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POROSITY OF THE EXTENDABLE CONNECTIVITY FUNCTION SPACE

Abstract

Let $I = [0,1]$, and let $Ext(I)$ or Ext denote the subspace of all extendable connectivity functions $f : I \to \mathbb{R}$ with the metric of uniform convergence on $I^{\mathbb{R}}$. We show that Ext is porous in the almost continuous function space AC by showing that the space $AC \cap PR$ of all almost continuous functions with perfect roads is porous in AC. We also show that for $n > 1$, the subspace $Ext(\mathbb{R}^n)$ of all extendable connectivity functions $f : \mathbb{R}^n \to \mathbb{R}$ is a boundary set in the Darboux function space $D(\mathbb{R}^n)$.

1 Introduction and Definitions

Whether fourteen Darboux-like real function spaces are porous or boundary sets in one another was examined in [8] and [9] to determine whether they are "thin". What the situation is for the following commutative diagram $(*)$, in which $\longrightarrow$ means proper inclusion, was left as an open problem in [9].

$$\begin{tikzcd} Ext \arrow[rr] \arrow[dr] \arrow[rr, phantom, shift right=4ex, "(*)"] & & AC \\ & AC \cap PR \arrow[ur] & \end{tikzcd}$$

Here we show Ext is porous in AC by first showing $AC \cap PR$ is porous in AC. Its proof depends on a recent result of Piotr Szuca [10].

Darboux-like function spaces are of interest for various reasons. Eleven of the fourteen, e.g., Ext and D [2], have the same intersection with the space

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B_1 of all Baire class 1 functions from $\mathbb{R}$ into $\mathbb{R}$. Consequently, any derivative $g' : \mathbb{R} \to \mathbb{R}$ must belong to Ext. A member $f : \mathbb{R} \to \mathbb{R}$ of the Darboux-like space $Conn$, consisting of all functions from $\mathbb{R}$ into $\mathbb{R}$ with connected graphs, must have a fixed point whenever the graph of f does not lie entirely above or entirely below the diagonal of $\mathbb{R} \times \mathbb{R}$.

Definitions. Let E denote I, $\mathbb{R}$, or $\mathbb{R}^n$. We abbreviate the classes to which the defined function $f : E \to \mathbb{R}$ belongs:

1. $D(E)$ – $f : E \to \mathbb{R}$ is a *Darboux* function if $f(J)$ is connected for each connected set $J \subset E$.

2. $Conn(E) - f$ is a *connectivity* function if the graph of $f \restriction J$ is a connected subset of $J \times \mathbb{R}$ for each connected subset J of E.

3. $AC(E) - f$ is *almost continuous* if each open subset of $E \times \mathbb{R}$ containing the graph of f also contains the graph of a continuous function $g : E \to \mathbb{R}$.

4. $Ext(E) - f$ is *extendable* if there is a connectivity function $F : E \times I \to \mathbb{R}$ such that $F(x, 0) = f(x)$ for every $x \in E$.

5. $PR - f : I \to \mathbb{R}$ has a *perfect road* if for each $x \in I$ there exists a perfect subset P of I having x as a two-sided limit point (one-sided limit point if x is an endpoint) such that $f \restriction P$ is continuous at x.

When $E = I$, we write, for example, Ext instead of $Ext(E)$. Each function space has on it the metric d of uniform convergence defined by

$$d(f, g) = \min\big\{1, \sup\{|f(x) - g(x)| \ : \ x \in E\}\big\}.$$

Suppose $E = I$ and $K \subset I \times \mathbb{R}$. For every $x \in I$, let

$$K_x = \big\{y \in \mathbb{R} \ : \ (x, y) \in K\big\}.$$

For $a \in \mathbb{R}$ and $A \subset \mathbb{R}$, $|a - A|$ denotes the distance between a and A. A closed set $K \subset I \times \mathbb{R}$ is a *blocking set* if $g \cap K \neq \emptyset$ for every continuous function $g : I \to \mathbb{R}$ but $h \cap K = \emptyset$ for some function $h : I \to \mathbb{R}$. A function $f : I \to \mathbb{R}$ is almost continuous if and only if $f \cap K \neq \emptyset$ for every blocking set K. On the other hand, if $h : I \to \mathbb{R}$ is not almost continuous, then there exists a minimal blocking set K in $I \times \mathbb{R}$ that misses the graph of h, and the x-projection of K is a non-degenerate connected set and K is a perfect set [7], [6], [5].

In a metric space (X, d), $B(x, r)$ denotes the open ball with center x and radius $r > 0$. Let $M \subset X$, $x \in X$, and $r > 0$, and let $\gamma(x, r, M)$ be the supremum of the set of all $s > 0$ for which there exists $z \in X$ such that $B(z,s) \subset B(x,r) \setminus M$. Then M is *porous at* x in X if

$$p(x,M) = \limsup_{r \to 0^+} \frac{\gamma(x,r,M)}{r} > 0\,.$$

M is *porous* in X if M is porous at every $x \in \overline{M}$. M is a *boundary set* in X if $\overline{X \setminus M} = X$. A set M which is porous in X must be a boundary set in X. Besides showing Ext is porous in AC, we show that for $n > 1$, $Ext(\mathbb{R}^n)$ is a boundary set in $D(\mathbb{R}^n)$. It would be nice to know whether $Ext(\mathbb{R}^n)$ is porous in $D(\mathbb{R}^n)$ for $n > 1$.

2 A Porous Set

Definition 6. $f : I \to \mathbb{R}$ is in *class* α if for every blocking set K and $\epsilon > 0$, either

(1) the set $\big\{x \in I \;:\; |f(x) - K_x| < \epsilon\big\}$ has cardinality c or

(2) there exists $x \in I$ such that $\big[f(x) - \epsilon, f(x) + \epsilon\big] \subset K_x$.

We need the following recent result of Szuca:

Proposition 1 (Szuca [10])**.** *$AC \subset \alpha$.*

Theorem 1. *$AC \cap PR$ is porous in AC.*

Proof. Suppose $f \in AC \cap PR$ and $0 < r \leq 1$. Note that $AC \cap PR$ is closed in AC because PR is closed in $\mathbb{R}^I$ [1]. According to Proposition 1, for each blocking set K of $I \times \mathbb{R}$ and each $r > 0$, either

(1) $\big\{x \in I \;:\; |f(x) - K_x| < r/2\big\}$ has cardinality c or

(2) there exists $x \in I$ such that $\Big[f(x) - \frac{r}{2}, f(x) + \frac{r}{2}\Big] \subset K_x$.

Let $\{K_\alpha \;:\; \alpha \in A\}$ denote the collection of all blocking sets in $I \times \mathbb{R}$ and $\{P_\alpha \;:\; \alpha \in A\}$ denote the collection of all perfect subsets of I, where A is well ordered with first element 1 and with each α in A preceded by less than c-many elements of A. We show how to use transfinite induction to obtain a function $g : I \to \mathbb{R}$ by redefining f just on a set $\{x_\alpha \;:\; \alpha \in A\}$ and on a set $\{y_\alpha, z_\alpha \;:\; \alpha \in A\}$ of distinct points in such a way that if $\alpha \in A$, then $(x_\alpha, g(x_\alpha)) \in K_\alpha$, $y_\alpha, z_\alpha \in P_\alpha$, $|f(x) - g(x)| < r/2$ for $x = x_\alpha, y_\alpha, z_\alpha$, but $|g(y_\alpha) - g(z_\alpha)| \geq r/2$.

If (1) holds for the blocking set $K = K_1$, choose

$$x_1 \in \left\{x \in I : |f(x) - (K_1)_x| < \frac{r}{2}\right\}$$

and pick $g(x_1) \in (K_1)_{x_1}$ with $\left|f(x_1) - g(x_1)\right| < \frac{r}{2}$.
But if (2), but not (1), holds for K_1, choose $x_1 \in I$ such that

$$\left[f(x_1) - \frac{r}{2}, f(x_1) + \frac{r}{2}\right] \subset (K_1)_{x_1}$$

and define $g(x_1) = f(x_1)$. Choose distinct points $y_1, z_1 \in P_1 \setminus \{x_1\}$ and define $g(y_1)$ and $g(z_1)$ so that $\left|f(x) - g(x)\right| < r/2$ for $x = y_1, z_1$ but so that $\left|g(y_1) - g(z_1)\right| \geq r/2$. Now suppose g has been defined on the set $\{x_\alpha \;:\; \alpha < \beta\}$ and on the set $\{y_\alpha, z_\alpha : \alpha < \beta\}$ of distinct points such that $g(x_\alpha) \in (K_\alpha)_{x_\alpha}$, $y_\alpha, z_\alpha \in P_\alpha$, $\left|f(x) - g(x)\right| < \frac{r}{2}$ for $x = x_\alpha$, y_α, z_α, but $\left|g(y_\alpha) - g(z_\alpha)\right| \geq \frac{r}{2}$. If (1) holds for the blocking set $K = K_\beta$, choose

$$x_\beta \in \left\{x \in I : \left|f(x) - (K_\beta)_x\right| < \frac{r}{2}\right\} \setminus \{x_\alpha, y_\alpha, z_\alpha : \alpha < \beta\}$$

and pick $g(x_\beta) \in (K_\beta)_{x_\beta}$ obeying $\left|f(x_\beta) - g(x_\beta)\right| < \frac{r}{2}$.
But if (2), but not (1), holds for K_β, choose $x_\beta \in I$ such that

$$\left[f(x_\beta) - \frac{r}{2}, f(x_\beta) + \frac{r}{2}\right] \subset (K_\beta)_{x_\beta}$$

and define

$$g(x_\beta) = \begin{cases} f(x_\beta) & \text{if (3) } x_\beta \notin \{x_\alpha, y_\alpha, z_\alpha\} \text{ for all } \alpha < \beta \\ g(x_\beta) & \text{if (4) } x_\beta \in \{x_\alpha, y_\alpha, z_\alpha\} \text{ for some } \alpha < \beta. \end{cases}$$

If (3) holds, then $\left|f(x_\beta) - g(x_\beta)\right| = 0 < \frac{r}{2}$ and $g(x_\beta) = f(x_\beta) \in (K_\beta)_{x_\beta}$. Suppose (4) holds for some $\alpha < \beta$. If $x_\beta = x_\alpha$, then $g(x_\beta) \in (K_\beta)_{x_\beta}$ because

$$\left|f(x_\beta) - g(x_\beta)\right| = \left|f(x_\alpha) - g(x_\alpha)\right| < \frac{r}{2} \quad \text{and} \quad \left[f(x_\beta) - \frac{r}{2}, f(x_\beta) + \frac{r}{2}\right] \subset (K_\beta)_{x_\beta}.$$

If $x_\beta =$ either y_α or z_α, say y_α, then

$$\left|f(x_\beta) - g(x_\beta)\right| = \left|f(y_\alpha) - g(y_\alpha)\right| < \frac{r}{2}$$

and so $g(x_\beta) \in (K_\beta)_{x_\beta}$. The case $x_\beta = z_\alpha$ is handled similarly. Choose distinct points y_β, $z_\beta \in P_\beta \setminus \{x_\alpha, y_\alpha, z_\alpha : \alpha < \beta\}$ and define $g(y_\beta)$ and $g(z_\beta)$ so that

$$\left|f(x) - g(x)\right| < \frac{r}{2} \quad \text{for} \quad x = y_\beta, z_\beta \quad \text{and} \quad \left|g(y_\beta) - g(z_\beta)\right| \geq \frac{r}{2}.$$

It follows from transfinite induction that $g : I \to \mathbb{R}$ can be obtained by redefining f on $\{x_\alpha, y_\alpha, z_\alpha \,:\, \alpha \in A\}$ in the above fashion.

Therefore $d(f,g) \le r/2$, and $g \in AC \setminus PR$ because $g \cap K_\alpha \neq \emptyset$ for every $\alpha \in A$ and because g is discontinuous on every perfect set P_α. Suppose $h \in AC$ and $d(h,g) < r/4$. Then $h \notin PR$ because h is also discontinuous on each perfect set P_α. Therefore $B\big(g, r/4\big) \subset B(f,r) \setminus PR$. $AC \cap PR$ is porous at f in AC since $\gamma(f, r, AC \cap PR) \ge \frac{r}{4}$ and $p(f, AC \cap PR) \ge \frac{1}{4} > 0$. □

If A is a subspace of B and B is porous in C, then A is porous in C [9]. According to this, the next result is an immediate consequence of Theorem 1 and the commutative diagram $(*)$.

Theorem 2. *Ext is porous in AC.*

3 A Boundary Set

This result is proved in [4]:

Proposition 2. *If $f : I^n \to I$, $n > 1$, is a Darboux and onto function and $g : I \to Y$, where Y is a metric space, is any function such that $g \circ f : I^n \to Y$ is a connectivity function, then g is continuous except perhaps at 0 or 1.*

We use the following version of it with $\mathbb{R}^n$ and $f(\mathbb{R}^n)$ replacing I^n and I respectively, and its proof is practically the same.

Proposition 3. *If $f : \mathbb{R}^n \to \mathbb{R}$, $n > 1$, is a Darboux non-constant function and $g : f(\mathbb{R}^n) \to Y$, where Y is a metric space, is any function such that $g \circ f : \mathbb{R}^n \to Y$ is a connectivity function, then g is continuous at every interior point of the interval $f(\mathbb{R}^n)$.*

For $n > 1$, $Ext(\mathbb{R}^n) = Conn(\mathbb{R}^n) \subset D(\mathbb{R}^n)$. $Ext(\mathbb{R}^n) \subset Conn(\mathbb{R}^n)$ and $Conn(\mathbb{R}^n) \subset D(\mathbb{R}^n)$ are evident from the definitions, and $Conn(\mathbb{R}^n) \subset Ext(\mathbb{R}^n)$ is shown in [3].

Theorem 3. *For $n > 1$, $Ext(\mathbb{R}^n)$ is a boundary set in $D(\mathbb{R}^n)$.*

Proof. Suppose $0 < r \le 1$ and $f : \mathbb{R}^n \to \mathbb{R}$ belongs to $Conn(\mathbb{R}^n)$. First suppose f is not a constant function. Let $i : f(\mathbb{R}^n) \to \mathbb{R}$ be the identity function on $f(\mathbb{R}^n)$, and take any Darboux function $g : f(\mathbb{R}^n) \to \mathbb{R}$ discontinuous at an interior point of $f(\mathbb{R}^n)$ such that $d(i,g) < r/2$. Then $d(f, g \circ f) = d(i,g) < r/2$. According to Proposition 3, $g \circ f \notin Conn(\mathbb{R}^n)$. The composition $g \circ f$ of two Darboux functions is Darboux, and so $g \circ f \in D(\mathbb{R}^n) \setminus Conn(\mathbb{R}^n)$. If $f = k$, a constant, then instead of to f, i, and g, we apply the previous argument to

$f_0(x,y) = k + \frac{r}{2}\sin x$ where $(x,y) \in \mathbb{R}\times\mathbb{R}^{n-1}$, to the identity $i_0 : f_0(\mathbb{R}^n) \to \mathbb{R}$, and to any Darboux discontinuous $g_0 : f_0(\mathbb{R}^n) \to \mathbb{R}$ with $d(i_0, g_0) < r/2$; note that

$$d(f, g_0 \circ f_0) \leq d(f, f_0) + d(f_0, g_0 \circ f_0) < \frac{r}{2} + \frac{r}{2} = r$$

and $g_0 \circ f_0 \in D(\mathbb{R}^n) \setminus Conn(\mathbb{R}^n)$. This shows $Ext(\mathbb{R}^n)$ is a boundary set in $D(\mathbb{R}^n)$. $\square$

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