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DIEUDONNÉ-TYPE THEOREMS FOR SET FUNCTIONS WITH VALUES IN (l)-GROUPS

Abstract

Some versions of Dieudonné theorems are given for set functions, not necessarily positive, taking values in Dedekind complete (l)-groups, relatively to the "(D)-convergence".

1 Introduction.

In a previous paper ([4]), we gave some versions of Vitali - Hahn - Saks and Nikodým theorems for set functions with values in suitable Dedekind complete (l)-groups. In this paper, we prove some versions of Dieudonné theorems, for (l)-group-valued finitely additive regular maps. In the literature, there exist several versions of theorems of this kind, for maps taking values in topological groups and/or Banach spaces. Among the authors, we specifically mention Brooks and Chacon ([5], [6]), Candeloro and Letta ([8]).

In the previous paper [2] similar results were proved with respect to order convergence for *positive* means taking values in spaces of the type $L^0(X, \mathcal{B}, \mu)$, where μ is a σ -additive locally finite positive $\widetilde{\mathbb{R}}$-valued measure.

2 Preliminaries.

We begin with the following.

Definitions 2.1. An Abelian group $(R, +)$ is called (l)-*group* if it is endowed with a compatible ordering $\leq$, and is a lattice with respect to it.

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An (l)-group R is said to be *Dedekind complete* if every nonempty subset of R, bounded from above, has a supremum in R. A sequence $(r_n)_n$ in R is said to be *order-convergent* (or (o)*-convergent*) to r if there exists a sequence $(p_n)_n$ in R such that $|r_n - r| \leq p_n$, $\forall\, n \in \mathbb{N}$ and $p_n \downarrow 0$ (see also [11]), and we will write $(o) \lim_n r_n = r$. A bounded double sequence $(a_{i,l})_{i,l}$ in R is called (D)*-sequence* or *regulator* if for all $i \in \mathbb{N}$ we have $a_{i,l} \downarrow 0$ as $l \to +\infty$. An (l)-group R is said to be *weakly* σ*-distributive* if for every (D)-sequence $(a_{i,l})_{i,l}$ we have $\bigwedge_{\varphi \in \mathbb{N}^{\mathbb{N}}} \left(\bigvee_{i=1}^{\infty} a_{i,\varphi(i)} \right) = 0$. From now on we assume that R is a weakly σ-distributive Dedekind complete (l)-group. We say that $b \in R$, $b \geq 0$, *dominates* a sequence $(r_n)_n$ of elements of R if $\exists\, n_0 \in \mathbb{N}$ such that $|r_n| \leq b$ for $n \geq n_0$. Moreover, given a regulator $(a_{i,l})_{i,l}$, we call a *bound* of $(a_{i,l})_{i,l}$ any element b of the type $b = \bigvee_{i=1}^{\infty} a_{i,\varphi(i)}$, with $\varphi \in \mathbb{N}^{\mathbb{N}}$. A sequence $(r_n)_n$ in R is said to be (D)*-convergent* to $r \in R$ (and we write $(D) \lim_n r_n = r$) if there exists a regulator $(a_{i,l})_{i,l}$ every bound of which dominates the sequence $(r_n - r)_n$. We note that, since R is weakly σ-distributive, the (D)limit is unique, and (o)- and (D)-convergence coincide (see [9]). If E is any nonempty set, we say that a sequence $(f_n)_n$ of elements of R^E (RD)*-converges to* $f \in R^E$ if there exists a regulator every bound of which dominates every sequence of the type $(f_n(x) - f(x))_n$, with $x \in E$. Analogously, we say that $(f_n)_n$ (UD)*-converges to* f if there exists a regulator every bound of which dominates the sequence $\left(\bigvee_{x \in E} |f_n(x) - f(x)| \right)_n$. Lemma 2.2 presents a relationship between simple D-convergence and (RD)-convergence, at least when E is countable. A sequence $(r_n)_n$ of elements of R is said to be (D)*-Cauchy* if the sequence of functions $(f_n : \mathbb{N} \to R)_n$, defined by setting $f_n(p) = r_n - r_{n+p}$, $p \in \mathbb{N}$, (UD)-converges to 0.

We now recall the following result (see [14], pp. 42-43), which will be useful in the sequel.

Lemma 2.2. *Let R be a Dedekind complete (l)-group (not necessarily weakly σ-distributive), and let $(a_{i,l}^{(n)})_{i,l}$, $n \in \mathbb{N}$, be a sequence of regulators in R. Then for every $u \in R$, $u \geq 0$ there exists a (D)-sequence $(a_{i,l})_{i,l}$ in R such that*

$$u \wedge \left[\sum_{n=1}^{\infty} \left(\bigvee_{i=1}^{\infty} a_{i,\varphi(i+n)}^{(n)} \right) \right] \leq \bigvee_{i=1}^{\infty} a_{i,\varphi(i)} \quad \forall\, \varphi \in \mathbb{N}^{\mathbb{N}}.$$

We now introduce the following definitions.

Definitions 2.3. Let Ω be any infinite set, $\mathcal{A} \subset \mathcal{P}(\Omega)$ be an algebra, R be a Dedekind complete weakly σ-distributive (l)-group. We say that $m : \mathcal{A} \to R$ is *bounded* if $\exists\, w \in R$, $w \geq 0$: $|m(A)| \leq w$, $\forall\, A \in \mathcal{A}$. The maps $m_j : \mathcal{A} \to R$, $j \in \mathbb{N}$, are *equibounded* if there exists an element $u \in R$, $u \geq 0$, such that

$$|m_j(A)| \leq u \quad \forall\, j \in \mathbb{N}, \forall\, A \in \mathcal{A}.$$

Given a finitely additive bounded measure (or, simply, *mean*) $m : \mathcal{A} \to R$, define the *semivariation* of m, $v_{\mathcal{A}}(m) : \mathcal{A} \to R$, or simply $v(m) : \mathcal{A} \to R$, by setting

$$v_{\mathcal{A}}(m)(A) = \sup_{B \in \mathcal{A}, B \subset A} |m(B)|,\ \forall\, A \in \mathcal{A},$$

and, if $\emptyset \neq \mathcal{E} \subset \mathcal{A}$, define $v_{\mathcal{E}}(m) : \mathcal{A} \to R$, by setting

$$v_{\mathcal{E}}(m)(A) = \sup_{B \in \mathcal{E}, B \subset A} |m(B)|,\ \forall\, A \in \mathcal{A}.$$

A mean $m : \mathcal{A} \to R$ is said to be σ*-additive* (or, briefly, *measure*) if there exists a (D)-sequence $(u_{i,l})_{i,l}$ such that, $\forall\, \varphi \in \mathbb{N}^{\mathbb{N}}$ and for every decreasing sequence $(H_s)_s$ in $\mathcal{A}$, $H_s \downarrow \emptyset$, there exists $\overline{s}$ such that $v_{\mathcal{A}}(m)(H_{\overline{s}}) \leq \displaystyle\bigvee_{i=1}^{\infty} u_{i,\varphi(i)}$. If a sequence of measures $m_j : \mathcal{A} \to R$, $j \in \mathbb{N}$, is given, *uniform* σ*-additivity* is defined as above, but with $\overline{s}$ *independent* of j (See also [4]).

A finitely additive measure $m : \mathcal{A} \to R$ is said to be (s)*-bounded in* $\emptyset \neq \mathcal{E} \subset \mathcal{A}$, or simply $\mathcal{E} - (s)$*-bounded,* if there exists a (D)-sequence $(w_{i,l})_{i,l}$ such that, $\forall\, \varphi \in \mathbb{N}^{\mathbb{N}}$ and for every disjoint sequence $(H_s)_s$ in $\mathcal{E}$ there exists $\overline{s}$: $\forall\, s \geq \overline{s}$, $v_{\mathcal{E}}(m)(H_s) \leq \displaystyle\bigvee_{i=1}^{\infty} w_{i,\varphi(i)}$.

If $\mathcal{E}$ is as above, we say that the maps $m_j : \mathcal{A} \to R$, $j \in \mathbb{N}$, are $\mathcal{E}$*-uniformly* (s)*-bounded* if the above condition holds, but with $\overline{s}$ independent of j (see also [4]). When $\mathcal{E} = \mathcal{A}$ we simply speak of (s)*-boundedness* or *uniform* (s)*-boundedness*.

A typical consequence of (s)-boundedness of a mean m is that the limit $m(A_n)$ exists for monotone sequences $(A_n)_n$ in $\mathcal{A}$ (see [4]). As to uniformly (s)-bounded measures, we shall report here a slight modification of Proposition 3.4 of [4], which will be used later.

Proposition 2.4. *Assume that* $(m_j)_j$ *is a sequence of* R*-valued uniformly* (s)*-bounded finitely additive measures on* $\mathcal{A}$*, and let* $(e_{i,l})_{i,l}$ *be a regulator related to this property. In correspondence with every decreasing sequence* $(A_n)_n$ *in* $\mathcal{A}$ *, for all* $n \in \mathbb{N}$ *let us define* $\xi_n : \mathbb{N} \to R$ *by* $\xi_n(j) = v(m_j)(A_n)$, $j \in \mathbb{N}$*. Then there exists* $\xi \in R^{\mathbb{N}}$ *such that* $(\xi_n)_n$ (UD)*-converges to* ξ*, and the regulator* $(e_{i,l})_{i,l}$ *works for this property.*

Given a sequence of means $(m_j)_{j\in\mathbb{N}\cup\{0\}}$, $m_j : \mathcal{A} \to R$, and a nonempty subfamily $\mathcal{E} \subset \mathcal{A}$, we say that the m_j's (D)-*converge to* m_0 *pointwise with respect to the same regulator,* or briefly $(RD)\lim_j m_j = m_0$ *in* $\emptyset \neq \mathcal{E} \subset \mathcal{A}$, if the sequence of functions $(m_j : \mathcal{E} \to R)_j$ (RD)-converges to m_0. We note that this condition is equivalent to the classical pointwise convergence of the involved set functions in the case of metrizable groups.

Now let Ω, R and $\mathcal{A}$ be as above. From now on, we assume that $\mathcal{F}$, $\mathcal{G} \subset \mathcal{A}$ are two fixed lattices, such that the complement (with respect to Ω) of every element of $\mathcal{F}$ belongs to $\mathcal{G}$. In the sequel we will not say it explicitly. If Ω is a topological normal space [resp. locally compact Hausdorff space], examples of $\mathcal{A}$, $\mathcal{F}$ and $\mathcal{G}$, satisfying the above properties, are the following: $\mathcal{A} = \{$Borel subsets of $\Omega\}$, $\mathcal{F} = \{$closed sets $\}$[resp.$\{$compact sets$\}$], $\mathcal{G} = \{$ open sets $\}$.

Definitions 2.5. A mean $m : \mathcal{A} \to R$ is said to be *regular* if there exists a (D)-sequence $(\gamma_{i,l})_{i,l}$ in R such that for each $A \in \mathcal{A}$ and $W \in \mathcal{F}$ there exists sequences $(F_n)_n$, $(F'_n)_n$ in $\mathcal{F}$, $(G_n)_n$, $(G'_n)_n$ in $\mathcal{G}$, such that

$$F_n \subset F_{n+1} \subset A \subset G_{n+1} \subset G_n \quad \forall\, n, \tag{1}$$

$$W \subset F'_{n+1} \subset G'_n \subset F'_n \quad \forall\, n, \tag{2}$$

and the sequences $\left(v_{\mathcal{A}}(m)(G_n \setminus F_n)\right)_n$ and $\left(v_{\mathcal{A}}(m)(G'_n \setminus W)\right)_n$ (D)-converge to 0 with respect to $(\gamma_{i,l})_{i,l}$.

The means $m_j : \mathcal{A} \to R$, $j \in \mathbb{N}$, are said to be *uniformly regular* if there exists a (D)-sequence $(\gamma_{i,l})_{i,l}$ in R such that $\forall\, A \in \mathcal{A}$ and $\forall\, W \in \mathcal{F}$ there exist sequences $(F_n)_n, (G_n)_n, (F'_n)_n, (G'_n)_n$ satisfying (1) and (2), and such that the sequences $(\psi_n)_n$, $(\omega_n)_n$ of elements of $R^{\mathbb{N}}$, defined by setting

$$\begin{aligned} \psi_n(j) &= v_{\mathcal{A}}(m_j)(G_n \setminus F_n), \\ \omega_n(j) &= v_{\mathcal{A}}(m_j)(G'_n \setminus W), \quad n, j \in \mathbb{N}, \end{aligned} \tag{3}$$

(UD)-converge to 0 with respect to $(\gamma_{i,l})_{i,l}$.

The following proposition shows that, if $(m_j : \mathcal{A} \to R)_j$ is a sequence of equibounded regular means, even if they are not uniformly regular, the sequences $(\gamma_{i,l})_{i,l}$, $(F_n)_n$, $(G_n)_n$, $(F'_n)_n$, $(G'_n)_n$ above can be taken independently of j , satisfying the definition of regularity.

Proposition 2.6. *Let R be as above, $\mathcal{A}$ be any algebra, and $(m_j : \mathcal{A} \to R)_j$ be a sequence of equibounded regular means. Then there exists a regulator $(p_{i,l})_{i,l}$ such that, for every $A \in \mathcal{A}$ and every $W \in \mathcal{F}$, there exist sequences*

$(F_n)_n$, $(F'_n)_n$ in $\mathcal{F}$, $(G_n)_n$, $(G'_n)_n$ in $\mathcal{G}$, satisfying (1) and (2) and such that the sequence of functions $(\psi_n)_n$, $(\omega_n)_n$ defined in (3) (RD)-converge to 0 with respect to $(p_{i,l})_{i,l}$.

PROOF. Set

$$u \equiv \sup_j \left[\sup_{A \in \mathcal{A}} |m_j(A)| \right]. \tag{4}$$

By hypothesis, for every $\mathcal{A} \in \mathcal{A}$ and $j \in \mathbb{N}$, there exist a regulator $(\gamma^{(j)}_{i,l})_{i,l}$ and two sequences $(G^{(j)}_n)_n$, $(F^{(j)}_n)_n$ such that $F^{(j)}_n \in \mathcal{F}$, $G^{(j)}_n \in \mathcal{G}$ $\forall\, j, n \in \mathbb{N}$, and

$$F^{(j)}_n \subset F^{(j)}_{n+1} \subset A \subset G^{(j)}_{n+1} \subset G^{(j)}_n \quad \forall\, j, n \in \mathbb{N}, \tag{5}$$

and $\forall\, \varphi \in \mathbb{N}^{\mathbb{N}}$ and $j \in \mathbb{N}$ there exists $n_0 \in \mathbb{N}$, $n_0(\varphi, j)$, such that $\forall\, n \geq n_0$ we have $v_{\mathcal{A}}(m_j)(G^{(j)}_n \setminus F^{(j)}_n) \leq \bigvee_{i=1}^{\infty} \gamma^{(j)}_{i,\varphi(i)}$. Moreover, by Lemma 2.2, there exists a regulator $(p_{i,l})_{i,l}$ such that

$$u \wedge \left[\sum_{j=1}^{\infty} \left(\bigvee_{i=1}^{\infty} \gamma^{(j)}_{i,\varphi(i+j)} \right) \right] \leq \bigvee_{i=1}^{\infty} p_{i,\varphi(i)} \qquad \forall\, \varphi \in \mathbb{N}^{\mathbb{N}}, \tag{6}$$

where u is as in (4).

For every $n \in \mathbb{N}$, set $G_n \equiv \cap_{j \leq n} G^{(j)}_n$ and $F_n \equiv \cup_{j \leq n} F^{(j)}_n$. Then $G_n \in \mathcal{G}$, $F_n \in \mathcal{F}$, and $A \subset G_n$, $A \supset F_n$ $\forall\, n \in \mathbb{N}$. Furthermore it is easy to check that $G_{n+1} \subset G_n$ and $F_{n+1} \supset F_n$ $\forall\, n$. Since $G_n \setminus F_n \subset G^{(j)}_n \setminus F^{(j)}_n$ $\forall\, j, n \in \mathbb{N}$, $\forall\, \varphi \in \mathbb{N}^{\mathbb{N}}$ and $\forall\, j$ there exists $n_0 \in \mathbb{N}$, $n_0(\varphi, j)$, such that

$$v_{\mathcal{A}}(m_j)(G_n \setminus F_n) \leq \bigvee_{i=1}^{\infty} p_{i,\varphi(i)} \quad \forall\, n \geq n_0.$$

Moreover, by hypothesis and Lemma 2.2, the regulator $(p_{i,l})_{i,l}$ in (6) is such that for all $W \in \mathcal{F}$ and $j \in \mathbb{N}$ there exist two sequences $({G'_n}^{(j)})_n$, $({F'_n}^{(j)})_n$ such that ${F'_n}^{(j)} \in \mathcal{F}$, ${G'_n}^{(j)} \in \mathcal{G}$ $\forall\, j, n \in \mathbb{N}$, and

$$W \subset {F'_{n+1}}^{(j)} \subset {G'_n}^{(j)} \subset {F'_n}^{(j)} \quad \forall\, j, n \in \mathbb{N},$$

and $\forall\, \varphi \in \mathbb{N}^{\mathbb{N}}$, $\forall\, j \in \mathbb{N}$, there exists $n_0 \in \mathbb{N}$, $n_0(\varphi, j)$, such that

$$v_{\mathcal{A}}(m_j)(G^{(j)}_n \setminus W) \leq \bigvee_{i=1}^{\infty} p_{i,\varphi(i)} \quad \forall\, n \geq n_0.$$

It is readily seen that $F'_n \equiv \cap_{j \le n} {F'_n}^{(j)}$, $G'_n \equiv \cap_{j \le n} {G'_n}^{(j)}$, $n \in \mathbb{N}$, satisfy the conclusion of the proposition. $\square$

We now introduce the concept of absolute continuity in our setting.

Definition 2.7. Let m be any R-valued finitely additive measure on $\mathcal{A}$. Given any other finitely additive measure $\nu : \mathcal{A} \to \mathbb{R}_0^+$, we say that m is *absolutely continuous* with respect to ν (and write $m \ll \nu$) if there exists a (D)-sequence $(a_{i,l})_{i,l}$ such that, whenever $(H_k)_k$ is a sequence from $\mathcal{A}$, satisfying $\lim_k \nu(H_k) = 0$, for every $\varphi \in \mathbb{N}^{\mathbb{N}}$ an integer $\overline{k}$ can be found, such that $|m(H_k)| \le \bigvee_{i=1}^{\infty} a_{i,\varphi(i)}$, for all $k \ge \overline{k}$.
In case ν is fixed, and $(m_j)_j$ is a sequence of finitely additive measures on $\mathcal{A}$, *uniform absolute continuity* of the m_j's with respect to ν can be defined in a similar way, but clearly the integer $\overline{k}$ must be independent of j.

3 The Dieudonné Theorem.

We shall prove a version of Dieudonné 's Theorem (see also [5], [7]).We begin with the following:

Lemma 3.1. *Let R, Ω, $\mathcal{A}$, $\mathcal{F}$, $\mathcal{G}$ be as in Proposition 2.6, and suppose that $m : \mathcal{A} \to R$ is any regular bounded finitely additive measure. Then, for each $A \in \mathcal{A}$, and every $V \in \mathcal{G}$, one has:*

$$v_{\mathcal{A}}(m)(A) = v_{\mathcal{F}}(m)(A), \tag{7}$$

$$v_{\mathcal{A}}(m)(V) = v_{\mathcal{G}}(m)(V). \tag{8}$$

Proof. The relation (7) is a direct consequence of regularity, and weak σ-distributivity. So, fix $V \in \mathcal{G}$. Let $(\gamma_{i,l})_{i,l}$ be the (D) -sequence related to regularity, let B be any element from $\mathcal{A}$, $B \subset V$, and fix $\varphi \in \mathbb{N}^{\mathbb{N}}$. Thanks to regularity of m, there exists a set $G \in \mathcal{G}$, $G \supset B$, such that

$$v(m)(G \setminus B) \le \bigvee_{i=1}^{\infty} \gamma_{i,\varphi(i)}.$$

Hence $|m(B)| \le |m(G)| + \bigvee_{i=1}^{\infty} \gamma_{i,\varphi(i)}$. Without loss of generality, we may assume $G \subset V$.Thus $|m(B)| \le v_{\mathcal{G}}(m)(V) + \bigvee_{i=1}^{\infty} \gamma_{i,\varphi(i)}$. As B is arbitrary, we get

$$v_{\mathcal{A}}(m)(V) \le v_{\mathcal{G}}(m)(V) + \bigvee_{i=1}^{\infty} \gamma_{i,\varphi(i)}.$$

Finally, as R is weakly σ-distributive, we deduce

$$v_{\mathcal{A}}(m)(V) \le v_{\mathcal{G}}(m)(V)$$

and then, obviously, the two elements coincide, and so (8) is proved. $\square$

We now prove the following lemma.

Lemma 3.2. *Under the same hypotheses and notation as above, let $(m_j : \mathcal{A} \to R)_j$ be a sequence of equibounded, regular and $\mathcal{G}$-uniformly (s)-bounded means (with respect to a (D)-sequence $(b_{i,l})_{i,l}$). Then the m_j's are $\mathcal{A}$-uniformly (s)-bounded, and uniformly regular.*

PROOF. Let $(K_n)_n$ be any disjoint sequence in $\mathcal{A}$. First of all, we note that the hypotheses of Proposition 2.6 are fulfilled. Let $(p_{i,l})_{i,l}$ be the same regulator as in that Proposition, define u as in (4), and let $(d_{i,l})_{i,l}$ be a (D)-sequence such that:

$$u \wedge \left[\sum_{h=1}^{\infty}\left(\bigvee_{i=1}^{\infty} p_{i,\varphi(i+h)}\right)\right] \le \bigvee_{i=1}^{\infty} d_{i,\varphi(i)} \quad \forall\, \varphi \in \mathbb{N}^{\mathbb{N}}.$$

Finally, let $e_{i,l} = 2(b_{i,l} + d_{i,l})$, $i, l \in \mathbb{N}$. We will prove that

$$(D)\lim_n\{\sup_j[v_{\mathcal{A}}(m_j)(K_n)]\} = 0$$

with respect to the regulator $(e_{i,l})_{i,l}$. If we deny this, then there exists $\varphi \in \mathbb{N}^{\mathbb{N}}$ such that $\forall\, k \in \mathbb{N}$, $\exists\, n_k \ge k$, $\exists\, j_k \in \mathbb{N}$, $\exists\, A_k \in \mathcal{A}$ with $A_k \subset K_{n_k}$ and

$$|m_{j_k}(A_k)| \not\le \bigvee_{i=1}^{\infty} e_{i,\varphi(i)}. \tag{9}$$

Moreover, thanks to (7), we can assume $A_k \in \mathcal{F}\ \forall\, k$.

Fix $k \in \mathbb{N}$, and put $b = \bigvee_{i=1}^{\infty} b_{i,\varphi(i)}$ and $e = \bigvee_{i=1}^{\infty} e_{i,\varphi(i)}$. We note that, by virtue of the regularity of the set functions m_j, $j \in \mathbb{N}$, there exist $G_k \in \mathcal{G}$, $F_k \in \mathcal{F}$ such that $A_k \subset G_k \subset F_k$, and

$$[v_{\mathcal{A}}(m_1) \vee \ldots \vee v_{\mathcal{A}}(m_{j_k})](F_k \setminus A_k) \le \bigvee_{i=1}^{\infty} p_{i,\varphi(i+k)}.$$

Now, we set

$$G_1^* = G_1, G_2^* = G_2 \setminus F_1, \ldots, G_{k+1}^* = G_{k+1} \setminus \left(\bigcup_{h=1}^{k} F_h\right), \ldots.$$

These sets are pairwise disjoint elements of $\mathcal{G}$. Hence there exists $k_0 \in \mathbb{N}$ such that $\sup_j v(m_j)(G_k^*) \leq b$ for all $k \geq k_0$. Now, as $A_{k+1} \setminus G_{k+1}^* \subset \bigcup_{h=1}^{k}(F_h \setminus A_h)$ holds for all k, we get

$$\begin{aligned} |m_{j_k}(A_k)| &\leq |m_{j_k}(A_k \cap G_k^*)| + |m_{j_k}(A_k \setminus G_k^*)| \\ &\leq b + u \wedge \left[\sum_{h=1}^{k}\left(\bigvee_{i=1}^{\infty} p_{i,\varphi(i+h)}\right)\right] \leq e, \quad \forall\, k \geq k_0. \end{aligned}$$

This is contrary to (9). So, the set functions m_j are $\mathcal{A}$-uniformly (s)-bounded.

We now turn to uniform regularity. The regulator $(p_{i,l})_{i,l}$ above is such that, for every $A \in \mathcal{A}$, two sequences can be found, $(F_n)_n$ and $(G_n)_n$ in $\mathcal{F}$ and $\mathcal{G}$ respectively, satisfying the conditions of the definition of regularity. As the sequence $(G_n \setminus F_n)_n$ is decreasing, by 2.6, 2.4 and weak σ-distributivity of R the sequence $(\sup_j[v_{\mathcal{A}}(m_j)(G_n \setminus F_n)])_n$ (D)-converges to 0. Similarly, for each $W \in \mathcal{F}$, we can find $(F_n')_n$ and $(G_n')_n$ in $\mathcal{F}$ and $\mathcal{G}$ respectively, satisfying the conditions of the definition of regularity, and we get that the sequence

$$(\sup_j[v_{\mathcal{A}}(m_j)(G_n' \setminus W)])_n$$

(D)-converges to 0. This concludes the proof of the lemma. $\square$

Theorem 3.3. (Dieudonné) *Let Ω, R, $\mathcal{G}$, $\mathcal{F}$ be as above, and assume that $\mathcal{A} \subset \mathcal{P}(\Omega)$ is a σ-algebra, and $\mathcal{G}$ is stable under countable disjoint unions. Suppose that $(m_j : \mathcal{A} \to R)_j$ is a sequence of equibounded regular σ-additive measures such that $m_0 = (RD)\lim_j m_j$ in $\mathcal{G}$ exists. Then we have:*

i) *The measures m_j, $j \in \mathbb{N}$, are $\mathcal{A}$-uniformly (s)-bounded and uniformly regular.*

ii) *The limit $m_0 = (RD)\lim_j m_j$ in $\mathcal{A}$ exists in R.*

iii) *The m_j's are uniformly σ-additive.*

iv) *m_0 is regular and σ-additive.*

Proof. i) Thanks to [4], Theorem 5.4, the set functions m_j are $\mathcal{G}$-uniformly (s)-bounded; hence, from Lemma 3.2 we get $\mathcal{A}$-uniform (s)-boundedness and uniform regularity.

ii) Fix $A \in \mathcal{A}$, and let $(y_{i,l})_{i,l}$ be the regulator related with uniform regularity. For each $\varphi \in \mathbb{N}^{\mathbb{N}}$ there exists $G \in \mathcal{G}$ such that $A \subset G$ and

$$v_{\mathcal{A}}(m_j)(G \setminus A) \leq \bigvee_{i=1}^{\infty} y_{i,\varphi(i)} \quad \forall\, j.$$

Corresponding to G, there exists $j_0 \in \mathbb{N}$ such that

$$|m_j(G) - m_{j+p}(G)| \leq \bigvee_{i=1}^{\infty} \alpha_{i,\varphi(i)} \quad \forall\, j \geq j_0, \quad \forall\, p \in \mathbb{N},$$

where $(\alpha_{i,l})_{i,l}$ is the regulator for (RD)-convergence in $\mathcal{G}$. So we have

$$|m_j(A) - m_{j+p}(A)| \leq 2 \bigvee_{i=1}^{\infty} y_{i,\varphi(i)} + \bigvee_{i=1}^{\infty} \alpha_{i,\varphi(i)} \quad \forall\, j \geq j_0, \quad \forall\, p \in \mathbb{N}. \tag{10}$$

From (10) it follows that the sequence $(m_j(A))_j$ is (D)-Cauchy in R. Since R is a Dedekind complete (l)-group, the sequence $(m_j(A))_j$ is (D)-convergent (see also [3], Theorem 2.16; [9]). Thus ii) is proved.

iii) follows from ii) and [4], Corollary 5.5.

iv) follows easily from i), ii), iii) and weak σ -distributivity of R. □

Under suitable additional conditions, it's also possible to state a finitely additive version of Dieudonné's theorem.

Theorem 3.4. *Let Ω, R, $\mathcal{A}$, $\mathcal{G}$, $\mathcal{F}$ be as in Proposition 2.6, and assume that $\mathcal{G}$ is stable under countable disjoint unions. Suppose that $(m_j : \mathcal{A} \to R)_j$ is a sequence of equibounded regular finitely additive measures, absolutely continuous with respect to a real-valued, nonnegative, finitely additive measure ν on $\mathcal{A}$. Assume that $m_0 = (RD)\lim_j m_j$ in $\mathcal{G}$ exists. Then we have:*

i) *The means m_j, $j \in \mathbb{N}$, are $\mathcal{A}$-uniformly (s)-bounded, uniformly regular and uniformly absolutely continuous with respect to ν.*

ii) *The limit $m_0 = (RD)\lim_j m_j$ in $\mathcal{A}$ exists in R.*

iii) *m_0 is (s)-bounded, regular and absolutely continuous with respect to ν.*

Proof. (i) Let $(\alpha_{i,l})_{i,l}$ be the regulator related to (RD)-convergence in $\mathcal{G}$ and let $(\beta_{i,l})_{i,l}$ be a regulator such that, for every disjoint sequence $(H_k)_k$ in $\mathcal{A}$, for every $j \in \mathbb{N}$ and every $\varphi \in \mathbb{N}^{\mathbb{N}}$, there exists $k_0 \in \mathbb{N}$ such that

$$v(m_j)(H_k) \leq \bigvee_{i=1}^{\infty} \beta_{i,\varphi(i)}$$

as soon as $k \geq k_0$. (Such a regulator exists, because of absolute continuity and Lemma 2.2.) Setting $c_{i,l} = \alpha_{i,l} \vee \beta_{i,l}$, $i, l \in \mathbb{N}$, we claim that $(6c_{i,l})_{i,l}$ works as a regulator for $\mathcal{G}$-uniform (s)-boundedness of the means m_j. Indeed, if this is not the case, there exist a disjoint sequence $(G_k)_k$ in $\mathcal{G}$, a mapping $\varphi \in \mathbb{N}^{\mathbb{N}}$ and a subsequence $(j_k)_k$ in $\mathbb{N}$ such that

$$|m_{j_k}(G_k)| \not\leq 6 \bigvee_{i=1}^{\infty} c_{i,\varphi(i)} \tag{11}$$

for each $k \in \mathbb{N}$. Now, denote by V the union of all $G_k{}'s$, and by $\mathcal{B}$ the σ-algebra in V generated by the sets G_k. Hence the sequence $(m_j : \mathcal{B} \to R)_j$ (RD)-converges to m_0. Then, we can apply Corollary 5.7 of [4], and deduce $\mathcal{B}$-uniform (s)-boundedness of the $m_j{}'s$, with respect to the regulator $(6c_{i,l})_{i,l}$, and this clearly is contrary to (11). Thus, the $m_j{}'s$ are $\mathcal{G}$-uniformly (s)-bounded, and therefore they are $\mathcal{A}$-uniformly (s)-bounded and uniformly regular, by 3.2, and uniformly absolutely continuous with respect to ν, by virtue of [4], Theorem 4.8.

(ii) can be proved as in the previous theorem.

(iii) The properties of (s)-boundedness, regularity and absolute continuity are easy consequences of the previous ones and of weak σ-distributivity of R. $\square$

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