$, if, given $\varepsilon > 0$, there exists $\gamma$ so that

$$\left|(\mathcal{E}_\gamma) \sum h(x, N, I) - \alpha\right| < \varepsilon$$

for every $\mathcal{E}_\gamma$. We then write

$$(\mathrm{H}) \int_E h(x, N, I) = \alpha.$$

If $h(x, N, I) = f(x)m(I)$ where $f$ is a point function on $\mathbb{R}^B$ extended by 0 on $\overline{\mathbb{R}}^B$, $m$ is an interval function (measure), and $I[N]$ is attached to $x$ in the above sense, we simply say that $f$ is H-integrable in $E$ with respect to $m$.

Given an elementary set $E$, $P$ is a *partial set* of $E$ if $P$ is the union of a subset of the intervals comprising $E$. It is known (see [6], Proposition 9, page 27) that if $(\mathrm{H})\int_E h$ exists, then $(\mathrm{H})\int_P h$ exists as a finitely additive function of the partial sets $P$ of $E$.

The following result is proved in [6], Proposition 10, page 28.

**Saks-Henstock Lemma.** *If $h(x, N, I)$ is* H-*integrable in an elementary set $E$, $P$ is a partial set of $E$, and, with a gauge $\gamma$ given, we have*

$$\left|(\mathcal{E}_\gamma) \sum h(x, N, I[N]) - (\mathrm{H}) \int_E h\right| < \varepsilon$$

*for all $\gamma$-fine divisions $\mathcal{E}_\gamma$ of $E$, then*

$$\left|(\mathcal{P}_\gamma)\sum h(x,N,I[N]) - (\mathrm{H})\int_P h\right| < 2\varepsilon$$

*for all $\gamma$-fine divisions $\mathcal{P}_\gamma$ of $P$, and*

$$(\mathcal{E}_\gamma)\sum\left|h(x,N,I[N]) - (\mathrm{H})\int_{I[N]} h\right| < 4\varepsilon$$

*for all $\gamma$-fine divisions $\mathcal{E}_\gamma$ of $E$.*

## 3 Main Results

Our main results in this paper are related to the integration of functions $f(x)m(I)$ in $\mathbb{R}^{]0,1]}$ where the interval function $m$ is defined on cylindrical intervals

$$I = \{x \in \mathbb{R}^{]0,1]} : u_j \leq x(t_j) < v_j, 0 < t_1 < t_2 < \cdots < t_j < \cdots < t_n \leq 1\}, \quad (2)$$

and $m(I)$ denotes

$$\prod_{j=1}^{n}(2\pi(t_j - t_{j-1}))^{-\frac{1}{2}}\int_{u_1}^{v_1}\!\!\ldots\int_{u_n}^{v_n}\exp\left(\frac{-1}{2}\sum_{j=1}^{n}\frac{(x_j - x_{j-1})^2}{t_j - t_{j-1}}\right)dx_1\ldots dx_n, \quad (3)$$

where $t_0 = x_0 = 0$. This means that $m$ coincides on $I$ with the Wiener measure $m_W$ induced by (3) on the cylindrical intervals, analogous to those in (2), in the Wiener space $\mathbf{C}_W$ of real-valued continuous functions $x$ on $[0,1]$ with $x(0) = 0$, that is, on the intervals

$$J = \{x \in \mathbf{C}_W : u_j \leq x(t_j) < v_j,\ 0 < t_1 < t_2 < \cdots < t_j < \cdots < t_n \leq 1\}.$$

Note that

$$J = I \cap \mathbf{C}_W. \quad (4)$$

(Strictly speaking, $\mathbf{C}_W$ is a subset of $X = \{0\} \times \mathbb{R}^{]0,1]}$, but we do not distinguish between corresponding subsets of $\mathbb{R}^{]0,1]}$ and $X$.)

Both $m$ and $m_W$ can be extended to probability measures defined on the $\sigma$-algebra of Borel subsets of $\mathbb{R}^{]0,1]}$ and $\mathbf{C}_W$ respectively. Then we can define the Carathéodory outer measure $m^*$ on all subsets of $\mathbb{R}^{]0,1]}$. It is known that, although $\mathbf{C}_W$ is not a Borel subset of $\mathbb{R}^{]0,1]}$ (see [5]), we have $m^*(\mathbf{C}_W) = 1$

(see [8]), and the Wiener measure $m_W$ defined on intervals (4) by (3) is equal to $m^*$ on the Borel subsets of $\mathbf{C}_W$, considered as subsets of $\mathbb{R}^{]0,1]}$.

The outer measure $m^*$ is a complete measure on the $\sigma$-algebra of the Carathodory $m^*$-measurable subsets of $\mathbb{R}^{]0,1]}$, and we can consider Lebesgue integration with respect to this measure. We shall write $m(S)$ instead of $m^*(S)$ if $S$ is measurable with respect to $m^*$.

Directly from the definition of the Carathodory outer measure and measurability we get the following lemma.

**Lemma 1.** *If $S$ is $m^*$-measurable in $\mathbb{R}^{]0,1]}$ then, for every $\varepsilon_0 > 0$, there exists a sequence $\{I^{(j)}\}_{j=1}^{\infty}$ of cylindrical intervals, such that*

$$S \subseteq \cup_{j=1}^{\infty} I^{(j)}, \;\; m(\cup I^{(j)} \setminus S) < \varepsilon_0.$$

We use the following notation for Hölder classes of functions in $\mathbf{C}_W$. Given $h > 0$, $0 < \lambda \leq 1$, let

$$\mathbf{C}_h^{\lambda} := \{x \in \mathbf{C}_W : |x(t) - x(t')| \leq h|t - t'|^{\lambda} \text{ for all } t, t' \in [0,1]\}.$$

It is clear that $\mathbf{C}_{h_1}^{\lambda} \subset \mathbf{C}_{h_2}^{\lambda}$ if $h_1 < h_2$ and so the class $\mathbf{C}^{\lambda}$ of functions from $\mathbf{C}_W$ which are Hölder continuous of order $\lambda$ can be written as the limit

$$\mathbf{C}^{\lambda} = \lim_{h \to \infty} \mathbf{C}_h^{\lambda}.$$

It is known (see [5]) that $\mathbf{C}^{\lambda}$ and $\mathbf{C}_h^{\lambda}$ are closed and hence measurable subsets of $\mathbf{C}_W$ with respect to Wiener measure $m_W$, and that $m_W(\mathbf{C}^{\lambda}) = 1$ if $0 < \lambda < \frac{1}{2}$. So for such $\lambda$,

$$\lim_{h \to \infty} m^*(\mathbf{C}_W \setminus \mathbf{C}_h^{\lambda}) = \lim_{h \to \infty} m_W(\mathbf{C}_W \setminus \mathbf{C}_h^{\lambda}) = 1 - \lim_{h \to \infty} m_W(\mathbf{C}_h^{\lambda}) = 0$$

and we can state the following consequence.

**Lemma 2.** *If $0 < \lambda < \frac{1}{2}$, then for any $\eta > 0$ there exists $h > 0$ such that $m^*(\mathbf{C}_W \setminus \mathbf{C}_h^{\lambda}) < \eta$.*

For any $t \in ]0,1]$ and $\alpha > 0$, consider cylinder sets

$$K_t(\alpha) := \{x \in \mathbf{C}_W : |x(t)| > \alpha\}. \tag{5}$$

In this notation we prove the following lemma.

**Lemma 3.** *For any $\eta > 0$ there exists $\alpha > 0$ such that $m^*\left(\cup_{t \in ]0,1]} K_t(\alpha)\right) < \eta$.*

PROOF. Choose $h$ according to Lemma 2, so that $m^*(\mathbf{C}_W \setminus \mathbf{C}_h^\lambda) < \eta$ for fixed $\lambda < \frac{1}{2}$. We show that we can take $\alpha = h$. Indeed, if $x \in K_t(h)$ for some $t \in ]0,1]$, then $|x(t) - x(0)| > h \geq ht^\lambda$ and $x \in \mathbf{C}_W \setminus \mathbf{C}_h^\lambda$. As this is true for any $t \in ]0,1]$ we get $\cup_{t\in]0,1]} K_t(h) \subset \mathbf{C}_W \setminus \mathbf{C}_h^\lambda$ and so by Lemma 2 $m^*\left(\cup_{t\in]0,1]} K_t(h)\right) \leq m^*\left(\mathbf{C}_W \setminus \mathbf{C}_h^\lambda\right) < \eta$ giving the result. □

**Theorem 1.** *If $f$ is a finite, real-valued function on $\mathbb{R}^{]0,1]}$, which is Lebesgue integrable with respect to $m^*$ then $f(x)$ is generalized Riemann integrable in $\mathbb{R}^{]0,1]}$ with respect to $m$, and the two integrals are equal.*

PROOF. Fix $\varepsilon > 0$ and choose $\eta$ so that, for $Y$ measurable, $m(Y) < \eta$ implies $\int_Y |f|\, dm < \frac{\varepsilon}{3}$. Let $\xi = \varepsilon/(3\eta + 3)$, and let $S_k = \{x : (k-1)\xi < f(x) \leq k\xi\}$, where $k$ is any integer. Apply Lemma 1 with $S = S_k$ and $\varepsilon_0 = \dfrac{\eta}{2^{|k|+3}(|k|+1)}$, and find $I^{(i,k)}$ such that

$$m^*\left((\cup_{i=1}^\infty I^{(i,k)}) \setminus S_k\right) = m(\cup_i I^{(i,k)}) - m(S_k) \leq \varepsilon_0.$$

Now we define a gauge $\gamma$ for the given $\varepsilon$. For an interval $I = I[N]$ having $N$ as its set of restricted dimensions, we write $N = N_I$. Put $A = \cup_{i,k} N_{I^{(i,k)}}$. To define $\delta_N(x(N))$, take, first, $x \in \mathbb{R}^{]0,1]}$. Suppose $x \in S_k$. Then $x \in I^{(i,k)}$ for some $i = i(x)$ and we let

$$L(x) = N_{I^{(i(x),k)}},\ \text{and}\ \delta_N(x(N)) = \rho\left(x(N), \Gamma\left(I^{(i(x),k)}(N)\right)\right)$$

for any $N \supset L(x)$, where $\Gamma(I(N))$ is the boundary of $I(N)$ in $\mathbb{R}^N$ and $\rho(y, Y)$ is the distance from the point $y$ to the set $Y$ in $\mathbb{R}^N$.

If $x \in \overline{\mathbb{R}}^{]0,1]} \setminus \mathbb{R}^{]0,1]}$, then for any $N$ we put $\delta_N(x(N)) = \alpha^{-1}$ where $\alpha$ is taken from Lemma 3 with $\eta$ substituted by $\eta/2$; and we let $L(x)$ be arbitrary. Now we take any $\gamma$-fine division $\mathcal{E} = \{(x^{(j)}, J^{(j)})\}$ of $\mathbb{R}^{]0,1]}$ and, bearing in mind that the integral sign in the following estimation is understood in the Lebesgue sense, we get

$$\begin{aligned}\left|(\mathcal{E})\sum_j f(x^{(j)})m(J^{(j)}) - \int_{\mathbb{R}^{]0,1]}} f\, dm\right| &= \left|(\mathcal{E})\sum_j \int_{J^{(j)}} \left(f(x^{(j)}) - f(x)\right) dm\right| \\ &\leq (\mathcal{E})\sum_j \int_{J^{(j)}} |f(x^{(j)} - f(x)|\, dm \quad (6) \\ &\leq p + q + r + s\end{aligned}$$

where, with $x^{(j)} \in S_{k(j)}$,

$$
\begin{aligned}
p &= \sum_j \left\{ \int_{J^{(j)} \cap S_{k(j)}} |f(x^{(j)}) - f(x)|\, dm : x^{(j)} \in \mathbb{R}^{]0,1]} \right\}\\
q &= \sum_j \left\{ \int_{J^{(j)} \setminus S_{k(j)}} |f(x^{(j)})|\, dm : x^{(j)} \in \mathbb{R}^{]0,1]} \right\},\\
r &= \sum_j \left\{ \int_{J^{(j)} \setminus S_{k(j)}} |f(x)|\, dm : x^{(j)} \in \mathbb{R}^{]0,1]} \right\},\\
s &= \sum_j \left\{ \int_{J^{(j)}} |f(x)|\, dm : x^{(j)} \in \overline{\mathbb{R}}^{]0,1]} \setminus \mathbb{R}^{]0,1]} \right\}.
\end{aligned}
$$

Now $x \in J^{(j)} \cap S_{k(j)}$ implies $(k(j) - 1)\xi < f(x)$ and $f(x^{(j)}) \leq k(j)\xi$; so $|f(x^{(j)}) - f(x)| \leq \xi$ and

$$
p \leq \sum_{\substack{j \\ x^{(j)} \in \mathbb{R}^{]0,1]}}} \left\{ \int_{J^{(j)} \cap S_{k(j)}} \xi\, dm \right\} \leq \xi \left( \sum_{\substack{j \\ x^{(j)} \in \mathbb{R}^{]0,1]}}} \left\{ m(J^{(j)}) \right\} \right) = \xi < \frac{\varepsilon}{3}. \quad (7)
$$

Next, $x^{(j)} \in S_{k(j)}$ and $k(j) = k$ imply $f(x_j) \leq k\xi$; so $|f(x^{(j)})| \leq |k|\xi \leq (|k|+1)\xi$; giving

$$
\begin{aligned}
q &= \sum_{k=-\infty}^{\infty} \left( \sum_{\substack{j \\ k(j)=k}} \int_{J^{(j)} \setminus S_k} |f(x^{(j)})|\, dm \right) \leq \sum_{k=-\infty}^{\infty} \left( \sum_{\substack{j \\ k(j)=k}} (|k|+1)\xi m(J^{(j)} \setminus S_k) \right)\\
&\leq \sum_{k=-\infty}^{\infty} (|k|+1)\xi m(\cup_{j=1}^{\infty} I^{(j,k)} \setminus S_k) = \sum_{k=-\infty}^{\infty} (|k|+1)\xi \frac{\eta}{2^{|k|+3}(|k|+1)} \qquad (8)\\
&= \xi\eta \frac{1}{2^3} \left( \sum_{-\infty}^{0} \frac{1}{2^{|k|}} + \sum_{1}^{\infty} \frac{1}{2^{|k|}} \right) = \frac{\varepsilon\eta}{3\eta+3} \left( \frac{3}{8} \right) < \frac{\varepsilon}{3}.
\end{aligned}
$$

The sets $J^{(j)}\backslash S_{k(j)}$ are disjoint; their union $Y_1$ satisfies

$$\begin{aligned} m(Y_1) =& \sum_{k=-\infty}^{\infty}\left(\sum_{j:k(j)=k} m(J^{(j)}\backslash S_k)\right) \\ \leq& \sum_{k=-\infty}^{\infty} m(\cup_{j=1}^{\infty} I^{(j,k)}\backslash S_k) \\ <&\eta \sum_{k=-\infty}^{\infty} \frac{1}{2^{|k|+3}(|k|+1)} < \frac{\eta}{2}. \end{aligned} \tag{9}$$

To estimate $s$ we define cylinder sets $\tilde{K}_t(\alpha) := \{x \in \mathbb{R}^{]0,1]} : x(t) > \alpha\}$. It is clear that $K_t(\alpha) = \tilde{K}_t(\alpha) \cap \mathbf{C}_W$, (see (5)), and according to the definitions of $m$ and $m_W$ on the cylindrical intervals, $m\left(\tilde{K}_t(\alpha)\right) = m_W\left(K_t(\alpha)\right)$. Similarly, these measures coincide for any finite union of such sets.

If $x^{(j)} \in \overline{\mathbb{R}}^{]0,1]} \setminus \mathbb{R}^{]0,1]}$ then $\delta_N(x^{(j)}(N)) = \alpha^{-1}$, and $J^{(j)} \subset \cup_{t\in N_{J^{(j)}}} \tilde{K}_t(\alpha)$. Let $\tilde{N} = \cup_j N_{J^{(j)}}$ where the union is taken over all $j$ such that $x^{(j)} \in \overline{\mathbb{R}}^{]0,1]} \setminus \mathbb{R}^{]0,1]}$. Note that $\tilde{N}$ is a finite set. Putting $Y_2 = \cup_j J^{(j)}$, the union being taken over the same set of $j$, we get, by the choice of $\alpha$,

$$\begin{aligned} m(Y_2) \leq& m\left(\cup_{t\in\tilde{N}}\tilde{K}_t(\alpha)\right) = m_W\left(\cup_{t\in\tilde{N}}K_t(\alpha)\right) = m^*\left(\cup_{t\in\tilde{N}}K_t(\alpha)\right) \\ \leq& m^*\left(\cup_{t\in]0,1]}K_t(\alpha)\right) < \frac{\eta}{2}. \end{aligned}$$

So, by (9), $m(Y_1 \cup Y_2) < \eta$. Taking into account the choice of $\eta$ we get

$$r + s = \int_{Y_1\cup Y_2} |f(x)|\, dm < \frac{\varepsilon}{3}. \tag{10}$$

Therefore, from (6), (7), (8) and (10) we have

$$\left|(\mathcal{E})\sum_j f(x^{(j)})m(J^{(j)}) - (\mathrm{L})\int_{\mathbb{R}^{]0,1]}} f\, dm\right| < \varepsilon,$$

giving the required result. □

Given the associated integration elements $x$, $N$, $I[N]$, we can define a generalized Riemann integrand $h(x, N, I[N]) = m(I[N])$.

We can view this integrand as $f(x)m(I)$, with $f(x) \equiv 1$ obviously Lebesgue integrable. Then, from the above theorem, we have the following result.

**Corollary 1.** *$m(I[N])$ is generalized Riemann integrable over $\mathbb{R}^{]0,1]}$, with integral 1.*

In the opposite direction, we consider conditions under which generalized Riemann integrability implies Lebesgue integrability with respect to the measure $m$. In the finite dimensional case it is known [2] that if a function is Henstock integrable and non-negative it is Lebesgue integrable with the same integral value. We prove here a partial analogue of this result, under the additional assumption that the integrand is measurable. We must assume this because the following problem seems to be open.

**Problem 1.** *If a function $f(x)$ is generalized Riemann integrable over $\mathbb{R}^{]0,1]}$, is $f$ measurable with respect to the outer measure $m^*$?*

We need the following extension of the B. Levy theorem.

**Theorem 2.** *Let $\{f_n\}$ be a sequence of H-integrable functions for which:*

1. $f_n(x) \leq f_{n+1}(x)$ *for all* $x \in \mathbb{R}^{]0,1]}$ *and* $n = 1, 2, 3, \ldots$.
2. $\lim_{n\to\infty} f_n(x) = f(x)$ *for all* $x \in \mathbb{R}^{]0,1]}$.
3. $\lim_{n\to\infty} (\mathrm{H}) \int_{\mathbb{R}^{]0,1]}} f_n = \beta$.

*Then $f$ is H-integrable in $\mathbb{R}^{]0,1]}$ and* $(\mathrm{H}) \int_{\mathbb{R}^{]0,1]}} f = \beta$.

PROOF. We repeat the argument used in finite dimensional cases (see [4]) taking additional care of the points with infinite coordinates. For a given $\varepsilon > 0$ take $n_0$ such that $0 < \beta - \int_{\mathbb{R}^{]0,1]}} f_n < \varepsilon$ for all $n \geq n_0$. For each $f_n$ we can find a gauge $\gamma_n$ such that for any $\gamma_n$-fine division $\mathcal{E}_n$

$$\left| (\mathcal{E}_n) \sum f_n(x) m(I) - \int_{\mathbb{R}^{]0,1]}} f_n \right| < \frac{\varepsilon}{2^n}. \tag{11}$$

For each $x \in \mathbb{R}^{]0,1]}$ we also find $m = m(x) \geq n_0$ so that

$$0 \leq f(x) - f_{m(x)}(x) < \varepsilon. \tag{12}$$

For $x \in \overline{\mathbb{R}}^{]0,1]} \setminus \mathbb{R}^{]0,1]}$ we put $m(x) = n_0$.

Now we are ready to define a gauge $\gamma$ for $f$. We put $A^\gamma = \cup_n A^{\gamma_n}$, where $A^\gamma$ is the countable set $A$ in the definition (1) of the gauge $\gamma$. If $x \in \mathbb{R}^{]0,1]}$ we let $\delta_N(x(N)) = \delta_N^{(m(x))}(x(N))$ and $L(x) = L^{(m(x))}(x)$, where $\delta_N^{(m(x))}$ and

$L(x) = L^{(m(x))}$ are respectively elements of (1) relating to the gauge $\gamma_{m(x)}$. Then, for any $\gamma$-fine division $\mathcal{E}$, we have

$$\begin{aligned}\left|(\mathcal{E})\sum f(x)m(I) - \beta\right| \leq & |(\mathcal{E}) \sum_{x\in\mathbb{R}^{]0,1]}} \left(f(x) - f_{m(x)}(x)m(I)\right)| \\ & + |(\mathcal{E})\sum \left(f_{m(x)}(x)m(I) - \int_I f_{m(x)}\right)| \\ & + |(\mathcal{E})\sum \int_I f_{m(x)} - \beta| = p + q + r\end{aligned}$$

By (12), $p < \varepsilon$. To estimate $q$ we note that this sum can be split into a finite number of sums, each of them related to some value of $m(x) = m$. We can apply the Saks-Henstock Lemma to each of these subsidiary sums and to $f_m$, and we get by (11)

$$q \leq \sum_{m=1}^{\infty} (\mathcal{E}_{\gamma_{m(x)}}) \sum_{\substack{x\in\mathbb{R}^{]0,1]} \\ m(x)=m}} \left|f_{m(x)}(x)m(I) - \int_I f_{m(x)}\right| \leq \sum_{m=1}^{\infty} 4\frac{\varepsilon}{2^m} = 4\varepsilon.$$

Turning to $r$, we note that with $\tilde{m} = \max\{m(x) : x\}$,

$$\int_{\mathbb{R}^{]0,1]}} f_{n_0} \leq (\mathcal{E})\sum \int_I f_{m(x)} \leq \int_{\mathbb{R}^{]0,1]}} f_{\tilde{m}} \leq \beta.$$

Then by (11)

$$r \leq \left|\int f_n - \beta\right| < \varepsilon.$$

Summing, we get finally

$$\left|(\mathcal{E})\sum f(x)m(I) - \beta\right| < 6\varepsilon.$$

This completes the proof. □

**Theorem 3.** *If the non-negative function $f(x)$ is $m$-measurable and generalized Riemann integrable over $\mathbb{R}^{]0,1]}$, then it is Lebesgue integrable over $\mathbb{R}^{]0,1]}$ with respect to $m$, and the two integrals are equal.*

Proof. For any positive integer $n$ consider the truncated function $f_n(x) = \min\{f(x), n\}$. Being measurable and bounded it is Lebesgue integrable over $\mathbb{R}^{]0,1]}$. Then by Theorem 1, it is also generalized Riemann integrable, and $(\mathrm{H})\int f_n = (\mathrm{L})\int f_n$. Now the result follows from Theorem 2 and the B. Levy theorem for Lebesgue integration. □

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