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COMPARING FAMILIES OF THIN SETS

Abstract

Recently the first author investigated a generalization of families of trigonometric thin sets replacing the sine function by a continuous function. In this paper we shall partially solve the problem of the relationship between such families obtained from different functions. In several cases we present conditions for equality of such a family with corresponding trigonometric family. Moreover we show that every basis of any of the families of B_0–sets, N_0–sets or A–sets has cardinality at least that of the continuum.

1 Families of Thin Sets

In [BZ] the first author introduced and studied a natural generalization of trigonometric thin sets replacing sine function by a sequence $\boldsymbol{f} = \{f_k\}_{k=0}^{\infty}$ of continuous functions defined on the unit circle $\mathbb{T}$ with non-negative reals values. We shall deal with the special case when the sequence $\{f_k\}_{k=0}^{\infty}$ is generated by a continuous function f; i.e., when $f_k(x) = f(kx)$.

We work with the topological group the unit circle $\mathbb{T}$. We may identify $\mathbb{T}$ with the interval $\langle -1/2, 1/2\rangle$ identifying $-1/2$ and $1/2$ with the operation of addition mod 1. If $f : \mathbb{T} \longrightarrow \mathbb{R}$ is a real-valued function, we denote the *zero set* of f by

$$\mathrm{Z}(f) = \{x \in \mathbb{T}; f(x) = 0\}.$$

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Throughout the paper, $f, g : \mathbb{T} \longrightarrow \langle 0, +\infty)$ are continuous functions with $f(0) = g(0) = 0$, and $f(x) > 0$, $g(y) > 0$ for some $x, y \in \mathbb{T}$; i.e.,

$$0 \in \mathrm{Z}(f) \neq \mathbb{T},\ 0 \in \mathrm{Z}(g) \neq \mathbb{T}.$$

We can assume that the functions f, g are periodically extended to the whole set $\mathbb{R}$ with the period 1. Since $\mathbb{T}$ is compact, the functions f, g are uniformly continuous and therefore there exists a non-increasing sequence $\{\delta_k\}_{k=0}^{\infty}$ of positive reals (fixed for the remainder of the paper) converging to 0, $\delta_0 \leq 1/2$ such that

$$(\forall x, y)\ \big(|x-y| < \delta_k \to \big(|f(x) - f(y)| < 2^{-k} \wedge |g(x) - g(y)| < 2^{-k}\big)\big). \quad (1)$$

Let us recall that a sequence $\{f_n\}_{n=0}^{\infty}$ of real-valued functions is said to *converge quasinormally*[1] to a function f on the set X, written $f_n \xrightarrow{QN} f$ on X, if there exists a sequence (*a control*) of positive reals $\{\varepsilon_n\}_{n=0}^{\infty}$ converging to zero such that

$$(\forall x \in X)(\exists n_0)(\forall n \geq n_0)\,(|f_n(x) - f(x)| < \varepsilon_n).$$

Let us remark that much as in the case of uniform convergence, if $\{\eta_k\}_{k=0}^{\infty}$ is a sequence of positive reals converging to 0 and $f_n \xrightarrow{QN} f$ on X, then one can find an increasing sequence of natural numbers $\{n_k\}_{k=0}^{\infty}$ such that $f_{n_k} \xrightarrow{QN} f$ with the control $\{\eta_k\}_{k=0}^{\infty}$. We shall use this fact without any comment.

To avoid subindices and subsubindices, we shall sometimes denote the n-th element a_n of the sequence $\{a_n\}_{n=0}^{\infty}$ by $a(n)$. We shall similarly do so for other sequences.

We recall the definitions of thin sets introduced in [BZ]. A subset A of $\mathbb{T}$ is called an *f-Dirichlet set* (briefly D_f-*set*), a *pseudo f-Dirichlet set* (briefly pD_f-*set*), an A_f-*set* if there exists an increasing sequence of natural numbers $\{n_k\}_{k=0}^{\infty}$ such that the sequence $\{f(n_k x)\}_{k=0}^{\infty}$ converges uniformly, quasinormally, pointwise to 0 on the set A, respectively. A subset A of $\mathbb{T}$ is called an N_{0f}-*set* (a B_{0f}-*set*) if there exists an increasing sequence of natural numbers $\{n_k\}_{k=0}^{\infty}$ (and a positive real d) such that the series $\sum_{k=0}^{\infty} f(n_k x)$ converges ($\sum_{k=0}^{\infty} f(n_k x) \leq d$) for every $x \in A$. A subset A of $\mathbb{T}$ is called an N_f-*set* (a B_f-*set*) if there exists a sequence $\{a_n\}_{n=0}^{\infty}$ of non-negative reals (and a positive real d) such that $\sum_{n=0}^{\infty} a_n = \infty$ and the series $\sum_{n=0}^{\infty} a_n f(nx)$ converges ($\sum_{n=0}^{\infty} a_n f(nx) \leq d$) for every $x \in A$. Finally, a subset A of $\mathbb{T}$ is called a *weak f-Dirichlet set* (briefly wD_f-*set*) if there exists an analytic set B, $A \subseteq B \subseteq \mathbb{T}$

[1] In [CL], the authors call this type of convergence equal convergence.

such that for every positive Borel measure μ on $\mathbb{T}$ there exists an increasing sequence of natural numbers $\{n_k\}_{k=0}^{\infty}$ such that

$$\lim_{k\to\infty} \int_B f(n_k x)\, d\mu(x) = 0.$$

The corresponding families will be denoted by $\mathcal{D}_f$, $p\mathcal{D}_f$, $\mathcal{A}_f$, $\mathcal{N}_{0f}$, $\mathcal{B}_{0f}$, $\mathcal{N}_f$, $\mathcal{B}_f$, and $w\mathcal{D}_f$, respectively. If $f(x) = \|x\|$ ($\|x\|$ is the distance of the real x to the nearest integer[2]) or equivalently $f(x) = |\sin(\pi x)|$, then we obtain the classical *trigonometric families* $\mathcal{D}$, $p\mathcal{D}$, $\mathcal{A}$, $\mathcal{N}_0$, $\mathcal{B}_0$, $\mathcal{N}$, $\mathcal{B}$, and $w\mathcal{D}$.

Recall (see [BL]) that a family $\mathcal{F} \subseteq \mathcal{P}(\mathbb{T})$ is said to be *a family of thin sets*, if the family $\mathcal{F}$ contains every singleton $\{x\}$, $x \in \mathbb{T}$, with every set $A \in \mathcal{F}$ the family $\mathcal{F}$ contains also every subset $B \subseteq A$, and $\mathcal{F}$ does not contain any non-trivial open interval. Moreover if for any A, $B \in \mathcal{F}$ also $A \cup B \in \mathcal{F}$, then $\mathcal{F}$ is called *an ideal*. It is well known (see [Ma]) that none of the trigonometric families is an ideal. A family $\mathcal{G} \subseteq \mathcal{F}$ is called *a basis of* $\mathcal{F}$ if for any $A \in \mathcal{F}$ there is a set $B \in \mathcal{G}$ such that $A \subseteq B$. A Borel basis is a basis consisting of Borel sets.

In [BZ] (Corollary 13) the following result has been proved.

Theorem 1. *Every family $\mathcal{D}_f$, $p\mathcal{D}_f$, $\mathcal{B}_{0f}$, $\mathcal{N}_{0f}$, $\mathcal{B}_f$, $\mathcal{N}_f$, $\mathcal{A}_f$, $w\mathcal{D}_f$ is a family of thin sets with a Borel basis and the following inclusions hold true (an arrow '$\rightarrow$' means the inclusion '$\subseteq$')*

$$\begin{tikzcd}
 & \mathcal{A}_f \arrow[r] & w\mathcal{D}_f \\
p\mathcal{D}_f \arrow[r] & \mathcal{N}_{0f} \arrow[r] \arrow[u] & \mathcal{N}_f \arrow[u] \\
\mathcal{D}_f \arrow[r] \arrow[u] & \mathcal{B}_{0f} \arrow[r] \arrow[u] & \mathcal{B}_f \arrow[u]
\end{tikzcd}$$

Moreover every A_f–set is σ–porous and therefore meager and of measure zero.

The following theorem follows almost immediately from the definitions.

Theorem 2. *Assume that there are positive reals $K > 0$, $\eta > 0$ such that*

$$(\forall x \in \mathbb{T})\,(f(x) < \eta \rightarrow g(x) < K \cdot f(x)).$$

Then $\mathcal{F}_f \subseteq \mathcal{F}_g$ if $\mathcal{F}$ is any of the symbols $\mathcal{D}$, $p\mathcal{D}$, $\mathcal{N}_0$, $\mathcal{B}_0$, $\mathcal{N}$, $\mathcal{B}$, $\mathcal{A}$, $w\mathcal{D}$.

[2] Let us remark that $\|x\| = |x|$ for $x \in \langle -1/2, 1/2\rangle$. Moreover, $\|x + y\| \le \|x\| + \|y\|$, for any $x, y \in \mathbb{R}$.

PROOF. The proof is rather standard. We sketch it only for the two cases $\mathcal{F} = \mathcal{B}$ and $w\mathcal{D}$. We set $d = \max\{g(x); x \in \mathbb{T}\}$. Suppose that

$$A = \{x \in \mathbb{T}; \sum_{n=0}^{\infty} a_n f(nx) \leq c\} \in \mathcal{B}_f,$$

where $\sum_{n=0}^{\infty} a_n = \infty$ and $a_n \geq 0$ for each $n \in \mathbb{N}$. Let $x \in A$. We set $L = \{n \in \mathbb{N}; f(nx) \geq \eta\}$. Then $\eta \cdot \sum_{n \in L} a_n \leq c$. Thus

$$\sum_{n=0}^{\infty} a_n g(nx) = \sum_{n \in L} a_n g(nx) + \sum_{n \in \mathbb{N} \setminus L} a_n g(nx) \leq d \cdot \frac{c}{\eta} + K \cdot c.$$

Hence

$$A \subseteq \{x \in \mathbb{T}; \sum_{n=0}^{\infty} a_n g(nx) \leq \left(\frac{d}{\eta} + K\right) \cdot c\} \in \mathcal{B}_g.$$

Now we assume that $A \in w\mathcal{D}_f$ is analytic, μ is a Borel measure on $\mathbb{T}$ and $\lim_{k \to \infty} \int_A f(n_k x)\, d\mu(x) = 0$. For given $\varepsilon > 0$ and $k \in \mathbb{N}$ we set

$$\varepsilon_1 = \frac{\varepsilon \eta}{K\eta + d}, \quad B_k = \{x \in \mathbb{T}; f(n_k x) < \eta\}.$$

Let k_0 be such that $\int_A f(n_k x)\, d\mu(x) < \varepsilon_1$ for $k \geq k_0$. Then $\mu(A \setminus B_k) < \varepsilon_1/\eta$ and

$$\int_A g(n_k x)\, d\mu(x) = \int_{A \setminus B_k} g(n_k x)\, d\mu(x) + \int_{B_k} g(n_k x)\, d\mu(x) < d\frac{\varepsilon_1}{\eta} + K\varepsilon_1 = \varepsilon.$$

Thus $A \in w\mathcal{D}_g$. □

Corollary 3. *If $M > 0$ is a positive real, then $\mathcal{F}_f = \mathcal{F}_{M \cdot f}$ if $\mathcal{F}$ is any of the symbols $\mathcal{D}$, $p\mathcal{D}$, $\mathcal{N}_0$, $\mathcal{B}_0$, $\mathcal{N}$, $\mathcal{B}$, $\mathcal{A}$, $w\mathcal{D}$.*

Using a more complicated argument we can prove a stronger result.

Theorem 4.

a) *If $\mathrm{Z}(f)$ is a g-Dirichlet set, then $\mathcal{D}_f \subseteq \mathcal{D}_g$.*

b) *If $\mathrm{Z}(f)$ is a pseudo g-Dirichlet set, then $p\mathcal{D}_f \subseteq p\mathcal{D}_g$.*

PROOF. Assume that the set $\mathrm{Z}(f)$ is a g-Dirichlet (pseudo g-Dirichlet) set, the sequence $\{n_k\}_{k=0}^{\infty}$ being such that $g(n_k x) \leq 2^{-k}$ for every $x \in \mathrm{Z}(f)$ and for every (for almost every) k. Let $A \in \mathcal{D}_f$ $(A \in p\mathcal{D}_f)$, $\{m_k\}_{k=0}^\infty$ being such that $f(m_k x) \rightrightarrows 0$ $(f(m_k x) \xrightarrow{QN} 0)$ on A with the control 2^{-k}. We set

$$\eta_k = \min\{f(x); (\forall z \in \mathrm{Z}(f))\, |x - z| \geq \delta_k/n_k\}.$$

Since the set $\mathrm{Z}(f)$ is closed, $\mathrm{Z}(f) \neq \mathbb{T}$ and $\lim_{k\to\infty} \delta_k/n_k = 0$, we can assume that η_k is defined and positive for every k. Moreover, $\lim_{k\to\infty} \eta_k = 0$. For every k let l_k be such that $2^{-l(k)} < \eta_k$.

If $x \in A$, then $f(m_{l(k)}x) \leq 2^{-l(k)-1} < \eta_k$ for every k (for almost every k) and $|m_{l(k)}x - z| < \delta_l/n_k$ for some $z \in Z$. Then $|n_k m_{l(k)}x - n_k z| < \delta_k$ and $|g(n_k m_{l(k)}x) - g(n_k z)| < 2^{-k}$ for every k (for almost every k). Thus $g(n_k m_{l(k)}x) < 2^{-k+1}$ for every k (for almost every k). So $g(n_k m_{l(k)}x) \rightrightarrows 0$ $(g(n_k m_{l(k)})x \xrightarrow{QN} 0)$ on A with the control $\{2^{-k+1}\}_{k=0}^\infty$. $\square$

The main aim of the paper is to investigate when those inclusions are proper and when the equalities hold true. We shall present first results in this direction. The main technical result of the paper, Theorem 5, is then applied to the problem of cardinalities of bases of some families.

The paper is organized as follows. The main result is Theorem 5, which is a generalization of the key lemma by J. Arbault [Ar] and which plays crucial role in proving several results. In section 3 we shall compare the families $\mathcal{F}_f$ with corresponding trigonometric families $\mathcal{F}$. Then, in section 4, we partially solve the question of properness of the inclusions for $\mathcal{B}_0$, $\mathcal{N}_0$ and $\mathcal{A}$ families. Using some infinite combinatorics we estimate the cardinality of bases and towers of families $\mathcal{B}_{0f}$, $\mathcal{N}_{0f}$ and $\mathcal{A}$ from below, solving so a problem, which was open even in the trigonometric case (section 5). Finally, in section 6 we shall apply results of sections 3, 4 and 5 and formulate main open problems.

2 Arbault Lemma

J. Arbault [Ar] has shown that the set

$$\{x \in \mathbb{T}; \sum_{k=0}^{\infty} (\sin(2^{2^k}\pi x))^2 < \infty\}$$

is not an N_0-set. Modifying his proof we can show several important results. Let $\{p_k\}_k^\infty$ be an increasing sequence of integers greater than 1 such that $3/p_k < \delta_k$ for every $k \in \mathbb{N}$. We let $q_k = p_0 \cdot \cdots \cdot p_k$. Starting from the Cantor expansion of a real $x \in \langle 0, 1\rangle$ with natural numbers $y_k = 0, 1, \ldots, p_k - 1$, $k \geq 0$

$$x = \sum_{k=0}^{\infty} \frac{y_k}{p_0 \cdot \cdots \cdot p_k} = \sum_{k=0}^{\infty} \frac{y_k}{q_k}$$

one can easily construct integers x_k, $k \in \mathbb{N}$ such that

$$x = \sum_{k=0}^{\infty} \frac{x_k}{p_0 \cdot \dots \cdot p_k}, \quad |x_k| \leq \frac{p_k}{2} \text{ for } k > 0, \quad x_0 = 0, \dots, p_0.$$

One can easily see that

$$q_n x = \frac{x_{n+1}}{p_{n+1}} + \theta_n \mod 1, \quad |\theta_n| \leq 1/p_{n+1} \tag{2}$$

and therefore

$$\frac{|x_{n+1}| - 1}{p_{n+1}} \leq \|q_n x\| \leq \frac{|x_{n+1}| + 1}{p_{n+1}}.$$

More generally, if $m \geq n+1$ and $x_i = 0$ for $n+2 \leq i \leq m$, then

$$q_n x = \frac{x_{n+1}}{p_{n+1}} + \theta_n \mod 1, \quad |\theta_n| \leq \frac{q_n}{q_m} \leq \frac{1}{p_m}. \tag{3}$$

Let us remark that J. Arbault [Ar] worked with $p_k = 2^{2^k}$.

If $\{n_k\}_{k=0}^{\infty}$ is an increasing sequence of natural numbers, we put

$$\mathbf{A}(\{n_k\}_{k=0}^{\infty}) = \{x \in \mathbb{T}; \lim_{k \to \infty} \|n_k x\| = 0\}.$$

If $\{k_i\}_{i=0}^{\infty}$ is an increasing sequence, then $\mathbf{A}(\{n_k\}_{k=0}^{\infty}) \subseteq \mathbf{A}(\{n_{k_i}\}_{i=0}^{\infty})$. In other words, if $\lim_{k \to \infty} \|n_k x\| = 0$ for every $x \in A$, then that is also true for any subsequence of the sequence $\{n_k\}_{k=0}^{\infty}$. Similarly for other considered properties. Therefore, for the sake of brevity, in the next a (chosen) subsequence of a given sequence will be often denoted by the same letters and indices.

We begin with a result that is the promised strengthening of the key lemma by J. Arbault [Ar].

Theorem 5. *Let $\{m_k\}_{k=0}^{\infty}$ be any increasing sequence of natural numbers such that*

$$B_0 = \{x \in \mathbb{T}; \sum_{k=0}^{\infty} f(q_k x) \leq 4\} \subseteq \mathbf{A}(\{m_k\}_{k=0}^{\infty}). \tag{4}$$

Then there are sequences $\{l_k\}_{k=0}^{\infty}$, $\{s_k\}_{k=0}^{\infty}$ of non-negative integers and an integer $r \neq 0$ such that the sequence $\{(s_k p_{l(k)+1} + r) q_{l(k)}\}_{k=0}^{\infty}$ is a subsequence of the sequence $\{m_k\}_{k=0}^{\infty}$. Moreover, by passing to a subsequence of the sequence $\{m_k\}_{k=0}^{\infty}$, we may assume that

(a) $l_k + l_0 \leq l_{k+1}$ *for every k, $l_0 > 1$,*

(b) $m_k \cdot p_{l(k)+1} \leq p_{l(k+1)} \cdot q_{l(k)}$ *for every* k,

(c) $(1+r)p_{l(k)} \leq p_{l(k+1)+1}$ *for every* k.

Proof. If $k \geq l$, then q_k/q_l is an integer and $f(q_k/q_l) = 0$. For $k < l$ we have $q_k/q_l \leq 1/p_l < \delta_l$. Thus

$$\sum_{k=0}^{\infty} f(q_k/q_l) = \sum_{k=0}^{l-1} f(q_k/q_l) \leq l\frac{1}{2^l} \leq 4.$$

Therefore $1/q_l \in B_0$ for every l and by (4), q_l divides m_k for all but finitely many $k \in \mathbb{N}$. Omitting finitely many members of the sequence $\{m_k\}_{k=0}^{\infty}$ we can assume that every m_k is divisible by some q_l. Let l_k be the greatest l such that q_l divides m_k; i.e., m_k is divisible by $q_{l(k)}$ and $q_{l(k)+1} = q_{l(k)} \cdot p_{l(k)+1}$ does not divide m_k. Thus, there are integers s_k, r_k such that

$$m_k = (s_k p_{l(k)+1} + r_k) q_{l(k)}, \quad 0 \neq |r_k| \leq \frac{p_{l(k)+1}}{2}.$$

Evidently the sequence $\{l_k\}_{k=0}^{\infty}$ is unbounded and we can pass to such a subsequence of $\{m_k\}_{k=0}^{\infty}$ and accordingly to a sequence of $\{l_k\}_{k=0}^{\infty}$ that even conditions (a) and (b) are satisfied.

Toward a contradiction, assume that the sequence $\{|r_k|\}_{k=0}^{\infty}$ is unbounded. Then (again by passing to subsequences) we can assume that $\dfrac{1}{|4r_k|} \leq \dfrac{1}{p_{k+1}}$. We shall construct a real $z \in B_0$ such that $\{\|m_k z\|\}_{k=0}^{\infty}$ does not converge to 0 contradicting the inclusion (4). Let $z_{l(k)+1} > 0$ be the smallest natural number for which

$$z_{l(k)+1}|r_k|/p_{l(k)+1} \geq 1/4. \tag{5}$$

Evidently $z_{l(k)+1} \leq 1/2p_{l(k)+1}$. Since $(z_{k(l)+1}-1)|r_k|/p_{l(k)+1} < 1/4$, we obtain

$$\frac{z_{l(k)+1}}{p_{l(k)+1}} < \frac{1}{|4r_k|} + \frac{1}{p_{l(k)+1}} \leq \frac{2}{p_{k+1}}.$$

We set $z_i = 0$ for the other indices i and we show that the real $z = \sum_{i=0}^{\infty} z_i/q_i$ provides the excepted contradiction.

By (2), for any k we obtain

$$\|q_{l(k)}z\| \leq \frac{z_{l(k)+1}+1}{p_{l(k)+1}} \leq \frac{3}{p_{k+1}} < \delta_k.$$

If i is not a value of the sequence $\{l_k\}_{k=0}^{\infty}$, then $\|q_i z\| \leq 1/p_{i+1} < \delta_i$. So by (1) we obtain $\sum_{i=0}^{\infty} f(q_i z) \leq 4$ and therefore $z \in B_0$.

On the other hand by (2) we obtain mod 1

$$m_k z = (s_k p_{l(k)+1} + r_k)\left(\frac{z_{l(k)+1}}{p_{l(k)+1}} + \theta_{l(k)}\right) = r_k \frac{z_{l(k)+1}}{p_{l(k)+1}} + \frac{m_k}{q_{l(k)}}\theta_{l(k)}. \tag{6}$$

Since $z_i = 0$ for $l_k + 1 < i \le l_{k+1}$, by (3) we have $\theta_{l(k)} \le \frac{1}{p_{l(k+1)}}$ and therefore by (b) we obtain

$$\left|\frac{m_k}{q_{l(k)}}\theta_{l(k)}\right| \le \frac{m_k}{q_{l(k)}p_{l(k+1)}} \le \frac{1}{p_{l(k)+1}}.$$

So the last term in (6) goes to zero and by inequality (5), for sufficiently large k we have $\|m_k z\| > 1/8$. Consequently, $\lim_{k\to\infty} \|m_k z\| \neq 0$; a contradiction. Since the sequence r_k, $k = 0, 1, \ldots$ is bounded, there exists an integer r such that $r = r_k$ for infinitely many k. So we can choose a subsequence satisfying the assertion of the theorem including the condition (c). □

We shall need the following simple result.

Lemma 6. *Let y_k be real, $k \in \mathbb{N}$, the sequence $\{l_k\}_{k=0}^{\infty}$ being increasing. Then there exists a real z such that $\|q_{l(k)}z - y_k\| \le \frac{2}{p_{l(k)+1}}$ and $\|q_i z\| \le \frac{1}{p_{i+1}}$ if i is not a value of the sequence $\{l_k\}_{k=0}^{\infty}$.*

PROOF. Evidently, for every k there exists an integer $|z_{l(k)+1}| \le 1/2p_{l(k)+1}$ such that $\left\|\frac{z_{l(k)+1}}{p_{l(k)+1}} - y_k\right\| \le \frac{1}{p_{l(k)+1}}$. Set $z_i = 0$ if i is not a value of the sequence $\{l_k + 1\}_{k=0}^{\infty}$. Then $z = \sum_{i=0}^{\infty} \frac{z_i}{q_i}$ is the desired one. □

3 The Families $\mathcal{F}$ and $\mathcal{F}_f$

Using the continuity of the function f one can show that in some cases the trigonometric families are the smallest one. Actually, from the definitions one immediately obtains the following.

Theorem 7. *$\mathcal{D} \subseteq \mathcal{D}_f$, $p\mathcal{D} \subseteq p\mathcal{D}_f$ and $\mathcal{A} \subseteq \mathcal{A}_f$.*

According to Theorem 4 we obtain the next assertion.

Corollary 8. *If $\mathrm{Z}(f)$ is a Dirichlet set (a pseudo Dirichlet set), then $\mathcal{D} = \mathcal{D}_f$ ($p\mathcal{D} = p\mathcal{D}_f$).*

The inverse inclusions need not hold true.

Theorem 9. *There exists a continuous function $f : \mathbb{T} \to \langle 0, 1\rangle$, $f(0) = 0$ such that $\mathcal{D} \neq \mathcal{D}_f$, $p\mathcal{D} \neq p\mathcal{D}_f$ and $\mathcal{A} \neq \mathcal{A}_f$.*

Proof. Let $\mathbf{C} \subseteq \mathbb{T}$ be the Cantor middle-third set; i.e.,

$$\mathbf{C} = \left\{ x \in \mathbb{T}; (\exists \{x_i\}_{i=1}^{\infty},\, x_i = 0, 2)\, x = \sum_{i=1}^{\infty} \frac{x_i}{3^i} \right\}.$$

It is well known that the Cantor set is not an A-set (see e.g. [Ba]). Let $f : \mathbb{T} \to \langle 0, 1 \rangle$ be a continuous function such that $\mathrm{Z}(f) = \mathbf{C}$. Since $3^k x \in \mathbf{C}$ for any $x \in \mathbf{C}$ and any $k \in \mathbb{N}$, we get $\mathbf{C} \subseteq \{x \in \mathbb{T}; f(3^k x) \rightrightarrows 0\}$. Thus $\mathbf{C} \in \mathcal{D}_f$. □

For A-sets we can prove an even better result than Theorem 7.

Theorem 10.

a) *If the zero set* $\mathrm{Z}(f)$ *is a finite set of rationals, then* $\mathcal{A}_f \subseteq \mathcal{A}$.

b) *If* $\mathcal{B}_{0f} \subseteq \mathcal{A}$, *then the zero set* $\mathrm{Z}(f)$ *is a finite set of rationals.*

Proof. Let $\mathrm{Z}(f)$ be a finite set of rationals. Then there exists a natural number m such that mz is an integer for any $z \in \mathrm{Z}(f)$. Assume that

$$A = \{x \in \mathbb{T}; \lim_{k\to\infty} f(n_k x) = 0\} \in \mathcal{A}_f.$$

We claim that $\lim_{k\to\infty} \|mn_k x\| = 0$ for any $x \in A$. Let $x \in A$ and assume that $\lim_{k\to\infty} \|mn_k x\| \neq 0$. Let $\eta > 0$ be such that $\|mn_k x\| \geq \eta$ for infinitely many k's. We can assume that η is such that

$$U = \{y \in \mathbb{T}; (\exists z \in \mathrm{Z}(f))\, |y - z| < \eta/m\} \neq \mathbb{T}.$$

Let $\beta = \min\{f(y); y \in \mathbb{T} \setminus U\}$. Then $f(n_k x) \geq \beta$ for infinitely many k's; a contradiction.

Now assume that $\mathrm{Z}(f)$ is not a finite set of rationals. Then for any integer m there exists a real $z \in \mathrm{Z}(f)$ such that mz is not an integer. We show that the B_{0f}-set $B_1 = \{x \in \mathbb{T}; \sum_{k=0}^{\infty} f(q_k x) \leq 5\}$ is not an A-set. Toward a contradiction, assume there exists an increasing sequence $\{m_k\}_{k=0}^{\infty}$ of natural numbers such that $B_1 \subseteq \mathbf{A}(\{m_k\}_{k=0}^{\infty})$. By Theorem 5 we may suppose that $m_k = (s_k p_{l(k)+1} + r) q_{l(k)}$ for corresponding s_k, l_k and r.

Let $y \in \mathrm{Z}(f)$ be such that ry is not an integer. One can easily find reals y_k such that $\sum_{k=0}^{\infty} f(y_k) \leq 1$ and $\lim_{k\to\infty} y_k = y$. By Lemma 6 there exists a real z such that $\|q_{l(k)} z - y_k\| \leq \dfrac{2}{p_{l(k)+1}}$ for any k and $\|q_i z\| \leq \dfrac{1}{p_{i+1}}$ for any i which is not a value of the sequence $\{l_k\}_{k=0}^{\infty}$. Then one can easily see that for any k we have $f(q_{l(k)}z) \leq 2^{-l(k)-1} + f(y_k)$ and for i which is not a value of the sequence $\{l(k)\}_{k=0}^{\infty}$ we obtain $f(q_i z) \leq 2^{-i-1}$. Thus

$$\sum_{k=0}^{\infty} f(q_k z) \leq 4 + \sum_{k=0}^{\infty} f(y_k) \leq 5$$

and therefore $z \in B_1$.

As in the proof of Theorem 16 we can show that $\|m_k z - r y_k\| < \delta_{k-1}$ and therefore $\lim_{k\to\infty} \|m_k z\| = \lim_{k\to\infty} \|r y_k\| = \|ry\| \neq 0$. □

Corollary 11. *$\mathcal{A} = \mathcal{A}_f$ if and only if $\mathrm{Z}(f)$ is a finite set of rationals.*

Generalizing the result by R. Salem [Sa] for $p = 2$, J. Arbault [Ar] showed that $\mathcal{N} = \mathcal{N}_f$ for $f(x) = |\sin \pi x|^p$, $p > 0$, or equivalently for $f(x) = \|x\|^p$. We present a further strengthening of this result.

Lemma 12. *Assume that $f(x) = f(-x)$ for every $x \in \mathbb{T}$, the function f is convex in the interval $\langle -1/2, 1/2\rangle$ and $\mathrm{Z}(f) = \{0\}$. Then $\mathcal{N}_f \subseteq \mathcal{N}$ and $\mathcal{B}_f \subseteq \mathcal{B}$.*

PROOF. The convex function f has a right derivative $\varphi(x) = f'_+(x)$ in every point $0 \leq |x| < 1/2$. The function φ is non-decreasing and therefore measurable. Moreover $f(x) = \int_0^x \varphi(t)\,dt$ for $x \in \langle -1/2, 1/2\rangle$. If $f'_+(0) > 0$, then $\|x\| \leq f(x)/f'_+(0)$ for every $|x| < 1/2$. Thus, by Theorem 2 we obtain $\mathcal{N}_f \subseteq \mathcal{N}$.

Now, assume that $\lim_{x\to 0+} f(x)/x = 0$. Then we define

$$\psi(t) = \psi(-t) = \sup\{z; \varphi(z) \leq t\} \text{ for } t \in \langle 0, \beta\rangle,\ \beta = \sup\{\varphi(x), x \in \langle 0, 1/2\rangle\}.$$

The conjugate function h is defined by $h(x) = \int_0^x \psi(t)\,dt$ for $|x| < \beta$ and Young inequality (see e.g. [KR, Ro][3])

$$|xy| \leq f(x) + h(y) \text{ for any } x \in \langle -1/2, 1/2\rangle,\ y \in (-\beta, \beta) \tag{7}$$

holds. Moreover, one can easily see that

$$\lim_{x\to 0+} \frac{h(x)}{x} = 0. \tag{8}$$

Let $A = \{x \in \mathbb{T}; \sum_{n=0}^{\infty} a_n f(nx) < \infty\}$ be an $\mathcal{N}_f$–set, where $a_n \geq 0$ and $\sum_{n=0}^{\infty} a_n = \infty$. Using (8) one can easily find a sequence of reals $y_n \in (0, \beta)$

[3] In [Ro], this inequality is called Felchel inequality.

such that $\sum_{n=0}^{\infty} a_n y_n = \infty$ and $\sum_{n=0}^{\infty} a_n h(y_n) < \infty$. By the inequality (7) we obtain

$$\sum_{n=0}^{\infty} a_n y_n \|nx\| \leq \sum_{k=0}^{\infty} a_n f(nx) + \sum_{k=0}^{\infty} a_n h(y_n) < \infty$$

for any $x \in A$. Thus, A is an $\mathcal{N}$–set.

Since the sum $\sum_{k=0}^{\infty} a_n h(y_n)$ does not depend on x, we have actually also proved the inclusion for B-sets. $\square$

Theorem 13. *If f is convex in the interval $\langle -1/2, 1/2\rangle$ and $\mathrm{Z}(f) = \{0\}$, then $\mathcal{N}_f = \mathcal{N}$ and $\mathcal{B}_f = \mathcal{B}$.*

Proof. It is easy to see that $f(x) \leq 2f(1/2)\|x\|$ for $x \in \mathbb{T}$. Thus, by Theorem 2 we obtain $\mathcal{N} \subseteq \mathcal{N}_f$ and $\mathcal{B} \subseteq \mathcal{B}_f$. $\square$

Theorem 14. *If $\mathrm{Z}(f) = \{0\}$, then $\mathcal{N}_f \subseteq \mathcal{N}$ and $\mathcal{B}_f \subseteq \mathcal{B}$.*

Proof. Let $C \subseteq \langle -1/2, 1/2\rangle \times \mathbb{R}$ be the closure of the convex hull of the set

$$\{[x, y] \in \langle -1/2, 1/2\rangle \times \mathbb{R}; y \geq f(x) \wedge y \geq f(-x)\}.$$

Then the function $h(x) = \min\{y \in \langle 0, +\infty); [x, y] \in C\}$ is a convex continuous function from $\mathbb{T}$ into $\langle 0, +\infty)$ (see e.g. [Ro]) and such that for every $x \in \mathbb{T}$ we have $h(x) = h(-x) \leq f(x)$. We show that $\mathrm{Z}(h) = \{0\}$. Assume that for some $0 < \xi < 1/2$ we have $h(\xi) = 0$. Set $m = \min\{f(x); \xi/2 \leq |x| \leq 1/2\}$. Then the prime line p going through points $[\xi/2, 0]$ and $[1/2, m]$ lies below the set C, contradicting the fact that the point $[\xi, 0]$ lying in the set C is below the line p. By Lemma 12 we obtain $\mathcal{N}_f \subseteq \mathcal{N}_h \subseteq \mathcal{N}$ and $\mathcal{B}_f \subseteq \mathcal{B}_h \subseteq \mathcal{B}$. $\square$

In [BZ], modifying a Marcinkiewicz construction [Ma], it is shown that none of the families $\mathcal{D}_f$, $p\mathcal{D}_f$, $\mathcal{A}_f$, $\mathcal{N}_{0f}$, $\mathcal{B}_{0f}$, $\mathcal{N}_f$, $\mathcal{B}_f$, and $w\mathcal{D}_f$ is an ideal provided that $f(x+y) \leq f(x) + f(y)$ for any x, $y \in \mathbb{T}$. We show related result under another assumptions.

Let us recall that the arithmetic sum of two sets is the set

$$A + B = \{x + y \in \mathbb{T}; x \in A \wedge y \in B\}.$$

It is well known that for any trigonometric family $\mathcal{F}$, $A + A \in \mathcal{F}$ for any $A \in \mathcal{F}$ (see e.g. [Ar, BKR, BL]).

Theorem 15. a) *If $\mathrm{Z}(f)$ is a finite set of rationals, then none of the families $\mathcal{D}_f$, $p\mathcal{D}_f$, $\mathcal{B}_{0f}$, $\mathcal{N}_{0f}$, and $\mathcal{A}_f$ is an ideal.*

b) *If $\mathrm{Z}(f) = \{0\}$, then none of the families $\mathcal{B}_f$ and $\mathcal{N}_f$ is an ideal.*

Proof. Let $\{m_k\}_{k=0}^{\infty}$ be an increasing sequence of natural numbers such that $2^{-m_k} < \delta_k$. Then $f(x) < 2^{-k}$ for $|x| \leq 2^{-m_k}$. Denote $n_k = \sum_{i=0}^{k} m_i$.

Let us remind that every real $x \in \mathbb{T} = (0,1\rangle$ has a unique binary expansion $x = \sum_{i=1}^{\infty} x_i 2^{-i}$, where $x_i = 0, 1$ and there is an arbitrarily large i such that $x_i = 1$. We set

$$A = \{x \in \mathbb{T}; (\forall k)(\forall i)\,(n_{2k} < i \leq n_{2k+1} \to x_i = 0)\},$$
$$B = \{x \in \mathbb{T}; (\forall k)(\forall i)\,(n_{2k+1} < i \leq n_{2k} \to x_i = 0)\}.$$

One can easily see that

$$A \subseteq \left\{x \in \mathbb{T}; (\forall k)\, f(2^{n_{2k}}) < 2^{-2k-1}\right\},$$
$$B \subseteq \left\{x \in \mathbb{T}; (\forall k)\, f(2^{n_{2k+1}}) < 2^{-2k-2}\right\}.$$

Thus A, $B \in \mathcal{D}_f$. Since $A + B = \mathbb{T}$ and $A + B \subseteq (A \cup B) + (A \cup B)$ we obtain that $A \cup B \notin \mathcal{A}$.

If $\mathrm{Z}(f)$ is a finite set of rationals, then by Theorem 10 we obtain $A \cup B \notin \mathcal{A}_f$.

If $\mathrm{Z}(f) = \{0\}$, then by Theorem 14 we obtain $A \cup B \notin \mathcal{N}_f$. □

4 $\mathcal{N}_{0f}$ and $\mathcal{N}_{0g}$ for Different f and g

Let us consider the following relationship between functions f and g:

$$(\forall \{x_k\}_{k=0}^{\infty}) \left(\sum_{k=0}^{\infty} f(x_k) < \infty \to \sum_{k=0}^{\infty} g(x_k) < \infty\right), \tag{9}$$

Immediately from the definitions one obtains that (9) implies that $\mathcal{N}_{0f} \subseteq \mathcal{N}_{0g}$. We show that the opposite implication often holds.

Theorem 16. *Assume that $\mathcal{B}_{0f} \subseteq \mathcal{N}_{0g}$ and $\mathrm{Z}(g)$ is a finite set of rationals. If the function f satisfies the condition*

$$(\forall m > 0)(\forall x,\ |x| < 1/2)\, f(x/m) \leq f(x), \tag{10}$$

then (9) *holds.*

Proof. Assume that (9) does not hold true; i.e., there are reals x_k, $|x_k| < 1/2$, $k \in \mathbb{N}$ such that $\sum_{k=0}^{\infty} f(x_k) < \infty$ and $\sum_{k=0}^{\infty} g(x_k) = \infty$. We construct a set $B_2 \in \mathcal{B}_{0f}$ such that $B_2 \notin \mathcal{N}_{0g}$. Set

$$B_2 = \{x \in \mathbb{T}; \sum_{k=0}^{\infty} f(q_k x) \leq 4 + \sum_{k=0}^{\infty} f(x_k)\}.$$

Now, to get a contradiction, suppose that $B_2 \in \mathcal{N}_{0g}$. Then there exists an increasing sequence $\{m_k\}_{k=0}^{\infty}$ such that

$$B_2 \subseteq \{x \in \mathbb{T}; \sum_{k=0}^{\infty} g(m_k x) < \infty\}.$$

Since $\mathrm{Z}(g)$ is a finite set of rationals, by Theorem 10 we obtain that

$$B_2 \subseteq \mathbf{A}(\{m_k\}_{k=0}^{\infty}).$$

Since $B_0 \subseteq B_2$, by Theorem 5 we can assume that the sequence $\{m_k\}_{k=0}^{\infty}$ has the form $m_k = (s_k p_{l(k)+1} + r) q_{l(k)}$ satisfying conditions (a) – (c). Now we set $y_k = x_k / r$. By (10) we obtain $\sum_{k=0}^{\infty} f(y_k) \leq \sum_{k=0}^{\infty} f(x_k) < \infty$. Let the real z be that constructed in the proof of Lemma 6. Then for any k we have $\|q_{l(k)} z - y_k\| \leq 2/p_{l(k)+1} \leq 2/p_k < \delta_k$ and therefore by (1) we obtain $|f(q_{l(k)} z)| \leq 2^{-k} + f(y_k)$. If i is not a value of the sequence $\{l(k)\}_{k=0}^{\infty}$, then $|f(q_i z)| < 2^{-i}$. Thus

$$\sum_{k=0}^{\infty} f(q_k z) \leq 4 + \sum_{k=0}^{\infty} f(y_k) \leq 4 + \sum_{k=0}^{\infty} f(x_k)$$

and hence $z \in B_2$.

On the other side for any k we have mod 1

$$m_k z - r y_k \quad = \quad \frac{m_k}{q_{l(k)}} \theta_{l(k)} + r \left(\frac{z_{l(k)+1}}{p_{l(k)+1}} - y_k \right).$$

Again, since $z_i = 0$ for $l(k)+1 < i \leq l(k+1)$, by (3) we have $|\theta_{l(k)}| \leq 1/p_{l(k+1)}$ and by (b) and (c) for sufficiently large k we obtain

$$\|m_k z - r y_k\| \leq \frac{1}{p_{l(k)+1}} + r \frac{1}{p_{l(k)+1}} \leq \frac{1}{p_{l(k-1)}} < \delta_{k-1}.$$

Thus $|g(m_k z) - g(r y_k)| < 2^{-k+1}$ and therefore $\sum_{k=0}^{\infty} g(m_k z) = \infty$; a contradiction. $\square$

5 Bases and Towers

Let us recall that two infinite sets $A_1, A_2 \subseteq \mathbb{N}$ are called *almost disjoint* if the intersection $A_1 \cap A_2$ is a finite set. A family $\mathcal{A}$ of infinite subsets of $\mathbb{N}$ is called *a family of almost disjoint sets* if any A_1, $A_2 \in \mathcal{M}$, $A_1 \neq A_2$ are almost disjoint. It is well known that there exists a family $\mathcal{M} \subseteq \mathcal{P}(\mathbb{N})$ of almost disjoint sets such that $|\mathcal{M}| = \mathfrak{c}$ (see e.g. [vD]).

The cardinal number $\mathfrak{t}$ (compare also [vD]) is the smallest cardinal number for which there exists a sequence of infinite subsets of $\mathbb{N}$

$$\{N_\xi; \xi < \mathfrak{t}\} \tag{11}$$

such that $N_\xi \setminus N_\eta$ is finite for any $\eta < \xi < \mathfrak{t}$ and there exists no infinite set $N \subseteq \mathbb{N}$ such that $N \setminus N_\xi$ is finite for any $\xi < \mathfrak{t}$. It is well known that $\mathfrak{t}$ is a regular cardinal $\aleph_0 < \mathfrak{t} \leq \mathfrak{c}$. Moreover we can assume that $N_\eta \setminus N_\xi$ is infinite for any $\eta < \xi < \mathfrak{t}$. The sequence (11) is usually called *a tower*.

If $\mathcal{F}$ is a family of thin sets, then a sequence $\{Y_\xi; \xi < \alpha\}$, α being an ordinal, of sets of the family $\mathcal{F}$ is called *an α–tower of the family $\mathcal{F}$*, if

$$Y_\xi \subseteq Y_\eta,\ Y_\xi \neq Y_\eta \text{ for any } \xi < \eta < \alpha.$$

A tower $\{Y_\xi; \xi < \alpha\}$ of the family $\mathcal{F}$ is said to be *maximal* if there is no set $Y \in \mathcal{F}$ such that $A_\xi \subseteq Y$ for all $\xi < \alpha$. Let us remark that we deal with the inclusion opposite that in the case of a tower of subsets of $\mathbb{N}$.

According to the results of [BS] (compare also [Re]) there exists a $\mathfrak{t}$–tower of the family $\mathcal{N}_{0f}$ for suitable f. We show that there exists a maximal $\mathfrak{t}$–tower of this family. For an infinite set $E \subseteq \mathbb{N}$ we let

$$\mathbf{B}(E) = \{x \in \mathbb{T}; \sum_{k \in E} f(q_k x) \leq 4\}.$$

We begin with some auxiliary results.

Lemma 17. *If E, F, $E \setminus F$ are infinite subsets of $\mathbb{N}$, then $\mathbf{B}(F) \setminus \mathbf{B}(E)$ contains a perfect subset.*

PROOF. Let $G \subseteq E \setminus F$ be an infinite set. We construct a real $x(G)$ such that $x(G) \in \mathbf{B}(F) \setminus \mathbf{B}(E)$. Since f is not identically equal to zero, there are reals α, β, γ such that $-1/2 < \alpha < \beta < 1/2$ and $f(x) \geq \gamma$ for any $x \in \langle \alpha, \beta \rangle$. We set x_i to be an integer such that $\alpha < (x_i - 1)/p_i < (x_i + 1)/p_i < \beta$ if $i - 1 \in G$ and $2/p_i < \beta - \alpha$ and $x_i = 0$ otherwise. Let $x(G) = \sum_{i=0}^\infty \dfrac{x_i}{q_i}$. For every $k \in G \subseteq E$ we have mod 1, $q_k x(G) = \frac{x_{k+1}}{p_{k+1}} + \theta_k$ and $|\theta_k| \leq 1/p_{k+1}$ and therefore for sufficiently large $k \in G$ we have $\alpha < \|q_k x(G)\| < \beta$. Hence

$$\sum_{k \in E} f(q_k x(G)) \geq \sum_{k \in G} f(q_k x(G)) = \infty.$$

Thus, $x(G) \notin \mathbf{B}(E)$. On the other side, if $k \in F$, then $x_{k+1} = 0$ and therefore $\|q_k x(G)\| \leq 1/p_{k+1}$ and $f(q_k x(G)) < 1/2^{k+1}$. Thus $x(G) \in \mathbf{B}(F)$.

Since we can find $2^{\aleph_0}$ infinite sets $G \subseteq E \setminus F$ and reals $x(G)$ that are different for different G's, the difference $\mathbf{B}(F) \setminus \mathbf{B}(E)$ has th power of the continuum. Being a Borel set it contains a perfect subset. $\square$

Lemma 18. *Assume that for every* $k \in \mathbb{N}$ $m_k = (s_{l(k)}p_{l(k)+1} + r)q_{l(k)}$, $r \neq 0$ *and* (a) – (c) *hold. Let* $L = \{l_k; k \in \mathbb{N};\}$. *If the set* $L \setminus E$ *is infinite, then*

$$\mathbf{B}(E) \nsubseteq \mathbf{A}(\{m_k\}_{k=0}^{\infty}).$$

PROOF. If $i = l_k + 1$, $l_k \notin E$, take an integer $x_i < \frac{1}{2}q_i$ such that $x_i > \frac{1}{4}p_{l(k)+1}$. Otherwise set $x_i = 0$. Let $x = \sum_{i=0}^{\infty} x_i/q_i$. If $i \in E$, then $x_{i+1} = 0$ and $q_i x = \theta_i$. By (1) and (2) $f(q_i x) < 1/2^i$ and therefore $x \in \mathbf{B}(E)$.

If $i - 1 \in L \setminus E$, $i = l_k + 1$, then we have mod 1

$$m_k x = (s_{l(k)}p_{l(k)+1} + r)q_{l(k)}x = r\frac{x_{l(k)+1}}{p_{l(k)+1}} + \frac{m_k}{q_{l(k)}}\theta_{l(k)}.$$

Since the last term is small, we obtain $\|m_k x\| \geq 1/8\gamma|r|$. Thus $\lim_{k\to\infty} m_k x \neq 0$ and therefore $x \notin \mathbf{A}(\{m_k\}_{k=0}^{\infty}$. $\square$

Theorem 19. *If* F, E *are infinite, almost disjoint subsets of* $\mathbb{N}$, *then there is no A–set containing both* $\mathbf{B}(E)$ *and* $\mathbf{B}(F)$.

PROOF. Assume that the sequence $\{m_k\}_{k=0}^{\infty}$ is such that $\mathbf{B}(E) \subseteq \mathbf{A}(\{m_k\}_{k=0}^{\infty})$. Let $E = \{e_0 < e_1 < \cdots < e_n < \dots\}$. We put

$$\bar{p}_0 = \prod_{j \leq e(0)} p_j, \quad \bar{p}_{i+1} = \prod_{e(i) < j \leq e(i+1)} p_j, \quad \bar{q}_i = \prod_{j \leq i} \bar{p}_i.$$

Then $\bar{q}_i = \prod_{j \leq e(i)} p_j = q_{e(i)}$ and $\mathbf{B}(E) = \{x \in \mathbb{T}; \sum_{k=0}^{\infty} f(\bar{q}_k x) \leq 4\}$. By Theorem 5 we can assume that $m_k = (\bar{s}_k \bar{p}_{l(k)+1} + r)\bar{q}_{l(k)}$, $r \neq 0$ and conditions (a) – (c) are satisfied. Then $L = \{e(l_k); k \in \mathbb{N}\} \subseteq E$. Thus $L \setminus F$ is infinite. By Lemma 18 we obtain $\mathbf{B}(F) \nsubseteq \mathbf{A}(\{m_k\}_{k=0}^{\infty})$; a contradiction. $\square$

Theorem 20. *Assume that* $\mathrm{Z}(f)$ *is a finite set of rationals. Then every basis of any of the families* $\mathcal{B}_{0f}$, $\mathcal{N}_{0f}$, $\mathcal{A}_f$ *has cardinality at least* $\mathfrak{c}$. *Especially, any basis of any of the trigonometric family* $\mathcal{B}_0$, $\mathcal{N}_0$, $\mathcal{A}$ *has cardinality at least* $\mathfrak{c}$.

PROOF. Take an almost disjoint family $\mathcal{M}$ of subsets $\mathbb{N}$ of cardinality $\mathfrak{c}$. Then by Theorem 19 the cardinality of any basis of any of the families $\mathcal{B}_{0f}$, $\mathcal{N}_{0f}$, $\mathcal{A}_f = \mathcal{A}$ must be greater than the cardinality of the family $\{\mathbf{B}(A); A \in \mathcal{M}\}$. $\square$

Let us remark that the families $\mathcal{B}_{0f}$, $\mathcal{N}_{0f}$, $\mathcal{A}_f = \mathcal{A}$ have Borel bases which are of cardinality $\mathfrak{c}$.

Theorem 21. *Assume that* $\mathrm{Z}(f)$ *is a finite set of rationals. Let* $\{R_\xi; \xi < \mathfrak{t}\}$ *be a tower of subsets of* $\mathbb{N}$. *Then* $\{\mathbf{B}(R_\xi); \xi < \mathfrak{t}\}$ *is a maximal tower of any of the families* $\mathcal{B}_{0f}$, $\mathcal{N}_{0f}$ *and* $\mathcal{A}$. *Moreover, for any* $\xi < \eta < \mathfrak{t}$ *there exists a perfect subset of* $\mathbf{B}(R_\eta) \setminus \mathbf{B}(R_\xi)$.

PROOF. By Lemma 17 for $\xi < \eta < \mathfrak{t}$ there exists a perfect subset of the set $\mathbf{B}(R_\eta) \setminus \mathbf{B}(R_\xi)$. Assume now that there exists an $\mathcal{N}_{0f}$ set A such that $\mathbf{B}(R_\xi) \subseteq A$ for every $\xi < \mathfrak{t}$. Let $A = \{x \in \mathbb{T}; \sum_{k=0}^{\infty} f(m_k x) < \infty\}$. Since $\mathbf{B}(R_0) \subseteq A$, by Theorem 5 there exist sequences of natural numbers $\{s_k\}_{k=0}^{\infty}$, $\{l_k\}_{k=0}^{\infty}$, an integer $r \neq 0$ and a subsequence of $\{m_k\}_{k=0}^{\infty}$ (denoted by same letters) such that $m_k = (s_k p_{l(k)+1} + r) q_{l(k)}$ for every k. Let $L = \{l_k; k \in \mathbb{N}\}$. Then there exits a $\xi < \mathfrak{t}$ such that $L \setminus R_\xi$ is infinite. Then by Lemma 18 we have $\mathbf{B}(R_\xi) \nsubseteq \mathbf{A}(\{m_k\}_{k=0}^{\infty})$; a contradiction. $\square$

6 Some Examples and Some Open Problems

According to the results of sections 3 and 4, we can find functions f, g such that $\mathcal{B}_{0f} \neq \mathcal{B}_{0g}$, $\mathcal{N}_{0f} \neq \mathcal{N}_{0g}$ or $\mathcal{A}_f \neq \mathcal{A}_g$. We present some examples.

Theorem 22. a) *Let* $f(x) = \|x\|^c$, $g(x) = \|x\|^d$ *for* $x \in \mathbb{T}$. *If* $0 < c < d$, *then*

$$\mathcal{N}_{0f} \subseteq \mathcal{N}_{0g},\ \mathcal{B}_{0g} \nsubseteq \mathcal{N}_{0f},\ \mathcal{N}_{0g} \nsubseteq \mathcal{N}_{0f}.$$

b) *For positive reals* c, d *we set*

$$f(x) = \begin{cases} 1/2^d \|x\|^c & \text{if } 0 \leq x \leq 1/2, \\ 1/2^c \|x\|^d & \text{if } 1/2 \leq x \leq 1, \end{cases}$$

and

$$g(x) = \begin{cases} 1/2^c \|x\|^d & \text{if } 0 \leq x \leq 1/2, \\ 1/2^d \|x\|^c & \text{if } 1/2 \leq x \leq 1, \end{cases}$$

If $c \neq d$, *then* $\mathcal{B}_{0f} \nsubseteq \mathcal{N}_{0g}$, $\mathcal{B}_{0g} \nsubseteq \mathcal{N}_{0f}$, $\mathcal{N}_{0f} \nsubseteq \mathcal{N}_{0g}$ *and* $\mathcal{N}_{0g} \nsubseteq \mathcal{N}_{0f}$.

The theorem immediately follows from Theorem 16. From Theorem 10 we obtain the following assertion.

Theorem 23. *Let* $f(x) = \|x\| \cdot |\sin(\pi/\|x\|)|$ *for* $x \neq 0$ *and* $f(0) = 0$. *Then* $\mathcal{B}_{0f} \nsubseteq \mathcal{A}$ *and* $\mathcal{A}_f \nsubseteq \mathcal{A}$.

In the proof of Theorem 9 we have constructed a continuous function f such that $\mathcal{D} \neq \mathcal{D}_f$ and $p\mathcal{D} \neq p\mathcal{D}_f$. However, we were not able to find such functions for distinguishing the families $\mathcal{B}_f$, $\mathcal{N}_f$ and $w\mathcal{D}_f$. If $\mathrm{Z}(f) = \{0\}$ and both $f'_+(0)$, $f'_-(0)$ are finite, then according to Theorems 2 and 14 the equalities $\mathcal{B}_f = \mathcal{B}$ and $\mathcal{N}_f = \mathcal{N}$ hold. The simplest case for which we do not know the answer is the case of the function $f(x) = \sqrt{\|x\|}$. Neither were we able to solve the problem of the cardinality of bases for the other families. Thus, we can formulate the main **open problems:**

a) Find continuous functions f, g such that $\mathcal{F}_f \neq \mathcal{F}_g$ for $\mathcal{F} = \mathcal{B}$ and/or $\mathcal{N}$, $w\mathcal{D}$.

b) Find a continuous function f such that $\mathcal{F}_f \neq \mathcal{F}$ for $\mathcal{F} = \mathcal{B}$ and/or $\mathcal{N}$, $w\mathcal{D}$.

c) Do the inclusions $\mathcal{B} \subseteq \mathcal{B}_f$, $\mathcal{N} \subseteq \mathcal{N}_f$ hold when $f(x) = \sqrt{\|x\|}$?

d) Does any basis of the family $\mathcal{D}$, $p\mathcal{D}$, $\mathcal{B}$, $\mathcal{N}$, $w\mathcal{D}$ have cardinality at least $\mathfrak{c}$?

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