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ON COMPLETENESS GENERATED BY CONVERGENCE WITH RESPECT TO A σ-IDEAL

Abstract

We consider convergence (introduced by E. Wagner in 1981) with respect to a σ-ideal of $\mathcal{S}$-measurable real valued functions on Y where $\mathcal{S} \subset \mathcal{P}(Y)$ is a σ-algebra containing a given σ-ideal $\mathcal{J}$. We check which operations preserve completeness generated by convergence with respect to a σ-ideal. We introduce uniform kinds of $\mathcal{J}$-convergence and $\mathcal{J}$-completeness and use them in a statement concerning the Fubini product of two σ-ideals.

1 Introduction.

It is well known that a sequence of real-valued measurable functions $(f_n)_{n\in\mathbb{N}}$ on $[0, 1]$ converges in measure to a function f if and only if each subsequence of $(f_n)_{n\in\mathbb{N}}$ contains a subsequence which converges to f almost everywhere. The same is true for functions defined on any space of finite measure. This fact was used by E. Wagner [W] to define an abstract kind of convergence with which we will be concerned. A similar convergence was considered earlier by D. Vladimirov [V] in another context. Throughout the paper, we assume that $\mathcal{S}$ is a σ-algebra of subsets of a given set Y and $\mathcal{J}$ is a proper σ-ideal (i.e., $Y \notin \mathcal{J}$) of subsets of Y. We briefly say that $\mathcal{S}$ and $\mathcal{J}$ are a σ-algebra and a σ-ideal on Y. In this section we assume additionally that $\mathcal{J} \subset \mathcal{S}$. (In the next sections we will drop this assumption to extend the sense of some notions.) By $\mathcal{F}(\mathcal{S})$ we denote the set of all $\mathcal{S}$-measurable real-valued functions on Y.

We say that a given property holds $\mathcal{J}$-*almost everywhere* (in short $\mathcal{J}$-a.e.) on Y if the set of all elements $y \in Y$ that do not have this property, belongs to $\mathcal{J}$. A sequence $(f_n)_{n \in \mathbb{N}}$ of functions from $\mathcal{F}(\mathcal{S})$ is called *convergent to $f \in \mathcal{F}(\mathcal{S})$ with respect to $\mathcal{J}$* if each subsequence of $(f_n)_{n \in \mathbb{N}}$ contains a subsequence convergent $\mathcal{J}$-almost everywhere to f. This is written as $f_n \xrightarrow{\mathcal{J}} f$. In our considerations, as in the classical measure case, we might assume that functions from $\mathcal{F}(\mathcal{S})$ are defined $\mathcal{J}$-almost everywhere and have values in $\mathbb{R} \cup \{-\infty, +\infty\}$ but are finite $\mathcal{J}$-almost everywhere. This however does not lead to essentially more general situation, since using the respective equivalence relation, we come back to the case of finite valued functions defined everywhere on Y. Note that a limit of a sequence $(f_n)_{n \in \mathbb{N}}$ in $\mathcal{F}(\mathcal{S})$ convergent with respect to $\mathcal{J}$ is not uniquely determined, however any two limits are $\mathcal{J}$-a.e. equal on Y. Indeed, assume that $f_n \xrightarrow{\mathcal{J}} f$ and $f_n \xrightarrow{\mathcal{J}} g$. Pick a subsequence (f_{n_k}) which is $\mathcal{J}$-a.e. convergent to f, and then pick a subsequence $(f_{n_{k_p}})$ which is $\mathcal{J}$-a.e. convergent to g. But $(f_{n_{k_p}})$ is also $\mathcal{J}$-a.e. convergent to f, so f and g are equal $\mathcal{J}$-a.e. on Y.

We use standard set theoretic notation. We will identify an ordinal with the set of its predecessors. Cardinals will be treated as initial ordinals. We denote by $\mathbb{N}^{\uparrow}$ the set of all increasing sequences of positive integers. Let $\mathcal{P}(E)$ stand for the power set of a set E, and let $[E]^{<\kappa}$, $[E]^{\leq \kappa}$ denote the families of subsets of E of cardinality $< \kappa$ and $\leq \kappa$, respectively.

Let us recall the classical example which will be useful further in the paper. In a sense, this example is universal since the same functions f, f_n, $n \in \mathbb{N}$, in $\mathcal{F}(\mathcal{S})$ witness that convergence "$\xrightarrow{\mathcal{J}}$" need not imply convergence "$\rightarrow, \mathcal{J}$-a.e.", for a large class of pairs $(\mathcal{S}, \mathcal{J})$ on $[0, 1]$.

Example 1. As usual, χ_A denotes the characteristic function of a set $A \subset Y$. Let $Y := [0, 1]$ and write the functions of the sequence

$$\chi_{[0,1/2]}, \chi_{[1/2,1]}, \chi_{[0,1/4]}, \chi_{[1/4,1/2]}, \chi_{[1/2,3/4]}, \chi_{[3/4,1]}, \chi_{[0,1/8]}, \ldots$$

as $f_1, f_2, f_3, \ldots$. Let $f \equiv 0$ on $[0,1]$. Then $f_n \rightarrow f$ in measure but $f_n(x) \rightarrow f(x)$ is false, for all $x \in [0, 1]$. We will show that this example works for a wide class of σ-algebras and σ-ideals. Clearly, the functions f_n, $n \in \mathbb{N}$, are Borel measurable. Let $\mathcal{S}$ be any σ-algebra on $[0, 1]$ containing Borel sets. Observe that $f_n \xrightarrow{\mathcal{J}} f$ where $\mathcal{J} \subset \mathcal{S}$ is any proper σ-ideal containing all countable subsets of Y. Indeed, if $(n_k) \in \mathbb{N}^{\uparrow}$ then we can find $(k_p) \in \mathbb{N}^{\uparrow}$ such that $f_{n_{k_p}} \rightarrow f$ everywhere except for at most one point from Y.

Due to the Riesz theorem, convergence in measure on $[0, 1]$ is equivalent to the respective Cauchy condition — if it is satisfied, a sequence of measurable functions is called fundamental in measure. As it was observed in [WBW], the

Cauchy condition in measure can be described without the use of measure. This leads to the notion of a sequence fundamental in category (when one uses as $\mathcal{J}$ the σ-ideal of meager subsets of $[0,1]$). The main result of [WBW] states that a sequence of functions with the Baire property is convergent in category on $[0,1]$ (i.e., convergent with respect to the σ-ideal of meager sets) if and only if it is fundamental in category. The proof of sufficiency is highly nontrivial — it yields a kind of completeness of the respective function space.

One can easily extend the considerations from [WBW] to the general case of sequences in $\mathcal{F}(\mathcal{S})$. Namely, we say that a sequence $(f_n)_{n \in \mathbb{N}}$ in $\mathcal{F}(\mathcal{S})$ is $\mathcal{J}$-*Cauchy* (or $\mathcal{J}$-*fundamental*) if, for any $(m_k)_{k \in \mathbb{N}}, (n_k)_{k \in \mathbb{N}} \in \mathbb{N}^{\uparrow}$, the sequence $(f_{m_k} - f_{n_k})_{k \in \mathbb{N}}$ converges, with respect to $\mathcal{J}$, to the constant function equal to zero on Y. Observe that (f_n) is $\mathcal{J}$-Cauchy if and only if for any $(m_k), (n_k) \in \mathbb{N}^{\uparrow}$ there is $(k_p) \in \mathbb{N}^{\uparrow}$ such that $(f_{m_{k_p}} - f_{n_{k_p}})_{p \in \mathbb{N}}$ tends to zero $\mathcal{J}$-a.e. on Y.

The following proposition can be proved similarly as in [WBW, Thm. 1].

Proposition 2. *For every sequence (f_n) of functions from $\mathcal{F}(\mathcal{S})$, if (f_n) converges with respect to $\mathcal{J}$ to a function $f \in \mathcal{F}(\mathcal{S})$, then (f_n) is a $\mathcal{J}$-Cauchy sequence.*

If every $\mathcal{J}$-Cauchy sequence in $\mathcal{F}(\mathcal{S})$ is convergent, with respect to $\mathcal{J}$, to a function $f \in \mathcal{F}(\mathcal{S})$ (i.e. the converse of implication from Proposition 2 holds), we say that $\mathcal{F}(\mathcal{S})$ is $\mathcal{J}$-*complete*.

Consider the following simple example.

Example 3. By $\{\emptyset\}_Y$ we denote the σ-ideal on Y consisting of $\emptyset$. Put $\mathcal{J} := \{\emptyset\}_Y$ and let $\mathcal{S}$ be an arbitrary σ-algebra on Y. Assume that $f_n \in \mathcal{F}(\mathcal{S})$, $n \in \mathbb{N}$, and $f \in \mathcal{F}(\mathcal{S})$. We will show that $f_n \xrightarrow{\mathcal{J}} f$ is equivalent to the usual pointwise convergence $f_n \to f$; this fact was mentioned in [W]. Let us start from an observation that a sequence $(a_n)_{n \in \mathbb{N}}$ of real numbers converges to $a \in \mathbb{R}$ if and only if every subsequence of (a_n) contains a subsequence convergent to a. Knowing this, assume that $f_n \xrightarrow{\mathcal{J}} f$. Thus for each $(n_k) \in \mathbb{N}^{\uparrow}$ there exists $(k_p) \in \mathbb{N}^{\uparrow}$ such that $f_{n_{k_p}}(x) \to f(x)$ for all $x \in Y$. By the above observation, if we put the quantifier "for all $x \in Y$" in front of the others, we obtain $f_n \to f$. Conversely, assume that $f_n \to f$ and suppose that $f_n \xrightarrow{\mathcal{J}} f$ is false. Hence there is $(n_k) \in \mathbb{N}^{\uparrow}$ such that for every $(k_p) \in \mathbb{N}^{\uparrow}$ we can find $x \in Y$ such that $f_{n_{k_p}}(x) \to f(x)$ is false. By considering $(k_p) = (p)$ this contradicts $f_n(x) \to f(x)$. Similarly, it can be shown that $(f_n)_{n \in \mathbb{N}}$ is $\mathcal{J}$-Cauchy if and only if $(f_n(x))_{n \in \mathbb{N}}$ is a Cauchy sequence for every $x \in Y$. Hence we conclude that $\mathcal{F}(\mathcal{S})$ is $\mathcal{J}$-complete, by the completeness of $\mathbb{R}$.

Let $\mathcal{M}$ and $\mathcal{N}$ stand for the σ-ideals of meager sets (i.e., of first category) in $\mathbb{R}$ and of Lebesgue null sets in $\mathbb{R}$. Their restrictions to an interval $I \subset \mathbb{R}$

will be denoted by $\mathcal{M}_I$ and $\mathcal{N}_I$. Let BAIRE and LEB stand for the σ-algebras of subsets of $\mathbb{R}$ with the Baire property and of Lebesgue measurable subsets of $\mathbb{R}$, respectively. Their restrictions to I will be written as BAIRE_I and LEB_I. Symbols $\mathcal{M}(Z)$ and $\text{BAIRE}(Z)$ have the same meanings as $\mathcal{M}$ and BAIRE but they concern an uncountable Polish space Z taken in place of $\mathbb{R}$.

Fact 4.   • $\mathcal{F}(\text{LEB}_{[0,1]})$ *is* $\mathcal{N}_{[0,1]}$*-complete (the classical Riesz theorem)*;
  • $\mathcal{F}(\text{BAIRE}_{[0,1]})$ *is* $\mathcal{M}_{[0,1]}$*-complete [WBW, Thm. 2]*;
  • $\mathcal{F}(\text{LEB}_{[0,1]} \cap \text{BAIRE}_{[0,1]})$ *is* $\mathcal{N}_{[0,1]} \cap \mathcal{M}_{[0,1]}$*-complete [WBW, Thm. 2a]*.

Our aim is to study, under which additional conditions, the completeness of $\mathcal{F}(\cdot)$ is preserved when various operations on σ-ideals are considered and the associated σ-algebras are respectively transformed. In Section 2 we obtain results concerning operations of restriction, intersection and direct sum. To obtain the respective statement (Proposition 24) for the Fubini product of two σ-ideals, we use the notion of uniform $\mathcal{J}$-completeness introduced in Section 3.

Several pairs $(\mathcal{S}, \mathcal{J})$ have not been investigated from the $\mathcal{J}$-completeness point of view. On the other hand, we know only one example of non-$\mathcal{J}$-completeness given in [V]. Let us recall it. Consider the partial order $\leq^*$ on $\mathbb{N}^\uparrow$ defined by $(m_j) \leq^* (n_j)$ if there exists $k \in \mathbb{N}$ such that $m_j \leq n_j$ for all $j \geq k$. A set $T \subset \mathbb{N}^\uparrow$ is called *bounded* if there exists $(n_j) \in \mathbb{N}^\uparrow$ such that $(m_j) \leq^* (n_j)$ for all $(m_j) \in T$. Let $\mathfrak{b}$ denote the minimal cardinality of an unbouned subset of $\mathbb{N}^\uparrow$, and let $\mathfrak{c} := |\mathbb{R}|$. (Originally, see [vD], the number $\mathfrak{b}$ is defined analogously for unbounded sets in $\mathbb{N}^\mathbb{N}$, however our approach is equivalent.) It is known that $\omega_1 \leq \mathfrak{b} \leq \mathfrak{c}$ and each of the conditions $\omega_1 = \mathfrak{b} < \mathfrak{c}$, $\omega_1 < \mathfrak{b} < \mathfrak{c}$, $\omega_1 < \mathfrak{b} = \mathfrak{c}$ is consistent (cf. [vD]). Obviously the Continuum Hypothesis (CH) implies $\mathfrak{b} = \mathfrak{c}$.

Fact 5. $\mathcal{F}(\mathcal{P}(\mathfrak{b}))$ *is not* $[\mathfrak{b}]^{<\mathfrak{b}}$*-complete. (See [V].)*

2   Some Operations Preserving $\mathcal{J}$-Completeness.

Let $\mathcal{S}$ and $\mathcal{J}$ be a σ-algebra and a σ-ideal on Y. Note that convergence with respect to $\mathcal{J}$, and also $\mathcal{J}$-Cauchy condition and $\mathcal{J}$-completeness, defined as in Section 1, make sense without assuming $\mathcal{J} \subset \mathcal{S}$. Namely, put

$$\mathcal{J}{\upharpoonright}\mathcal{S} := \{A \subset Y \colon (\exists B \in \mathcal{J} \cap \mathcal{S}) A \subset B\}.$$

Then $\mathcal{J}{\upharpoonright}\mathcal{S}$ is the σ-ideal generated by $\mathcal{J} \cap \mathcal{S}$. If $f, f_n \in \mathcal{F}(\mathcal{S})$, $n \in \mathbb{N}$, observe that "$f_n \to f$, $\mathcal{J}$-a.e.," is the same as the statement "$f_n \to f$, $\mathcal{J}{\upharpoonright}\mathcal{S}$-a.e.". Hence it follows that $f_n \xrightarrow{\mathcal{J}} f$ is equivalent to $f_n \xrightarrow{\mathcal{J}{\upharpoonright}\mathcal{S}} f$. So, our extended definition has been reduced to the case of the pair $(\mathcal{S}, \mathcal{J}{\upharpoonright}\mathcal{S})$ with $\mathcal{J}{\upharpoonright}\mathcal{S} \subset \mathcal{S}$. Similarly, we

interpret the extended notions of a $\mathcal{J}$-Cauchy sequence and $\mathcal{J}$-completeness of $\mathcal{F}(\mathcal{S})$. In fact this setting is not more general but it is useful from the technical standpoint.

We say that a family $\mathcal{B} \subset \mathcal{P}(Y)$ is a *basis* of a σ-ideal $\mathcal{J}$ on Y if $\mathcal{B} \subset \mathcal{J}$ and each set from $\mathcal{J}$ is contained in a set from $\mathcal{B}$. Let $A \triangle B := (A \setminus B) \cup (B \setminus A)$ for $A, B \subset Y$. Put

$$\mathcal{S} \triangle \mathcal{J} := \{A \triangle B \colon A \in \mathcal{S} \text{ and } B \in \mathcal{J}\};$$

this is the σ-algebra generated by $\mathcal{S} \cup \mathcal{J}$. It is known that

$$\mathcal{F}(\mathcal{S} \triangle \mathcal{J}) = \{f \in \mathbb{R}^Y : (\exists g \in \mathcal{F}(\mathcal{S}))f = g, \ \mathcal{J}\text{-a.e.}\}$$

provided $\mathcal{J}$ has a basis contained in $\mathcal{S}$ (equivalently, if $\mathcal{J} = \mathcal{J}{\upharpoonright}\mathcal{S}$), see [F, 1D(c), pp 7–8]. Hence one can easily check that, if $\mathcal{J}$ has a basis contained in $\mathcal{S}$, then $\mathcal{F}(\mathcal{S} \triangle \mathcal{J})$ is $\mathcal{J}$-complete if and only if $\mathcal{F}(\mathcal{S})$ is $\mathcal{J}$-complete. Therefore the case "$\mathcal{J}$ has a basis contained in $\mathcal{S}$" is principally not more general than the case "$\mathcal{J} \subset \mathcal{S}$". When we do not speak about a basis, it follows that $\mathcal{F}(\mathcal{S} \triangle (\mathcal{J}{\upharpoonright}\mathcal{S}))$ is $\mathcal{J}{\upharpoonright}\mathcal{S}$-complete iff $\mathcal{F}(\mathcal{S})$ is $\mathcal{J}{\upharpoonright}\mathcal{S}$-complete iff $\mathcal{F}(\mathcal{S})$ is $\mathcal{J}$-complete.

In this section we study several operations preserving $\mathcal{J}$-completeness. First, consider the operation of restriction.

Proposition 6. *Fix a σ-algebra $\mathcal{S}$ and a σ-ideal $\mathcal{J}$ on Y. Let $Y_0 \subset Y$, $Y_0 \in \mathcal{S} \setminus \mathcal{J}$ and put $\mathcal{J}{\upharpoonright}Y_0 := \mathcal{J} \cap \mathcal{P}(Y_0)$, $\mathcal{S}{\upharpoonright}Y_0 := \mathcal{S} \cap \mathcal{P}(Y_0)$. We then have:*
  (a) $\mathcal{J}{\upharpoonright}Y_0$ is a σ-ideal and $\mathcal{S}{\upharpoonright}Y_0$ is a σ-algebra on Y_0.
  (b) $\mathcal{F}(\mathcal{S}{\upharpoonright}Y_0) = \{f{\upharpoonright}Y_0 \colon f \in \mathcal{F}(\mathcal{S})\}$.
  (c) If $\mathcal{F}(\mathcal{S})$ is $\mathcal{J}$-complete, then $\mathcal{F}(\mathcal{S}{\upharpoonright}Y_0)$ is $\mathcal{J}{\upharpoonright}Y_0$-complete.
  (d) If $Y \setminus Y_0 \in \mathcal{J}$, then $\mathcal{F}(\mathcal{S})$ is $\mathcal{J}$-complete if and only if $\mathcal{F}(\mathcal{S}{\upharpoonright}Y_0)$ is $\mathcal{J}{\upharpoonright}Y_0$-complete.

Proof. The proofs of (a), (b) and (d) are self-evident. (In the proof of (d) we use (c).) To show (c) consider a $\mathcal{J}{\upharpoonright}Y_0$-Cauchy sequence $(f_n)_{n \in \mathbb{N}}$ in $\mathcal{F}(\mathcal{S}{\upharpoonright}Y_0)$. Extend each f_n to g_n defined on the whole Y by putting $g_n(x) := 0$ for $x \in Y \setminus Y_0$. Then $(g_n)_{n \in \mathbb{N}}$ is $\mathcal{J}$-Cauchy in $\mathcal{F}(\mathcal{S})$. By assumption there is $g \in \mathcal{F}(\mathcal{S})$ such that $g_n \overset{\mathcal{J}}{\to} g$. Hence $f_n \overset{\mathcal{J}{\upharpoonright}Y_0}{\to} f$ where $f := g{\upharpoonright}Y_0 \in \mathcal{F}(\mathcal{S}{\upharpoonright}Y_0)$. $\square$

Now, let us examine the operation of countable intersection. If a σ-algebra $\mathcal{S}$ on Y is fixed, we say that two σ-ideals $\mathcal{J}_1$ and $\mathcal{J}_2$ on Y are $\mathcal{S}$-*orthogonal*, if there is $A \in \mathcal{S}$ such that $A \in \mathcal{J}_1$ and $Y \setminus A \in \mathcal{J}_2$.

Proposition 7. *Assume that $\mathcal{S}_i$ is a σ-algebra and $\mathcal{J}_i$ is a σ-ideal on Y for $i \in M$ where $M \neq \emptyset$. Let $\mathcal{J} = \bigcap_{i \in M} \mathcal{J}_i$ and $\mathcal{S} = \bigcap_{i \in M} \mathcal{S}_i$. We then have:*

(a) $\mathcal{J}$ is a σ-ideal and $\mathcal{S}$ is a σ-algebra on Y. If $\mathcal{J}_i{\restriction}\mathcal{S} = \mathcal{J}_i$ for all $i \in M$, then $\mathcal{J}{\restriction}\mathcal{S} = \mathcal{J}$. If $\mathcal{J}_i \subset \mathcal{S}_i$ for all $i \in M$, then $\mathcal{J} \subset \mathcal{S}$.

(b) $\mathcal{F}(\bigcap_{i \in M} \mathcal{S}_i) = \bigcap_{i \in M} \mathcal{F}(\mathcal{S}_i)$.

(c) If M is countable, $\mathcal{J}_i{\restriction}\mathcal{S} = \mathcal{J}_i$ for $i \in M$ and $\mathcal{F}(\mathcal{S}_i)$ is $\mathcal{J}_i$-complete for $i \in M$, then $\mathcal{F}(\mathcal{S})$ is $\mathcal{J}$-complete.

(d) If M is countable, $\mathcal{S}_i = \mathcal{S}$ and $\mathcal{J}_i{\restriction}\mathcal{S} = \mathcal{J}_i$ for $i \in M$, and $\mathcal{J}_i$, $i \in M$, are pairwise $\mathcal{S}$-orthogonal, then $\mathcal{F}(\mathcal{S})$ is $\bigcap_{i \in M} \mathcal{J}_i$-complete if and only if $\mathcal{F}(\mathcal{S})$ is $\mathcal{J}_i$-complete for all $i \in \mathcal{J}_i$.

PROOF. Assertions (a) and (b) are clear. We will prove (c) in the case $M = \mathbb{N}$. Let $(f_n)_{n \in \mathbb{N}}$ be $\mathcal{J}$-Cauchy in $\mathcal{F}(\mathcal{S})$. Then it is $\mathcal{J}_i$-Cauchy in $\mathcal{F}(\mathcal{S}_i)$ for each $i \in \mathbb{N}$. Consider $(n_k) \in \mathbb{N}^{\uparrow}$. Put $A_0 := Y$ and $r(1) := 1$. Since $\mathcal{F}(\mathcal{S}_1)$ is $\mathcal{J}_1$-complete, pick $g^{(1)} \in \mathcal{F}(\mathcal{S}_1)$ such that $f_n \overset{\mathcal{J}_1}{\to} g^{(1)}$. Hence we find $(k_p^{(1)}) \in \mathbb{N}^{\uparrow}$ such that

$$A_1 := \left\{ x \in A_0 \colon \lim_{p \to \infty} f_{n_{k_p^{(1)}}}(x) \neq g^{(1)}(x) \right\} \in \mathcal{J}_{r(1)}.$$

Since $\mathcal{J}_1{\restriction}\mathcal{S} = \mathcal{J}_1$, we may include A_1 in $B_1 \in \mathcal{J}_1 \cap \mathcal{S}$. For simplicity assume $B_1 = A_1$. Then we proceed inductively. If $A_1 \in \mathcal{J}$, we are done. Otherwise, let $r(2) := \min\{j \in \mathbb{N} \colon A_1 \notin \mathcal{J}_j\}$. Hence $A_1 \in \bigcap_{i=1}^{r(2)-1} \mathcal{J}_i \backslash \mathcal{J}_{r(2)}$. Since $\mathcal{F}(\mathcal{S}_{r(2)}{\restriction}A_1)$ is $(\mathcal{J}_{r(2)}{\restriction}A_1)$-complete (cf. Proposition 6 (c)), pick $g^{(2)} \in \mathcal{F}(\mathcal{S}_{r(2)}{\restriction}A_1)$ such that $f_n{\restriction}A_1 \overset{\mathcal{J}_{r(2)}{\restriction}A_1}{\to} g^{(2)}$ on A_1. Hence we find a subsequence $(k_p^{(2)})_{p \in \mathbb{N}}$ of $(k_p^{(1)})_{p \in \mathbb{N}}$ such that

$$A_2 = \left\{ x \in A_1 \colon \lim_{p \to \infty} f_{n_{k_p^{(2)}}}(x) \neq g^{(2)}(x) \right\} \in \mathcal{J}_{r(2)}.$$

As before, we can enlarge A_2 to have $A_2 \in \mathcal{J}_{r(2)} \cap \mathcal{S}$. Then we pick $r(3)$ if possible. If our procedure is infinite, we obtain a sequence $r(1) < r(2) < \dots$ of positive integers and the respective sequence of sets $A_0 \supset A_1 \supset \dots$ with $A_i \in \mathcal{J}_{r(i)} \cap \mathcal{S}$ for all $i \in \mathbb{N}$. Define $f \in \mathcal{F}(\mathcal{S})$ by putting $f := g^{(i)}$ on $A_{i-1} \setminus A_i$, $i \in \mathbb{N}$ and $f := 0$ on $\bigcap_{i \in \mathbb{N}} A_i$ (if this set is nonempty). We conclude that the diagonal sequence $(f_{n_{k_p^{(p)}}})_{p \in \mathbb{N}}$ converges $\mathcal{J}$-a.e. (more exactly, everywhere except for the set $\bigcap_{i \in \mathbb{N}} A_i \in \mathcal{J}$) to f on Y. If the procedure needs only finitely many steps, the argument is similar.

(d) We assume that $M \subset \mathbb{N}$. There is a partition $\{Y_n : n \in M\} \subset \mathcal{S}$ of Y such that $Y \setminus Y_n \in \mathcal{J}_n$ and $\mathcal{J}{\restriction}Y_n = \mathcal{J}_n{\restriction}Y_n$. To see this let $A_{n,m} \in \mathcal{S}$ for $n < m$, $n, m \in M$ be such that $A_{n,m} \in \mathcal{J}_n$ and $Y \setminus A_{n,m} \in \mathcal{J}_m$ whenever $n < m$. Put $Y_n := \bigcap_{k<n} A_{k,n} \cap \bigcap_{k>n}(Y \setminus A_{n,k})$ for $n \in M$. Then $Y \setminus Y_n \in \mathcal{J}_n$, $n \in M$, and $Y_n \cap Y_m = \emptyset$ if $n \neq m$. Consequently, $Y_n \in \bigcap_{k \neq n} \mathcal{J}_k$ for all $n \in M$ and thus $\mathcal{J}_n{\restriction}Y_n = \mathcal{J}{\restriction}Y_n$, $n \in M$. We have $Y \setminus \bigcup_{n \in M} Y_n \in \mathcal{S} \cap \mathcal{J}$, so we may

add this set to one of Y_n's and then $\{Y_n \colon n \in M\}$ is as desired. Now, the implication "$\Leftarrow$" in (d) follows from (c) for $\mathcal{S}_n = \mathcal{S}$. We prove "$\Rightarrow$". Assume that $\mathcal{F}(\mathcal{S})$ is $\mathcal{J}$-complete and let $i \in M$. Then $\mathcal{F}(\mathcal{S}{\upharpoonright}Y_i)$ is $\mathcal{J}{\upharpoonright}Y_i$-complete by Proposition 6 (c), $\mathcal{F}(\mathcal{S}{\upharpoonright}Y_i)$ is $\mathcal{J}_i{\upharpoonright}Y_i$-complete because $\mathcal{J}_i{\upharpoonright}Y_i = \mathcal{J}{\upharpoonright}Y_i$, and finally, $\mathcal{F}(\mathcal{S})$ is $\mathcal{J}_i$-complete by Proposition 6 (d). $\qquad\square$

It is well known that each of the σ-ideals $\mathcal{M}$ and $\mathcal{N}$ has a basis consisting of Borel sets. (See [O].) Thus by Proposition 7 (c) we have the following corollary.

Corollary 8. $\mathcal{F}(\mathrm{BAIRE} \cap \mathrm{LEB})$ *is* $\mathcal{M} \cap \mathcal{N}$*-complete.*

Next, look at the operation of direct sum. The main property will follow from Proposition 7 since direct sum is closely related to a particular case of intersection.

Proposition 9. *For a nonempty set M, let $(Y_i)_{i \in M}$ be a family of pairwise disjoint sets, and for each $i \in M$, assume that $\mathcal{J}_i$ and $\mathcal{S}_i$ are a σ-ideal and a σ-algebra on Y_i. Put $Y := \bigcup_{i \in M} Y_i$, $\bigoplus_{i \in M} \mathcal{J}_i := \{A \subset Y \colon (\forall i \in M) A \cap Y_i \in \mathcal{J}_i\}$ and $\bigoplus_{i \in M} \mathcal{S}_i := \{A \subset Y \colon (\forall i \in M) A \cap Y_i \in \mathcal{S}_i\}$. We then have:*

(a) $\bigoplus_{i \in M} \mathcal{J}_i$ is a σ-ideal and $\bigoplus_{i \in M} \mathcal{S}_i$ is a σ-algebra on Y.

(b) $\mathcal{F}(\bigoplus_{i \in M} \mathcal{S}_i) = \{f \in \mathbb{R}^Y \colon (\forall i \in M) f{\upharpoonright}Y_i \in \mathcal{F}(\mathcal{S}_i)\}$.

(c) If M is countable, then all $\mathcal{F}(\mathcal{S}_i)$, $i \in M$, are $\mathcal{J}_i$-complete if and only if $\mathcal{F}(\bigoplus_{i \in M} \mathcal{S}_i)$ is $\bigoplus_{i \in M} \mathcal{J}_i$-complete.

PROOF. Assertions (a) and (b) are immediate. We will prove (c). Let $\mathcal{J}_i' = \{A \subset Y \colon A \cap Y_i \in \mathcal{J}_i\}$, $i \in M$. Then $\mathcal{J}_i = \mathcal{J}_i'{\upharpoonright}Y_i$, $Y \setminus Y_i \in \mathcal{J}_i'$, and $\mathcal{J}_i'$, $i \in M$, are pairwise $\mathcal{S}$-orthogonal. Put $\mathcal{J} := \bigoplus_{i \in M} \mathcal{J}_i$ and $\mathcal{S} := \bigoplus_{i \in M} \mathcal{S}_i$. Then $\mathcal{J} = \bigcap_{i \in M} \mathcal{J}_i'$. By Proposition 7 (d) and Proposition 6 (d) we have: $\mathcal{F}(\mathcal{S})$ is $\mathcal{J}$-complete iff $\mathcal{F}(\mathcal{S})$ is $\mathcal{J}_i'$-complete for all i iff $\mathcal{F}(\mathcal{S}{\upharpoonright}Y_i)$ is $\mathcal{J}_i$-complete for all i. $\qquad\square$

Let us explain why convergence in measure on $\mathbb{R}$ and convergence with respect to $\mathcal{N}$ are not the same notions. Assume that $f_n \in \mathcal{F}(\mathrm{LEB})$, $n \in \mathbb{N}$, and $f \in \mathcal{F}(\mathrm{LEB})$. Then $f_n \to f$ in measure implies $f_n \overset{\mathcal{N}}{\to} f$. Indeed, if $f_n \to f$ in measure, then $f_n{\upharpoonright}[m, m+1) \to f{\upharpoonright}[m, m+1)$ in measure for every $m \in \mathbb{Z}$. This, by Proposition 9, implies that $f_n \overset{\mathcal{N}}{\to} f$. However, condition $f_n \overset{\mathcal{N}}{\to} f$ need not imply $f_n \to f$ in measure. As a counterexample, it suffices to consider $\chi_{[n, +\infty)}$, $n \in \mathbb{N}$. So, $\mathcal{N}$-completeness of $\mathcal{F}(\mathrm{LEB})$, which will be proved below, is not a direct consequence of the Riesz theorem.

Corollary 10. *The space* $\mathcal{F}(\text{LEB})$ *is* $\mathcal{N}$*-complete.*

Proof. We have

$$\mathcal{N} = \bigoplus_{m \in \mathbb{Z}} \mathcal{N}_{[m,m+1)} \text{ and LEB} = \bigoplus_{m \in \mathbb{Z}} \text{LEB}_{[m,m+1)}$$

and $\mathcal{F}(\text{LEB}_{[m,m+1)})$ is $\mathcal{N}_{[m,m+1)}$-complete for every $m \in \mathbb{N}$ (by the Riesz theorem). So, it is enough to apply Proposition 9(c). $\qquad\square$

Analogously, from $\mathcal{M}_{[0,1]}$-completeness of $\mathcal{F}(\text{BAIRE}_{[0,1]})$ (cf. [WBW]) we derive that $\mathcal{F}(\text{BAIRE})$ is $\mathcal{M}$-complete. We can also apply Proposition 9 to mixed direct sums. For instance, we can deduce $\mathcal{J}$-completeness of $\mathcal{F}(\mathcal{S})$ where $\mathcal{J} := \mathcal{M}_{[0,1)} \oplus \mathcal{N}_{[1,2)}$ and $\mathcal{S} := \text{BAIRE}_{[0,1)} \oplus \text{LEB}_{[1,2)}$.

Now, we will show that bijections preserve $\mathcal{J}$-completeness in an appriopriate way.

Proposition 11. *Fix a* σ*-algebra* $\mathcal{S}$ *and a* σ*-ideal* $\mathcal{J} \subset \mathcal{S}$ *on* Y*. For any bijection* $h \colon Y \to Z$ *we have:*
  (a) $h * \mathcal{J} := \{h(A) \colon A \in \mathcal{J}\}$ *is a* σ*-ideal and* $h * \mathcal{S} := \{h(A) \colon A \in \mathcal{S}\}$ *is a* σ*-algebra on* Z*.*
  (b) $\mathcal{F}(h * \mathcal{S}) = \{f \circ h^{-1} \colon f \in \mathcal{F}(\mathcal{S})\}$*.*
  (c) If $\mathcal{F}(\mathcal{S})$ *is* $\mathcal{J}$*-complete, then* $\mathcal{F}(h * \mathcal{S})$ *is* $h * \mathcal{J}$*-complete.*

Proof. Assertion (a) is clear. To show inclusion "$\subset$" in (b), assume that $g \in \mathcal{F}(h * \mathcal{S})$. Then for an open $U \subset \mathbb{R}$ we have $g^{-1}(U) \in h * \mathcal{S}$. Hence $g^{-1}(U) = h(A)$ for some $A \in \mathcal{S}$ which yields $(g \circ h)^{-1}(U) \in \mathcal{S}$. Putting $f = g \circ h$ we have $g = f \circ h^{-1}$ and $f \in \mathcal{F}(\mathcal{S})$. Similarly, we demonstrate the reverse inclusion.

To prove (c) assume that $\mathcal{F}(\mathcal{S})$ is $\mathcal{J}$-complete and consider an $h * \mathcal{J}$-Cauchy sequence $(g_n)_{n \in \mathbb{N}}$ in $\mathcal{F}(h * \mathcal{S})$. Then $(g_n \circ h)_{n \in \mathbb{N}}$ is a $\mathcal{J}$-Cauchy sequence in $\mathcal{F}(\mathcal{S})$. Hence $g_n \circ h \xrightarrow{\mathcal{J}} f$ for some $f \in \mathcal{F}(\mathcal{S})$. Observe that $g_n \xrightarrow{h*\mathcal{J}} f \circ h^{-1}$ and $f \circ h^{-1} \in \mathcal{F}(h * \mathcal{S})$. $\qquad\square$

Corollary 12. $\mathcal{F}(\text{BAIRE}(Z))$ *is* $\mathcal{M}(Z)$*-complete for any uncountable Polish space* Z *without isolated points.*

Proof. By Proposition 11 (c), it suffices to apply the second assertion of Fact 4 and the fact that there is a Borel isomorphism $h \colon [0,1] \to Z$ (that is, h is a bijection and h, h^{-1} are Borel measurable) preserving the Baire category; see [CKW, Thm 3.15]. $\qquad\square$

Corollary 13. $\mathcal{F}(\mathcal{P}(\mathfrak{c}))$ *is not* $[\mathfrak{c}]^{<\mathfrak{b}}$*-complete. Hence if* $\mathfrak{b} = \omega_1$*, then* $\mathcal{F}(\mathcal{P}(\mathbb{R}))$ *is not* $[\mathbb{R}]^{\leq \omega}$*-complete.*

Proof. The first assertion follows from Fact 5 and Proposition 6 (c). To show the second assertion we use the first assertion and Proposition 11 with a bijection between $\mathfrak{c}$ and $\mathbb{R}$. $\qquad\square$

Problem 14. *Is the last consequence in Corollary 13 provable in ZFC?*

3 Uniform $\mathcal{J}$-Convergence and Uniform $\mathcal{J}$-Completeness.

Let $\mathcal{S}$ and $\mathcal{J}$ be a σ-algebra and a σ-ideal on Y. Additionally, fix a set $X \neq \emptyset$. For $f \colon X \times Y \to \mathbb{R}$ and $x \in X$ we write as $(f)_x \colon Y \to \mathbb{R}$ the function given by $(f)_x(y) := f(x,y)$, $y \in Y$. The set of all functions $f \colon X \times Y \to \mathbb{R}$ such that $(f)_x \in \mathcal{F}(\mathcal{S})$ for all $x \in X$ will be written as $\mathcal{F}_X(\mathcal{S})$. If $A \subset X \times Y$ and $x \in X$, we let $(A)_x := \{y \in Y \colon (x,y) \in A\}$.

We introduce two kinds of parametric convergence of $(f_n)_{n \in \mathbb{N}}$ to f in $\mathcal{F}_X(\mathcal{S})$. They are defined by

$$f_n \xrightarrow{X,\mathcal{J}} f \text{ iff } (\forall x \in X)(f_n)_x \xrightarrow{\mathcal{J}} (f)_x$$

and

$$f_n \xrightarrow{\mathcal{J},X} f \text{ iff } (\forall (n_k) \in \mathbb{N}^\uparrow)(\exists (k_p) \in \mathbb{N}^\uparrow)(\forall x \in X)(f_{n_{k_p}})_x \to (f)_x \; \mathcal{J}\text{-a.e. on } Y.$$

Of course, $f_n \xrightarrow{\mathcal{J},X} f$ implies $f_n \xrightarrow{X,\mathcal{J}} f$. The converse need not hold but sometimes it is true. For instance, if X is countable, then, using the "diagonal technique" applied in the proof of Proposition 7 (c), one can demonstrate that the both types of convergence are the same. Let us consider two examples with particular $\mathcal{J}$.

Example 15. Let $\mathcal{J} = \{\emptyset\}_Y$ and let $\mathcal{S} \subset \mathcal{P}(Y)$ be an arbitrary σ-algebra. Fix a set $X \neq \emptyset$. In this case, for any $f_n \in \mathcal{F}_X(\mathcal{S})$, $n \in \mathbb{N}$, and $f \in \mathcal{F}_X(\mathcal{S})$ we have

$$(f_n \xrightarrow{X,\mathcal{J}} f) \Rightarrow (f_n \xrightarrow{\mathcal{J},X} f).$$

Indeed, assume that $f_n \xrightarrow{X,\mathcal{J}} f$. Then $(f_n)_x \to (f)_x$ for each $x \in X$ (cf. Example 3). Suppose that $f_n \xrightarrow{\mathcal{J},X} f$ is false. Then there is $(n_k) \in \mathbb{N}^\uparrow$ such that for every $(k_p) \in \mathbb{N}^\uparrow$ we can find $x \in X$ violating $(f_{n_{k_p}})_x \to (f)_x$. By considering $(k_p) = (p)$, this last condition contradicts our assumption.

Example 16. We follow notation from Example 1. Consider the respective pair $(\mathcal{S}, \mathcal{J})$ on $[0,1]$ and pick $f_n, f \in \mathcal{F}(\mathcal{S})$, $n \in \mathbb{N}$, with $f_n \xrightarrow{\mathcal{J}} f$, chosen as

in Example 1. Let $X := \mathbb{N}^{\uparrow}$ and for an $x \in \mathbb{N}^{\uparrow}$ let $x = (m_n^x)_{n \in \mathbb{N}}$. Define $g_k \colon X \times [0,1] \to \mathbb{R}$, $k \in \mathbb{N}$, by putting $g_k(x,y) := f_n(y)$ if $k = m_n^x$, and $g_k(x,y) := 0$ otherwise. Let $g(x,y) := 0$ for all $(x,y) \in X \times [0,1]$. It is obvious that $(g_k)_x \xrightarrow{\mathcal{J}} (g)_x = f$ for all $\in X$. Hence $g_k \xrightarrow{X,\mathcal{J}} g$. To show that $g_k \xrightarrow{\mathcal{J},X} g$ is false, take an arbitrary subsequence $(r_n)_{n \in \mathbb{N}}$ of the sequence $(n)_{n \in \mathbb{N}}$. Put $x := (r_n)$. Hence $r_n = m_n^x$ for all $n \in \mathbb{N}$. Since $f_n \to f$, $\mathcal{J}$-a.e., does not hold, by the definition of $(g_k)_{k \in \mathbb{N}}$ we infer that $(g_{r_n})_x \to (g)_x = f$, $\mathcal{J}$-a.e., does not hold.

Convergence $\xrightarrow{\mathcal{J},X}$ in $\mathcal{F}_X(\mathbb{S})$ will be called *uniform convergence with respect to* $\mathcal{J}$, or simply *uniform $\mathcal{J}$-convergence*. We say that a sequence $(f_n)_{n \in \mathbb{N}}$ in $\mathcal{F}_X(\mathbb{S})$ is *uniformly $\mathcal{J}$-fundamental* if for any $(m_k), (n_k) \in \mathbb{N}^{\uparrow}$ there is $(k_p) \in \mathbb{N}^{\uparrow}$ such that $((f_{m_{k_p}})_x - (f_{n_{k_p}})_x)_{p \in \mathbb{N}}$ tends to zero $\mathcal{J}$-a.e. for all $x \in X$. It is not hard to prove that every uniformly $\mathcal{J}$-convergent sequence in $\mathcal{F}_X(\mathbb{S})$ is uniformly $\mathcal{J}$-fundamental (cf. Proposition 2 and [WBW, Thm 1]). If every uniformly $\mathcal{J}$-fundamental sequence in $\mathcal{F}_X(\mathbb{S})$ is uniformly $\mathcal{J}$-convergent to a function from $\mathcal{F}_X(\mathbb{S})$, we say that $\mathcal{F}_X(\mathbb{S})$ is *uniformly $\mathcal{J}$-complete*.

Proposition 17.   *(a) Let X be countable. Then $\mathcal{F}_X(\mathbb{S})$ is uniformly $\mathcal{J}$-complete if and only if $\mathcal{F}(\mathbb{S})$ is $\mathcal{J}$-complete.*
  (b) Let X_1 and X_2 be equinumerable. Then $\mathcal{F}_{X_1}(\mathbb{S})$ is uniformly $\mathcal{J}$-complete if and only if $\mathcal{F}_{X_2}(\mathbb{S})$ is uniformly $\mathcal{J}$-complete.
  (c) Let κ, λ be cardinals. If $\lambda \geq \kappa$ and $\mathcal{F}_{\lambda}(\mathbb{S})$ is uniformly $\mathcal{J}$-complete, then $\mathcal{F}_{\kappa}(\mathbb{S})$ is uniformly $\mathcal{J}$-complete.

PROOF. (a) Assume that $\mathcal{F}_X(\mathbb{S})$ is uniformly $\mathcal{J}$-complete. Fix $x \in X$. Then $\mathcal{F}_{\{x\}}(\mathbb{S})$ is (uniformly) $\mathcal{J}$-complete which is equivalent to the $\mathcal{J}$-completeness of $\mathcal{F}(\mathbb{S})$ by Proposition 11.

Now, assume that $\mathcal{F}(\mathbb{S})$ is $\mathcal{J}$-complete and let (g_n) be a uniformly $\mathcal{J}$-fundamental sequence in $\mathcal{F}_X(\mathbb{S})$. Then $((g_n)_x)_{n \in \mathbb{N}}$ is $\mathcal{J}$-fundamental for each $x \in X$. So for each $x \in X$ there is $g_x \in \mathcal{F}(\mathbb{S})$ such that $(g_n)_x \xrightarrow{\mathcal{J}} g_x$. Putting $g(x,y) := g_x(y)$ for $(x,y) \in X \times Y$ we see that $g \in \mathcal{F}_X(\mathbb{S})$ and $g_n \xrightarrow{X,\mathcal{J}} g$. But since X is countable, we have $g_n \xrightarrow{\mathcal{J},X} g$ (use the "diagonal argument").

(b) (Cf. Proposition 11.) Let $h \colon X_1 \to X_2$ be a bijection and assume that $\mathcal{F}_{X_1}(\mathbb{S})$ is uniformly $\mathcal{J}$-complete. Let (f_n) be uniformly $\mathcal{J}$-fundamental in $\mathcal{F}_{X_2}(\mathbb{S})$. Putting $g_n(x,y) := f_n(h(x),y)$ for $(x,y) \in X_1 \times Y$ and $n \in \mathbb{N}$ observe that (g_n) is uniformly $\mathcal{J}$-fundametal in $\mathcal{F}_{X_1}(\mathbb{S})$. Hence there is $g \in \mathcal{F}_{X_1}(\mathbb{S})$ such that $g_n \xrightarrow{\mathcal{J},X_1} g$. Putting $f(x,y) := g(h^{-1}(x),y)$ for $(x,y) \in X_2 \times Y$, we see that $f_n \xrightarrow{\mathcal{J},X_2} f$. The proof of the reverse implication is analogous.

(c) Assume that (f_n) is uniformly $\mathcal{J}$-fundamental in $\mathcal{F}_\kappa(\mathcal{S})$. Putting $g_n :=$ f_n on $\kappa \times Y$ and $g_n := 0$ on $(\lambda \setminus \kappa) \times Y$, observe that (g_n) is uniformly $\mathcal{J}$-fundamental in $\mathcal{F}_\lambda(\mathcal{S})$. So there is $g \in \mathcal{F}_\lambda(\mathcal{S})$ such that $g_n \overset{\mathcal{J},\lambda}{\to} g$. Hence

$$f_n = g_n{\upharpoonright}(\kappa \times Y) \overset{\mathcal{J},\kappa}{\to} g{\upharpoonright}(\kappa \times Y) =: f$$

and $f \in \mathcal{F}_\kappa(\mathcal{S})$. $\square$

Thanks to the referee's suggestion, we are able to give a strengthened version of Proposition 17 (a). To this aim we need some definitions. Let $\mathfrak{h}$ be the distributivity number of the Boolean algebra $\mathcal{P}(\mathbb{N})/[\mathbb{N}]^{<\omega}$ (see [BS], [Va]). For $A, B \in [\mathbb{N}]^\omega$ we write $B \subset^* A$ whenever $|B \setminus A| < \omega$. A set $H \subset [\mathbb{N}]^\omega$ is called *dense* if for every $A \in [\mathbb{N}]^\omega$ there exists $B \in H$ such that $B \subset^* A$, and H is called *open* if for every $A \in H$ and every $B \in [\mathbb{N}]^\omega$, condition $B \subset^* A$ implies $B \in H$. The number $\mathfrak{h}$ can be defined as the minimal cardinality of a family $\mathcal{D}$ consisting of dense open sets with $\bigcap \mathcal{D} = \emptyset$. It is known that $\omega_1 \leq \mathfrak{h} \leq \mathfrak{b}$ and the strict inequalities are consistent (cf. [Va]).

Proposition 18. *Let $|X| < \mathfrak{h}$. Then $\mathcal{F}_X(\mathcal{S})$ is uniformly $\mathcal{J}$-complete if and only if $\mathcal{F}(\mathcal{S})$ is $\mathcal{J}$-complete.*

PROOF. We show "$\Leftarrow$". Assume that $\mathcal{F}(\mathcal{S})$ is $\mathcal{J}$-complete and let (g_n) be a uniformly $\mathcal{J}$-fundamental sequence in $\mathcal{F}_X(\mathcal{S})$. Then there is $g \in \mathcal{F}_X(\mathcal{S})$ such that $g_n \overset{X,\mathcal{J}}{\to} g$ (cf. the proof of Proposition 17 (a)). We identify a sequence $(n_k) \in \mathbb{N}^\uparrow$ with $\{n_k : k \in \mathbb{N}\} \in [\mathbb{N}]^\omega$. For every $x \in X$ put

$$H_x = \{A \in [\mathbb{N}]^\omega : \{(g_n)_x\}_{n \in A} \to (g)_x, \ \mathcal{J} - \text{a.e.}\}$$

Then $\mathcal{D} := \{H_x : x \in X\}$ consists of dense open sets in $[\mathbb{N}]^\omega$. Since $|\mathcal{D}| < \mathfrak{h}$, we have $D := \bigcap \mathcal{D} \neq \emptyset$ and D is s dense open set which yields $g_n \overset{\mathcal{J},X}{\to} g$. $\square$

By Proposition 17 (b), the phenomenon of uniform $\mathcal{J}$-completeness of $\mathcal{F}_X(\mathcal{S})$ does not depend on the nature of X but on its cardinality. Thus we may only speak of uniform $\mathcal{J}$-completeness of $\mathcal{F}_\kappa(\mathcal{S})$ where κ is a cardinal; for $\kappa < \mathfrak{h}$ this notion is equivalent to $\mathcal{J}$-completeness of $\mathcal{F}(\mathcal{S})$, by Proposition 18.

Problem 19. Determine for which $\kappa \geq \mathfrak{h}$ the family $\mathcal{F}_\kappa(\text{BAIRE})$ is uniformly $\mathcal{M}$-complete and the family $\mathcal{F}_\kappa(\text{LEB})$ is uniformly $\mathcal{N}$-complete.

4   Completeness with Respect to the Fubini Product of σ-Ideals.

We will consider the operation of (the Fubini) product of two σ-ideals and we will associate with it the respective σ-algebra.

Lemma 20. *Let $\mathcal{J}$ and $\mathcal{S}$ be a σ-ideal and a σ-algebra on Y and let $\mathcal{I}$ be a σ-ideal on X. We then have*:

(1) $\mathcal{I} \otimes \mathcal{S} := \{A \subset X \times Y : \{x \in X : (A)_x \notin \mathcal{S}\} \in \mathcal{I}\}$ is a σ-algebra on $X \times Y$;

(2) $\mathcal{I} \otimes \mathcal{J} := \{A \subset X \times Y : \{x \in X : (A)_x \notin \mathcal{J}\} \in \mathcal{I}\}$ is a σ-ideal on $X \times Y$, and if $\mathcal{J} \subset \mathcal{S}$, then $\mathcal{I} \otimes \mathcal{J} \subset \mathcal{I} \otimes \mathcal{S}$.

PROOF. Straightforward. $\square$

The σ-ideal $\mathcal{I} \otimes \mathcal{J}$ is called the *Fubini product* of σ-ideals $\mathcal{I}$ and $\mathcal{J}$.

Let $\mathbb{Q}$ stand for the set of all rationals.

Lemma 21. *Assume that $\mathcal{J}$ and $\mathcal{S}$ are as in Lemma 20. For a function $f : X \times Y \to \mathbb{R}$, the following conditions are equivalent*:

(1) $f \in \mathcal{F}(\mathcal{I} \otimes \mathcal{S})$;

(2) $f^{-1}[(-\infty, a)] \in \mathcal{J} \otimes \mathcal{S}$ for all $a \in \mathbb{Q}$;

(3) $\{x \in X : (f)_x^{-1}[(-\infty, a)] \notin \mathcal{S}\} \in \mathcal{I}$ for all $a \in \mathbb{Q}$;

(4) $\bigcup_{a \in \mathbb{Q}} \{x \in X : (f)_x^{-1}[(-\infty, a)] \notin \mathcal{S}\} \in \mathcal{I}$;

(5) $\{x \in X : (f)_x \notin \mathcal{F}(\mathcal{S})\} \in \mathcal{I}$.

PROOF. Equivalences (1)⇔(2) and (4)⇔(5) are obvious. The implication (2)⇒(3) follows from the definition of $\mathcal{I} \otimes \mathcal{S}$. The implication (3)⇒(4) follows from the σ-additivity of $\mathcal{I}$. The implication (4)⇒(2) follows from the definition of $\mathcal{I} \otimes \mathcal{S}$ and the fact that $\mathcal{I}$ is a hereditary family. $\square$

Remark 22. Let $\mathcal{I} := \{\emptyset\}_X$. From Lemma 21 it follows that $\mathcal{F}_X(\mathcal{S}) = \mathcal{F}(\mathcal{I} \otimes \mathcal{S})$. Note that uniform $\mathcal{J}$-convergence in $\mathcal{F}_X(\mathcal{S})$ (respectively, uniform $\mathcal{J}$-completenss of $\mathcal{F}_X(\mathcal{S})$) is the same as $\mathcal{I} \otimes \mathcal{J}$-convergence in $\mathcal{F}(\mathcal{I} \otimes \mathcal{S})$ (respectively, $\mathcal{I} \otimes \mathcal{J}$-completeness of $\mathcal{F}(\mathcal{I} \otimes \mathcal{S})$). If $\mathcal{S} = \mathcal{P}(Y)$, we have $\mathcal{I} \otimes \mathcal{S} = \mathcal{P}(X \times Y)$.

Corollary 23. *Let $\mathcal{I} := \{\emptyset\}_X$. Then $\mathcal{F}(\mathcal{I} \otimes \mathcal{P}(\mathfrak{b}))$ is not $\mathcal{I} \otimes [\mathfrak{b}]^{<\mathfrak{b}}$-complete. Equivalently, $\mathcal{F}_X(\mathcal{P}(\mathfrak{b}))$ is not uniformly $[\mathfrak{b}]^{<\mathfrak{b}}$-complete.*

PROOF. By Remark 22 it suffices to show the first assertion. Suppose that $\mathcal{F}(\mathcal{I} \otimes \mathcal{P}(\mathfrak{b}))$ is $\mathcal{I} \otimes [\mathfrak{b}]^{<\mathfrak{b}}$-complete. Put $\mathcal{S} := \mathcal{I} \otimes \mathcal{P}(\mathfrak{b})$, $\mathcal{J} := \mathcal{I} \otimes [\mathfrak{b}]^{<\mathfrak{b}}$. Fix $x_0 \in X$ and let $Y_0 := \{x_0\} \times \mathfrak{b}$. Then $\mathcal{F}(\mathcal{S}{\restriction}Y_0)$ is $\mathcal{J}{\restriction}Y_0$-complete by Proposition 6. Observe that $\mathcal{S}{\restriction}Y_0 = \mathcal{P}(\{x_0\} \times \mathfrak{b})$ and $\mathcal{J}{\restriction}Y_0 = [\{x_0\} \times \mathfrak{b}]^{<\mathfrak{b}}$. Hence we easily deduce $[\mathfrak{b}]^{<\mathfrak{b}}$-completeness of $\mathcal{F}(\mathcal{P}(\mathfrak{b}))$ which contradicts Fact 5. $\square$

For an ideal $\mathfrak{I} \subset \mathcal{P}(X)$ we let

$$\text{add}(\mathfrak{I}) := \min\{|\mathfrak{U}| \colon \mathfrak{U} \subset \mathfrak{I} \text{ and } \bigcup \mathfrak{U} \notin \mathfrak{I}\}.$$

Proposition 24. *Assume that $|X| = \lambda > \mathfrak{c}$ and $\mathfrak{I}$ is a σ-ideal on X with* $\text{add}(\mathfrak{I}) > \mathfrak{c}$. *Let $\mathcal{S}$ and $\mathcal{J} \subset \mathcal{S}$ be a σ-algebra and a σ-ideal on Y. If $\mathcal{F}_\lambda(\mathcal{S})$ is uniformly $\mathcal{J}$-complete, then $\mathcal{F}(\mathfrak{I} \otimes \mathcal{S})$ is $\mathfrak{I} \otimes \mathcal{J}$-complete.*

PROOF. Let $(f_n)_{n \in \mathbb{N}}$ be an $\mathfrak{I} \otimes \mathcal{J}$-fundamental sequence in $\mathcal{F}(\mathfrak{I} \otimes \mathcal{S})$. For arbitrary $h_1, h_2 \in \mathbb{N}^\uparrow$, $h_1 = (m_k)$, $h_2 = (n_k)$, pick $z = z(h_1, h_2) \in \mathbb{N}^\uparrow$, $z = (k_p)$, such that $(f_{m_{k_p}} - f_{n_{k_p}})_{p \in \mathbb{N}}$ tends to zero $(\mathfrak{I} \otimes \mathcal{J})$-a.e. on $X \times Y$. Hence

$$E(h_1, h_2) := \{x \in X \colon \{y \in Y \colon \neg(f_{m_{k_p}}(x,y) - f_{n_{k_p}}(x,y) \to 0)\} \notin \mathcal{J}\} \in \mathfrak{I}.$$

Since $f_n \in \mathcal{F}(\mathfrak{I} \otimes \mathcal{S})$ for $n \in \mathbb{N}$, we have (cf. Lemma 21)

$$F_n := \{x \in X \colon (f_n)_x \notin \mathcal{F}(\mathcal{S})\} \in \mathfrak{I} \text{ for } n \in \mathbb{N}.$$

Put

$$E := \bigcup_{h_1, h_2 \in \mathbb{N}^\uparrow} E(h_1, h_2) \cup \bigcup_{n \in \mathbb{N}} F_n.$$

From $\text{add}(\mathfrak{I}) > \mathfrak{c}$ it follows that $E \in \mathfrak{I}$. Let $g_n := f_n \restriction (X \setminus E) \times Y$ for $n \in \mathbb{N}$, and observe that (g_n) is uniformly $\mathcal{J}$-fundamental in $\mathcal{F}_{X \setminus E}(\mathcal{S})$. Since $|X \setminus E| \leq \lambda$, by assumption and Proposition 17(c) we infer that $\mathcal{F}_{X \setminus E}(\mathcal{S})$ is uniformly $\mathcal{J}$-complete. Hence there is $g \in \mathcal{F}_{X \setminus E}(\mathcal{S})$ such that $g_n \overset{\mathcal{J}, X \setminus E}{\to} g$. Putting $f := g$ on $(X \setminus E) \times Y$ and $f := 0$ on $E \times Y$ note that f is $\mathfrak{I} \otimes \mathcal{S}$-measurable (cf. Lemma 21) and $f_n \overset{\mathfrak{I} \otimes \mathcal{J}}{\to} f$. $\square$

We could apply Proposition 24 to $\mathcal{J} \in \{\mathcal{M}, \mathcal{N}\}$ and the respective $\mathcal{S} \in \{\text{BAIRE}, \text{LEB}\}$ if we would know that $\mathcal{F}_\lambda(\mathcal{S})$ is uniformly $\mathcal{J}$-complete (cf. Problem 19). However, we have the following (rather simple) application of Proposition 24.

Corollary 25. *Let $\lambda > \mathfrak{c}$ be a cardinal and let $\mathcal{S} \subset \mathcal{P}(Y)$ be a σ-algebra. Then $\mathcal{F}([\lambda]^{\leq \mathfrak{c}} \otimes \mathcal{S})$ is $[\lambda]^{\leq \mathfrak{c}} \otimes \{\emptyset\}_Y$-complete.*

PROOF. It suffices to observe that $\text{add}([\lambda]^{\leq \mathfrak{c}}) > \mathfrak{c}$ (this follows from $\mathfrak{c} \cdot \mathfrak{c} = \mathfrak{c}$) and that $\mathcal{F}_\lambda(\mathcal{S})$ is uniformly $\{\emptyset\}_Y$-complete (see Examples 3 and 15). $\square$

Proposition 24 seems to be less than satisfactory since from one strong kind of completeness we derive another (related) strong kind of completeness. Probably better results can be obtained in particular cases.

5   Appendix: The Proof of Fact 5.

We follow the argument from [V].

Lemma 26. *Let $M \subset \mathbb{N}^\uparrow$ and $|M| < \mathfrak{b}$. There exists a sequence $(z_n)_{n \in \mathbb{N}}$ of nonnegative real numbers such that*

(a) $\sum_{i=1}^\infty z_i = \infty$,

(b) $\lim_{j \to \infty} \sum_{i=j+1}^{j+n_j} z_i = 0$ for each $(n_j) \in M$.

PROOF. By the definition of $\mathfrak{b}$, there is $(r_k) \in \mathbb{N}^\uparrow$ such that $(n_k) \leq^* (r_k)$ for all $(n_k) \in M$. Define $(m^{(j)}) \in \mathbb{N}^\uparrow$ by formulas $m^{(1)} := 1$ and $m^{(j+1)} := m^{(j)} + r_{m^{(j)}}$ for $j \in \mathbb{N}$. Obviously $(m^{(j)}) \in \mathbb{N}^\uparrow$. Let

$$z_i := \begin{cases} 1/j & \text{if } i = m^{(j)} \ (j \in \mathbb{N}) \\ 0 & \text{for the remaining } i \in \mathbb{N}. \end{cases}$$

Assertion (a) is clear. To show (b) fix $(n_k) \in M$. Pick $p \in \mathbb{N}$ such that $r_k \geq n_k$ for all $k \geq p$. Let $j \geq m^{(p)}$. There is a unique $k_j \in \mathbb{N}$ such that $m^{(k_j)} \leq j < m^{(k_j+1)}$. Then $j \geq p$ and $n_j \leq r_j < r_{m^{(k_j+1)}}$. Hence

$$\{j+1, j+2, \ldots, j+n_j\} \subset \{m^{(k_j)} + 1, \ldots, m^{(k_j)} + r_{m^{(k_j+1)}}\}$$

$$\subset \{m^{(k_j)} + 1, \ldots, m^{(k_j+1)} + r_{m^{(k_j+1)}}\} = \{m^{(k_j)} + 1, \ldots, m^{(k_j+2)}\}.$$

Consequently,

$$0 \leq \sum_{i=j+1}^{j+n_j} z_j \leq \frac{1}{k_j+1} + \frac{1}{k_j+2}.$$

Since $\lim_{j \to \infty} k_j = \infty$, we have (b). $\qquad\qquad\qquad\qquad\qquad\square$

To finish the proof of Fact 5, write $\mathcal{J} := [\mathfrak{b}]^{<\mathfrak{b}}$ and fix an unbounded set $T = \{t_\alpha : \alpha < \mathfrak{b}\} \subset \mathbb{N}^\uparrow$ with $t_\alpha = (n_k^{(\alpha)})_{k \in \mathbb{N}}$. If for every $\alpha < \mathfrak{b}$ we apply Lemma 26 with $M_\alpha := \{t_\beta : \beta \leq \alpha\}$, $|M_\alpha| < \mathfrak{b}$, we obtain a sequence $(f_i(\alpha))_{i \in \mathbb{N}}$ of nonnegative real numbers such that

(a') $\sum_{i=1}^\infty f_i(\alpha) = \infty$,

(b') $\lim_{j \to \infty} \sum_{i=j+1}^{j+n_j^{(\beta)}} f_i(\alpha) = 0$ for all $\beta \leq \alpha$.

In this way we have defined a sequence $f_i : \mathfrak{b} \to [0, \infty)$, $i \in \mathbb{N}$. Put $s_n := \sum_{i=1}^n f_i$ for $n \in \mathbb{N}$. By (a'), for every $\alpha < \mathfrak{b}$ we have $\lim_{n \to \infty} s_n(\alpha) = \infty$. In particular $s_n \xrightarrow{\mathcal{J}} s$ is false for any $s \in \mathcal{F}(\mathcal{P}(\mathfrak{b}))$. It suffices to show that (s_n) is $\mathcal{J}$-Cauchy. Let $(m_k), (n_k) \in \mathbb{N}^\uparrow$. We will find $(k_i) \in \mathbb{N}^\uparrow$ such that $\lim_{i \to \infty}(s_{m_{k_i}} - s_{n_{k_i}}) = 0$, $\mathcal{J}$-a.e. If $m_k = n_k$ for infinitely many $k \in \mathbb{N}$, we are

done. Otherwise, we may suppose that, for instance, $m_k > n_k$ for infinitely many $k \in \mathbb{N}$. For simplicity assume that $m_k > n_k$ for all $k \in \mathbb{N}$. Fix $(p_k) \in \mathbb{N}^{\uparrow}$ such that $p_k \geq m_k$ for all $k \in \mathbb{N}$ (e.g. let $p_k := k + \max_{1 \leq i \leq k} m_i$, $k \in \mathbb{N}$). Consequently,

(c) $0 \leq s_{m_k} - s_{n_k} = \sum_{j=n_k+1}^{m_k} f_j \leq \sum_{j=k+1}^{k+p_k} f_j$ for all $k \in \mathbb{N}$.

Because T is unbounded, the condition $(\forall \beta < \mathfrak{b}) \ t_\beta \leq^* (p_k)$ is false. So, there is $\beta < \mathfrak{b}$ such that $n_{k_i}^{(\beta)} > p_{k_i}$ for some $(k_i) \in \mathbb{N}^{\uparrow}$. By (b') we infer that

$$\lim_{k \to \infty} \sum_{j=k+1}^{k+n_k^{(\beta)}} f_j = 0, \quad \mathcal{J}\text{-a.e.}$$

In particular,

$$\lim_{i \to \infty} \sum_{j=k_i+1}^{k_i+n_{k_i}^{(\beta)}} f_j = 0, \quad \mathcal{J}\text{-a.e.}$$

Hence by (c) we have $\lim_{i \to \infty}(s_{m_{k_i}} - s_{n_{k_i}}) = 0$, $\mathcal{J}$-a.e., as desired.

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