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THE PRIMITIVE OF A KURZWEIL-HENSTOCK INTEGRABLE FUNCTION IN MULTIDIMENSIONAL SPACE

Abstract

In this paper, give a new, full characterization of the primitive of a Kurzweil-Henstock integrable function in multidimensional space.

1 Introduction

Recently, Lu and Lee [6] gave a characterization of the primitive of a Kurzweil-Henstock integral in $\mathbb{R}^m$. They showed that given a point function f and an additive interval function F defined on an m-dimensional interval E, f is Kurzweil-Henstock integrable on E with primitive F if and only if

(1) $F'(x)$ is equal to $f(x)$ except for a set of Γ_k-measure zero, and

(2) F is Γ_k-Strong Lusin for every $k \in \mathbb{N}$ where Γ_k is some inner cover involving F and f.

A question that arises is whether we can describe the primitive F without explicitly involving f. In this paper, we give an affirmative answer to this question. Furthermore, we provide a corresponding characterization of the primitive of a McShane integrable function.

2 Preliminaries

The set E always refers to a fixed, compact interval in the multidimensional space $\mathbb{R}^m$. The collection $\mathcal{I}(E)$ is the family of compact subintervals I of E.

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The Lebesgue measure of the interval I is denoted by $|I|$. A *Kurzweil-Henstock partial division* $\Delta = \{(x, I)\}$ of E is any *finite* family of point-interval pairs with x a vertex of I, $I \in \mathcal{I}(E)$ and the intervals I have disjoint interiors. Unless otherwise stated, a partial division in this paper always refers to a Kurzweil-Henstock partial division. Let T be a subset of E and $\Delta = \{(x, I)\}$ a Kurzweil-Henstock partial division. If $x \in T$ for any $(x, I) \in \Delta$, then Δ is said to be T-*tagged* or tagged in T. A partial division $\Delta = \{(x, I)\}$ of E is called a *division* of E if the union of the intervals I in Δ is equal to E. Given a function $\delta : E \to (0,1)$, a partial division Δ of E is said to be δ-*fine* if for every $(x, I) \in \Delta$, the interval I is contained in the open ball $B(x; \delta(x))$ centered at x and of radius $\delta(x)$. A real-valued point function f is *Kurzweil-Henstock integrable to a number* A on E if for every $\varepsilon > 0$ there is a function $\delta : E \to (0,1)$, also called a *gauge* on E, such that

$$\left|(\Delta) \sum f(x)|I| - A\right| < \varepsilon$$

whenever Δ is a δ-fine division of E. The number A, which is *unique*, is called the *Kurzweil-Henstock integral* of f on E. In symbols, $A = (KH)\int_E f$. If f is integrable on E, then f is also integrable on any $I \in \mathcal{I}(E)$ and for any adjacent subintervals $I_1, I_2 \in \mathcal{I}(E)$ with disjoint interiors, $(KH)\int_{I_1 \cup I_2} f = (KH)\int_{I_1} f + (KH)\int_{I_2} f$. This property is called *additivity*. Hence, an interval function F on $\mathcal{I}(E)$ can be defined with $F(I) = (KH)\int_I f$, $I \in \mathcal{I}(E)$. The function F is called the *primitive* and it possesses the additivity property. Hence, F may be described as *additive*. In other words, an additive function of interval F is a primitive if there is some f such that $(KH)\int_E f = F(E)$.

The following lemma is also known as *Henstock's Lemma*. Its proof can be found, for example, in [4, p. 144] or [5, p. 12]. The converse is certainly true and is useful in the proofs of Theorem 2.2 and Theorem 2.3 below.

Lemma 2.1. *If a real-valued function f is Kurzweil-Henstock integrable with primitive F, then for every $\varepsilon > 0$ there is a gauge δ on E such that*

$$\left|(\Delta) \sum f(x)|I| - (\Delta) \sum F(I)\right| < \varepsilon$$

and

$$(\Delta) \sum |f(x)|I| - F(I)| < 2\varepsilon$$

whenever Δ is a δ-fine division of E.

Theorem 2.2 and Theorem 2.3 were proved in [1, Thms 1 and 2 resp.].

Theorem 2.2. *Let f and F be such that f is a real-valued point function defined on E and F is a real-valued function of interval defined on $\mathcal{I}(E)$. Then f is Kurzweil-Henstock integrable on E and F is its primitive if and only if for every $\varepsilon > 0$ there exists a gauge δ on E such that*

$$(\Delta)\sum |f(x)|I|| < \varepsilon \text{ and } (\Delta)\sum |F(I)| < \varepsilon$$

whenever Δ is a δ-fine partial division in Γ_ε^H where

$$\Gamma_\varepsilon^H = \{(x, I) : x \text{ is a vertex of } I \text{ and } |F(I) - f(x)|I|| \geq \varepsilon |I|\}.$$

Theorem 2.3. *Let f and F be such that f is a real-valued point function defined on E and F is a real-valued function of intervals defined on $\mathcal{I}(E)$. Then f is Kurzweil-Henstock integrable on E and F is its primitive if and only if for every $\varepsilon > 0$ there exists a gauge δ on E such that*

$$(\Delta)\sum |I| < \varepsilon \text{ and } (\Delta)\sum |F(I)| < \varepsilon$$

whenever Δ is a δ-fine partial division in Γ_ε^H.

3 The Kurzweil-Henstock Primitive

A consequence of the Lebesgue Differentiation Theorem [3, p. 98] is the following. *Suppose f is Lebesgue integrable on E. Then for almost every $x \in E$, we have $\lim_{|I_c|\longrightarrow 0} \frac{1}{|I_c|}\int_{I_c} f = f(x)$ where I_c is a compact cubic subinterval of E and x is a vertex of I_c.* (Integration above is in the Lebesgue sense.)

We adopt the convention in [4, p. 57] that

$$D_c F(x) = \begin{cases} \lim\limits_{|I_c|\to 0} \frac{F(I_c)}{|I_c|} & \text{if the limit exists,} \\ 0 & \text{if the limit does not exist.} \end{cases}$$

Then, by the Lebesgue Differentiation Theorem, we have $D_c F(x) = f(x)$ almost everywhere and $(L)\int_E D_c F = (L)\int_E f$. The proof [3, p. 98], [2, pp. 39–40] of the Lebesgue Differentiation Theorem makes use of the fact that a Lebesgue integrable function is also absolutely integrable. Absolute integrability does not hold for the Kurzweil-Henstock integral. And so the question remains, whether we can define a corresponding point function $D_c F$ for a Kurzweil-Henstock primitive F.

Suppose f is Kurzweil-Henstock integrable on E. Then for any compact subinterval I of E, $F(I) = (KH)\int_I f$. Let

$$\Gamma_\varepsilon^H = \{(x, I) : x \text{ is a vertex of } I \text{ and } |F(I) - f(x)|I|| \geq \varepsilon |I|\}.$$

For the cubic subintervals I_c, we can define

$$\Gamma_{\frac{1}{k}}^{H,c} = \left\{(x, I_c) \in \Gamma_{\frac{1}{k}}^{H}\right\}$$

and

$$\Gamma_{\frac{1}{k},\frac{1}{j}}^{H,c} = \left\{(x, I_c) \in \Gamma_{\frac{1}{k}}^{H,c} : I_c \subset B\left(x; \frac{1}{j}\right)\right\}.$$

Also let

$$X_{\frac{1}{k},\frac{1}{j}}^{c} = \left\{x \in E : \text{there exists } I_c \text{ with } (x, I_c) \in \Gamma_{\frac{1}{k},\frac{1}{j}}^{H,c}\right\}.$$

Lemma 3.1. *Given a point function f on E and an interval function F on $\mathcal{I}(E)$, the set of all points in E where either $\lim\limits_{|I_c|\longrightarrow 0} \frac{F(I_c)}{|I_c|}$ exists but not equal to $f(x)$ or $\lim\limits_{|I_c|\longrightarrow 0} \frac{F(I_c)}{|I_c|}$ does not exist at all is given by $\bigcup\limits_{k=1}^{\infty} \bigcap\limits_{j=1}^{\infty} X_{\frac{1}{k},\frac{1}{j}}^{c}$.*

PROOF. We are looking at two possible situations: either (1) $\lim\limits_{|I_c|\longrightarrow 0} \frac{F(I_c)}{|I_c|}$ exists but is not equal to $f(x)$ or (2) $\lim\limits_{|I_c|\longrightarrow 0} \frac{F(I_c)}{|I_c|}$ does not exist. In either of the two possible situations, there exists $k \in \mathbb{N}$ such that for every $j \in \mathbb{N}$ there exists $I_c \subset B\left(x; \frac{1}{j}\right)$ and $|F(I_c) - f(x)|I_c|| \geq \frac{|I_c|}{k}$. Hence $x \in \bigcup\limits_{k=1}^{\infty} \bigcap\limits_{j=1}^{\infty} X_{\frac{1}{k},\frac{1}{j}}^{c}$.

On the other hand, suppose $x \in \bigcup\limits_{k=1}^{\infty} \bigcap\limits_{j=1}^{\infty} X_{\frac{1}{k},\frac{1}{j}}^{c}$. Then there exists a $k \in \mathbb{N}$ such that for every $j \in \mathbb{N}$, $x \in X_{\frac{1}{k},\frac{1}{j}}^{c}$. That is, for every $j \in \mathbb{N}$, there exists I_c such that $I_c \subset B\left(x; \frac{1}{j}\right)$ and $|F(I_c) - f(x)|I_c|| \geq \frac{|I_c|}{k}$. Therefore, either either (1) $\lim\limits_{|I_c|\longrightarrow 0} \frac{F(I_c)}{|I_c|}$ exists but is not equal to $f(x)$ or (2) $\lim\limits_{|I_c|\longrightarrow 0} \frac{F(I_c)}{|I_c|}$ does not exist at all. The proof is complete. □

In the next lemma and thereafter, let $X_{\frac{1}{k}}^{c} = \bigcap\limits_{j=1}^{\infty} X_{\frac{1}{k},\frac{1}{j}}^{c}$.

Lemma 3.2. *Suppose f is Kurzweil-Henstock integrable with primitive F and $k \in \mathbb{N}$. Then for every $\varepsilon > 0$ there exists a gauge δ on E such that*

$$(\Delta_c) \sum |I_c| < \varepsilon$$

whenever $\Delta_c = \{(x, I_c)\}$ is a δ-fine, $X_{\frac{1}{k}}^{c}$-tagged partial division of E in $\Gamma_{\frac{1}{k}}^{H,c}$.

PROOF. If f is Kurzweil-Henstock integrable with primitive F, then by Theorem 2.3, for every $n \in \mathbb{N}$ there exists a gauge δ_n on E such that

$$(\Delta) \sum |I| < \frac{1}{n}$$

whenever Δ is a δ_n-fine partial division of E in $\Gamma^H_{\frac{1}{n}}$. Now fix a natural number k. For every $\varepsilon > 0$ there exists $n(\varepsilon) \in \mathbb{N}$, $n(\varepsilon) \geq k$ such that $\frac{1}{n(\varepsilon)} < \varepsilon$. We can therefore choose a gauge δ on E such that $\delta(x) = \delta_{n(\varepsilon)}(x)$. Then for every δ-fine partial division $\Delta = \{(x, I)\}$ of E in $\Gamma^H_{\frac{1}{k}} \subset \Gamma^H_{\frac{1}{n(\varepsilon)}}$ we have

$$(\Delta) \sum |I| < \frac{1}{n(\varepsilon)} < \varepsilon.$$

The desired result follows easily; that is, for every δ-fine, $X^c_{\frac{1}{k}}$-tagged partial division $\Delta_c = \{(x, I_c)\}$ of E in $\Gamma^{H,c}_{\frac{1}{k}}$ we have $(\Delta_c) \sum |I_c| < \varepsilon$. $\square$

Lemma 3.3. *If f is Kurzweil-Henstock integrable with primitive F, then for each $k \in \mathbb{N}$, the set $X^c_{\frac{1}{k}}$ is of Lebesgue measure zero.*

PROOF. Given $k \in \mathbb{N}$, then by Lemma 3.2, for every $\varepsilon > 0$ there exists a gauge δ on E such that $(\Delta_c) \sum |I_c| < \varepsilon$ whenever $\Delta_c = \{(x, I_c)\}$ is a δ-fine, $X^c_{\frac{1}{k}}$-tagged partial division of E in $\Gamma^{H,c}_{\frac{1}{k}}$. Let

$$\Gamma^{H,c}_{\frac{1}{k}}\left(\delta, X^c_{\frac{1}{k}}\right) = \left\{(x, I_c) \in \Gamma^{H,c}_{\frac{1}{k}} : x \in X^c_{\frac{1}{k}} \text{ and } I_c \subset B(x; \delta(x))\right\}$$

The collection $\Gamma^{H,c}_{\frac{1}{k}}\left(\delta, X^c_{\frac{1}{k}}\right)$ is a Vitali cover for $X^c_{\frac{1}{k}}$. By the Vitali Covering Theorem [3, p. 110], there exists a finite collection of point-interval pairs $(x_i, I_{c,i}) \in \Gamma^{H,c}_{\frac{1}{k}}\left(\delta, X^c_{\frac{1}{k}}\right)$ such that the intervals $I_{c,i}$ are pairwise disjoint and $\left|X^c_{\frac{1}{k}} \setminus \bigcup_i I_{c,i}\right| < \varepsilon$. Therefore $\left|X^c_{\frac{1}{k}}\right| \leq \left|X^c_{\frac{1}{k}} \setminus \bigcup_i I_{c,i}\right| + \sum_i \left|I_{c,i}\right| < 2\varepsilon$. $\square$

The next lemma is a direct consequence of the previous one. Hence, the proof has been omitted.

Lemma 3.4. *If f is a Kurzweil-Henstock integrable function with primitive F, then $\lim_{|I_c| \to 0} \frac{F(I_c)}{|I_c|} = f(x)$ almost everywhere. (The interval I_c contains x as a vertex.)*

We can now answer the question of whether it is possible to define a function D_cF for a Kurzweil-Henstock primitive such that D_cF is Kurzweil-Henstock integrable on E and $(KH)\int_E D_cF = F(E)$. If F is a Kurzweil-Henstock primitive of some f, then as for the Lebesgue integral, we have $D_cF(x) = f(x)$ almost everywhere. So $(KH)\int_E D_cF = (KH)\int_E f$. Recall that $D_cF(x)$ is zero where the limit does not exist as defined earlier in this section.

Theorem 3.5. *An additive interval function F is a Kurzweil-Henstock primitive if and only if for every $\varepsilon > 0$ there exists a gauge δ on E such that*

$$(\Delta)\sum |(D_cF(x))\,|I|| < \varepsilon \textit{ and } (\Delta)\sum |F(I)| < \varepsilon$$

whenever Δ is a δ-fine partial division in Ω_ε^H where

$$\Omega_\varepsilon^H = \{(x,I) : x \textit{ is a vertex of } I \textit{ and } |F(I) - D_cF(x)\,|I|| \geq \varepsilon\,|I|\}.$$

PROOF. ($\Longrightarrow$) Suppose F is a Kurzweil-Henstock primitive, then there must be some f such that $(KH)\int_E f = F(E)$ and $D_cF(x) = f(x)$ almost everywhere. Hence, by Theorem 2.2, for every $\varepsilon > 0$ there exists a gauge δ on E such that

$$(\Delta)\sum |(D_cF(x))\,|I|| < \varepsilon \text{ and } (\Delta)\sum |F(I)| < \varepsilon$$

whenever Δ is a δ-fine partial division in Ω_ε^H. ($\Longrightarrow$) This part follows easily since we can let $f(x) = D_cF(x)$ itself. $\square$

Corollary 3.6. *An additive interval function F is a Kurzweil-Henstock primitive if and only if for every $\varepsilon > 0$ there exists a gauge δ on E such that*

$$(\Delta)\sum |I| < \varepsilon \textit{ and } (\Delta)\sum |F(I)| < \varepsilon$$

whenever Δ is a δ-fine partial division in Ω_ε^H.

In the next theorem,

$$IV\left(\Omega_{\frac{1}{k}}^H\right) = \inf_{\text{all } \delta} IV\left(\Omega_{\frac{1}{k}}^H, \delta\right) \text{ and } IV\left(F, \Omega_{\frac{1}{k}}^H\right) = \inf_{\text{all } \delta} IV\left(\Omega_{\frac{1}{k}}^H, \delta\right)$$

where

$$IV\left(\Omega_{\frac{1}{k}}^H, \delta\right) = \sup\left\{(\Delta)\sum |I| : \Delta \text{ a } \delta\text{-fine partial division of } E \text{ in } \Omega_{\frac{1}{k}}^H\right\}$$

and

$$IV\left(F, \Omega_{\frac{1}{k}}^H, \delta\right) = \sup\left\{(\Delta)\sum |F(I)| : \Delta \text{ a } \delta\text{-fine partial division of } E \text{ in } \Omega_{\frac{1}{k}}^H\right\}.$$

Theorem 3.7. *Let F be an additive interval function on $\mathcal{I}(E)$ and*

$$\Omega_{\varepsilon}^{H} = \{(x, I) : x \text{ is a vertex of } I \text{ and } |F(I) - (D_c F(x))|I|| \geq \varepsilon |I|\}.$$

Then the following are equivalent:
(1) for every $\varepsilon > 0$ there exists a gauge δ on E such that

$$(\Delta) \sum |I| < \varepsilon \text{ and } (\Delta) \sum |F(I)| < \varepsilon$$

whenever Δ is a δ-fine partial division of E in Ω_{ε}^{H}.
(2) for every $k \in \mathbb{N}$, $IV\left(\Omega_{\frac{1}{k}}^{H}\right) = 0$ and $IV\left(F, \Omega_{\frac{1}{k}}^{H}\right) = 0$.

PROOF. Suppose (1) holds. Then it follows that for every $k \in \mathbb{N}$ there exists a gauge δ_k on E such that $(\Delta) \sum |I| < \frac{1}{k}$ and $(\Delta) \sum |F(I)| < \frac{1}{k}$ whenever Δ is a δ_k-fine partial division of E in $\Omega_{\frac{1}{k}}^{H}$. Following the steps in the proof of Lemma 3.2, we arrive at (2).

Now suppose (2) holds. We fix $k \in \mathbb{N}$. Then (2) is the same as saying that for every $\varepsilon > 0$ there exists a gauge δ on E such that $(\Delta) \sum |I| < \varepsilon$ and $(\Delta) \sum |F(I)| < \varepsilon$ whenever Δ is a δ-fine partial division of E in $\Omega_{\frac{1}{k}}^{H}$. We arrive at (1) by letting $\varepsilon = \frac{1}{k}$ and then using the nested and increasing property of the sets $\Omega_{\frac{1}{k}}^{H}$. □

Corollary 3.8. *An additive interval function F is a Kurzweil-Henstock primitive if and only if for every $k \in \mathbb{N}$, $IV\left(\Omega_{\frac{1}{k}}^{H}\right) = 0$ and $IV\left(F, \Omega_{\frac{1}{k}}^{H}\right) = 0$.*

4 The Absolute Version

A function f is McShane integrable on E with primitive F if for every $\varepsilon > 0$ there is a gauge δ on E such that $|(\Delta) \sum f(x)|I| - A| < \varepsilon$ whenever Δ is a δ-fine McShane division of E. Here, a McShane partial division $\Delta = \{(x, I)\}$ of E is one where the tags x may not be contained in I.

Corresponding versions of Theorem 3.5, Corollary 3.6 and Corollary 3.8 can be formulated in exactly the same way for the McShane primitive except that Ω_{ε}^{H} is replaced with Ω_{ε}^{M} where

$$\Omega_{\varepsilon}^{M} = \{(x, I) : x \text{ is a vertex of } I \text{ or } x \notin I, \text{ and } |F(I) - (D_c F(x))|I|| \geq \varepsilon |I|\}.$$

The proofs follow the same line of reasoning with the Kurzweil-Henstock partial divisions changed to McShane partial divisions. For instance, we have the following theorem which corresponds to Corollary 3.8.

Theorem 4.1. *An additive interval function F is a McShane primitive if and only if for every $k \in \mathbb{N}$, $IV\left(\Omega_{\frac{1}{k}}^{M}\right) = 0$ and $IV\left(F, \Omega_{\frac{1}{k}}^{M}\right) = 0$.*

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