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GENERALIZATION OF THE BANACH INDICATRIX THEOREM

Abstract

In this paper we introduce a measure on an interval connected with variation of given function f. Next we use this measure to calculate variation of a composition.

Let $f : [0,1] \to [0,1]$ be a given continuous function. For every closed interval $[a,b] \subset [0,1]$ let $v^*([a,b]) = \bigvee_a^b f$. (We allow $[a,b]$ to be a degenerate interval, ; i.e., $a = b$. In this case $v^*([a,b]) = 0$.) Now for every set $A \subset [0,1]$ let

$$v_z(A) = \inf_{\{I_n\}} \{\sum_{n\in\mathbb{N}} v^*(I_n) : A \subset \bigcup_{n\in\mathbb{N}} I_n\}.$$

where $\{I_n\}$ denotes an arbitrary family of closed intervals covering A.

Proposition 1. *The function $v_z : 2^{[0,1]} \to \mathbb{R}_+ \cup \{+\infty\}$ is an outer measure in sense of Carathéodory.*

The proof is easy and hence is omitted.

Proposition 2. *Every closed interval $[a,b] \subset [0,1]$ satisfies Carathodory's condition.*

Proof. Let $[a,b] \subset [0,1]$ and let $W \subset [a,b]$, $Z \subset [0,1] \setminus [a,b]$. Let $\{I_n\}_{n\in\mathbb{N}}$ be a family of closed intervals covering $W \cup Z$. For every $n \in \mathbb{N}$, let $J_n = I_n \cap [a,b]$; $K_n = I_n \cap [0,a]$; $L_n = I_n \cap [b,1]$. Then the family $\{J_n\}$ covers set W, and the family $\{K_n\} \cup \{L_n\}$ covers set Z, moreover,

$$\sum_{n\in\mathbb{N}} v^*(I_n) = \sum_{n\in\mathbb{N}} v^*(J_n) + \sum_{n\in\mathbb{N}} v^*(K_n) + \sum_{n\in\mathbb{N}} v^*(L_n).$$

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So

$$v_z(W \cup Z) \geq v_z(W) + v_z(Z).$$

□

Corollary 1. *By virtue of the Carathodory's theorem, the function v_z cut to the σ-algebra of sets satisfying condition of Carathodory is a measure.*

Let's denote this measure by v, and this σ-algebra by S. Of course both S and v depend on function f. However by virtue of Proposition 2, S always contains the Borel sets.

Proposition 3. *For every interval $[a,b] \subset [0,1]$, $v([a,b]) = \bigvee_a^b f$.*

PROOF. The single-element family consisting of the interval $[a,b]$ covers this interval. So clearly $v([a,b]) \leq \bigvee_a^b f$. Now let $(I_n)_{n\in\mathbb{N}}$ be a family of closed intervals covering $[a,b]$. We show that $\sum_{n\in\mathbb{N}} \bigvee_{I_n} f \geq \bigvee_a^b f$. Let N_k be the indicatrix of function f cut to the interval I_k. Then $\bigvee_{I_k} f = \int_{[0,1]} N_k d\mu$. We show, that $\sum_{k\in\mathbb{N}} N_k \geq N$, where N stands for the indicatrix of function f cut to the interval $[a,b]$. Let $y \in [0,1]$ and let $q \in \mathbb{N}$ be a given number less than or equal to $N(y)$. Then there exists at least q different roots $x_1, x_2, ...x_q$ of the equation $f(x) = y$. For every $i \leq q$, there exists at least one number n such that $x_i \in (I_n)$. Let $n_0 \in \mathbb{N}$ be a number such that the set $\bigcup_{i=1}^{n_0} I_i$ contains every point $x_1, x_2, ...x_q$. Then$\sum_{k\in\mathbb{N}} N_k(y) \geq \sum_{i=1}^{n_0} N_i(y) \geq q$ so it follows that $\sum_{k\in\mathbb{N}} N_k(y) \geq N(y)$.

The series $\sum_{i\in\mathbb{N}} N_i$ of non-negative functions converges to some function $N^* : [0,1] \to \bar{\mathbb{R}}$, and $N^* \geq N$. So

$$\sum_{n\in\mathbb{N}} \bigvee_{I_n} f = \sum_{n\in\mathbb{N}} \int_{[0,1]} N_n = \int_{[0,1]} N^* \geq \int_{[0,1]} N = \bigvee_a^b f.$$

□

Theorem 1. *Let $f : [0,1] \to [0,1]$ be a continuous function, and let us define the measure v_f using f the same way as above. Let $g : [0,1] \to [0,1]$ be a continuous function, and $N_g : [0,1] \to \mathbb{N} \cup \{\infty\}$ stands for the indicatrix of function g. Then variation of composition $f \circ g$ can be expressed by following formula: $\bigvee_0^1 f \circ g = \int_0^1 N_g dv_f$.*

PROOF. Let the sequence $a_k^n = \frac{k}{2^n}$. divide the interval $[0,1]$ into 2^n equal parts. Let m_k^n and M_k^n stand for the minimum and maximum of the function g on interval $[a_k^n, a_{k+1}^n]$. Let $A_k^n = \bigvee_{m_k^n}^{M_k^n} f$ and let $A^n = \sum_{k=0}^{2^n-1} A_k^n$. At the beginning we observe, that the sequence A^n is increasing. In order to do this let us consider an arbitrary interval $[a,b] \subset [0,1]$. Let m and M stand for the minimum and maximum of function g on this interval. Now let m_1 and M_1 stand for the minimum and maximum of the function g on $[a, \frac{a+b}{2}]$, and m_2 and M_2 on $[\frac{a+b}{2}, b]$, respectively. Since function g is continuous, it has Darboux property, so

$$[m_1, M_1] \cup [m_2, M_2] = [m, M]$$

and then $\bigvee_{m_1}^{M_1} f + \bigvee_{m_2}^{M_2} f \geq \bigvee_{m}^{M} f$. As a result $A_{2k}^{n+1} + A_{2k+1}^{n+1} \geq A_k^n$ and at last $A^{n+1} \geq A^n$.

Now observe, that for every $n \in \mathbb{N}$ $A^n \leq \bigvee_0^1 f \circ g$. In fact, we only have to show that $A_k^n \leq \bigvee_{a_k}^{a_{k+1}} f \circ g$. Let us consider an arbitrary interval $[a,b] \subset [0,1]$. Let m and M stand for the minimum and maximum of the function g on this interval. Let $A = \bigvee_m^M f$. We show, that $A \leq \bigvee_a^b f \circ g$. Let $P : m = y_0 < y_1 < ... < y_n = M$ be a given division of interval $[m,M]$. Let $c, d \in [a,b]$, be numbers such that $g(c) = m$ and $g(d) = M$. For instance suppose that $c < d$. Since the function g has Darboux property, there exists $x_1 \in (c,d)$ such that $g(x_1) = y_1$. Next, there exists $x_2 \in (x_1, d)$ such that $g(x_2) = y_2$. And so on, to x_{n-1}. The sequence x_n is increasing, and commonly with c and d forms the division of the interval $[c,d]$. Therefore

$$\sum_{i=0}^{n-1} |f(y_i) - f(y_{i+1})| = \sum_{i=0}^{n-1} |f(g(x_i)) - f(g(x_{i+1}))| \leq \bigvee_c^d f \circ g \leq \bigvee_a^b f \circ g.$$

Since the division P was arbitrary, we obtain that $A = \bigvee_m^M f \leq \bigvee_a^b f \circ g$.

Now we show that $\lim_{n \to \infty} A^n = \bigvee_0^1 f \circ g$. Let $\epsilon > 0$. Let $P : 0 = x_0 < x_1 < ... < x_k = 1$ be a given division of the interval $[0,1]$. The function $f \circ g$ is uniformly continuous on $[0,1]$. Let us take $\delta > 0$ such that $|f(g(\alpha)) - f(g(\beta))| < \frac{\epsilon}{2k}$ if $|\alpha - \beta| < \delta$. Let $n \in \mathbb{N}$ be such number, that $\frac{1}{2^n} < \delta$, and at the same time $\frac{1}{2^n} < \min_{i=1...k}(x_i - x_{i-1})$. Let $a_i^n = \frac{i}{2^n}$, $i = 1...2^n - 1$.

Let us consider $Q : \alpha_0 < \alpha_1 < ... < \alpha_l$ constructed with points x_i and a_i^n. Hence

$$\sum_{i=0}^{k-1} |f(g(x_i)) - f(g(x_{i+1}))| \leq \sum_{i=0}^{l-1} |f(g(\alpha_i)) - f(g(\alpha_{i+1}))|.$$

The last sum contains $2k - 2$ components of type $|f(g(a_i^n)) - f(g(x_j))|$ or

$|f(g(x_j)) - f(g(a_i^n))|$. Each of them can be estimated by $\frac{\epsilon}{2k}$. Consequently,

$$\begin{aligned}\sum_{i=0}^{l-1} |f(g(\alpha_i)) - f(g(\alpha_{i+1}))| &< \sum_{i=0}^{2^n-1} |f(g(a_i^n)) - f(g(a_{i+1}^n))| + \epsilon \\ &\leq \sum_{i=0}^{2^n-1} A_i^n + \epsilon = A^n + \epsilon.\end{aligned}$$

It implies, that for every ϵ and every division P there exists n which satisfies the above inequality. Therefore $\lim_{n\to\infty} A^n = \bigvee_0^1 f \circ g$.

Now we show, that $\lim_{n\to\infty} A^n = \int_{[0,1]} N_g dv_f$. Let us fix an $n \in \mathbb{N}$. For $k = 0...2^n - 1$ let $N_k^n : [0,1] \to \infty$ be a characteristic function of interval $[m_k^n, M_k^n]$. Next let $N^n = \sum_{i=0}^{2^n-1} N_k^n$. Sequence N^n is increasing. We write $C = \{\frac{i}{2^n} : n \in \mathbb{N}, i = 1, ..., 2^n\}$. For every y such that $y \notin g(C)$ we have $N^n(y) \to N_g(y)$. Since $g(C)$ is at most countable, we obtain, that the sequence N^n is converging to function N_g v_f-almost everywhere. By virtue of proposition 2 every function N_k^n is measurable with respect to v_f. Therefore N^n is measurable, and consequently N_g is measurable too. Moreover, by virtue of Proposition 3, $\int_{[0,1]} N_k^n \, dv = \bigvee_{m_k^n}^{M_k^n} f = A_k^n$. So $\int_{[0,1]} N^n \, dv = A^n$, and finally, by Lebesgue's Monotone Convergence Theorem $\int_{[0,1]} N_g \, dv_f = \lim_{n\to\infty} A^n$. Hence $\bigvee_0^1 f \circ g = \int_0^1 N_g \, dv_f$. □

References

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