**Real Analysis Exchange**
Vol. 28(2), 2002/2003, pp. 549–562

Supriya Pal, Serampore College, Serampore, Hooghly, West-Bengal, India, PIN-712201. e-mail: palsupriya@yahoo.com

D. K. Ganguly, Department of Pure Mathematics, University of Calcutta, 35, Ballygunge Circular Road, Calcutta-700019, India.
e-mail: gangulydk@yahoo.co.in

Lee Peng Yee, MME, National Institute of Education, 1 Nanyang Walk, Singapore 637616. e-mail: pylee@nie.edu.sg.

# THE FUNDAMENTAL THEOREM OF CALCULUS FOR $GR_k$-TYPE INTEGRALS

**Abstract**

The $GR_k$ integral, a Stieltjes type integral, was introduced by the authors. In this paper we introduce a $GR_k$-type integral that admits the fundamental theorem of calculus. Also, a convergence theorem is proved using equi-integrability conditions.

## 1 Introduction

The Henstock-Stieltjes integral is well known [1]. Recently the $GR_k$-integral was introduced by the authors (see [3]). Some results on the $GR_k$ integral were established in [3]. The $GR_k$-integral is not covered by the Henstock-Stieltjes type integrals in the literature. The $GR_k$-integral was defined using a different type of division called $\delta^k$-fine division. For this division we had a different type of partial division and the Saks-Henstock lemma where the building blocks were point-division pairs rather than point-interval pairs as in the usual case.

It is observed that pointwise tagging is not helpful in studying the $GR_k$-integral and so we introduce the concept of what we call "local tagging". Further the concept of "regulated $\delta^k$-fine division" is introduced by means of local tagging and using this type of divisions we introduced the $GR_k^*$-integral. It was shown that the new integral includes the $GR_k$-integral provided the

Key Words: Henstock Integral, $\delta^k$-fine division, regulated $\delta^k$-fine division, Saks-Henstock lemma, $g^k$-variation, $GR_k$-integral, $GR_k^*$-integral, $g$ regular function, Equiintegrability, $BV^k$, $LBV^k$, nearly additive function.
Mathematical Reviews subject classification: 26A39
Received by the editors December 23, 2001

jump $J(g;x)$ exists at every point. We verify some fundamental prperties of $GR_k^*$ integrals and find that the primitive function for a $GR_k^*$ integral is also not additive; rather it is, what we call, nearly additive. We also introduce the concept of local $g^k$ variation which plays an important role in the development of the integral. We obtain a fundamental theorem of calculus for the $GR_k^*$ integral using the notion of '$g$-regularity' of primitive functions. Further, we prove the equi-integrability convergence theorem (see [2]) for the $GR_k^*$ integral.

## 2 Preliminaries

Let $k$ be a fixed positive integer and $\delta$ be a positive function defined on $[a,b]$. We shall call a division $D$ of $[a,b]$ given by $a = x_0 < x_1 < \ldots < x_n = b$ with associated points $\{\xi_0, \xi_1, \ldots, \xi_{n-k}\}$ satisfying

$$\xi_i \in [x_i, x_{i+k}] \subset (\xi_i - \delta(\xi_i), \xi_i + \delta(\xi_i)) \text{ for } i = 0, 1, \ldots, n-k$$

a *$\delta^k$-fine division of* $[a,b]$. For a given positive function $\delta$, we denote a $\delta^k$-fine division $D$ by $\{[x_i, x_{i+k}], \xi_i\}_{i=0,1,\ldots,n-k}$. Using compactness of $[a,b]$ it is easy to verify that such $\delta^k$-fine divisions exist. Note that $\delta^1$fine divisions coincide with the usual definition of $\delta$-fine divisions.

Let $g$ be a real-valued function defined on a closed interval $[a,b]^{k+1}$ in $(k+1)$-dimensional space, and $f$ a real-valued function defined on $[a,b]$. We say that *$f$ is $GR_k$-integrable with respect to $g$ to $I$ on* $[a,b]$ if for every $\epsilon > 0$ there is a function $\delta(\xi) > 0$ for $\xi \in [a,b]$ such that for any $\delta^k$-fine division $D = \{[x_i, x_{i+k}], \xi_i\}_{i=0,1,\ldots,n-k}$ we have

$$\left| \sum_{i=0}^{n-k} f(\xi_i) g(x_i, \ldots, x_{i+k}) - I \right| < \epsilon.$$

We shall denote the above Riemann sum by $s(f,g;D)$. If $f$ is integrable with respect to $g$ in the above sense, we write $(f,g) \in GR_k[a,b]$ and denote the integral by $\int_a^b f\,dg$.

Let $x \in [x_i, x_{i+k}]$ where $x_i < x_{i+1} < \ldots < x_{i+k}$. The *jump of $g$ at $x$*, denoted by $J(g;x)$, is defined by $J(g;x) = \lim_{x_i \to x, x_{i+k} \to x} g(x_i, \ldots, x_{i+k})$, if the limit exists finitely.

In [3] the following theorem was proved.

**Theorem 2.1.** *Let $(f,g) \in GR_k[a,c]$ and $(f,g) \in GR_k[c,b]$. If $J(g;c)$ exists, then $(f,g) \in GR_k[a,b]$ and*

$$\int_a^b f\,dg = \int_a^c f\,dg + \int_c^b f\,dg + (k-1) f(c) J(g;c).$$

**Remark 2.2.** We note that if we define $F(u,v) = \int_u^v f\,dg$ for $[u,v] \subset [a,b]$, then $F$ is not an additive function on the closed subintervals of $[a,b]$ for $k > 1$. But for $k = 1$ it is additive because the extra term vanishes.

Let the domain of $F$ be $\{[u,v] \subset [a,b] | u \leq v\}$. We say $F$ is *nearly additive* if for $a < c < b, F(a,b) = F(a,c) + F(c,c) + F(c,b)$. Further, $F$ is called *g-nearly additive with respect to f* if for all $x \in (a,b)$ we have that $F(x,x) = (k-1)f(x)J(g;x)$. So, the integral function $F$ of the $GR_k$-integral is $g$-nearly additive with respect to $f$ in $[a,b]$.

In [3] the following $\delta^k$-fine partial division of a special kind was introduced. Let $[a_i,b_i], i = 1,2,\ldots,p$ be pairwise non-overlapping, and $\cup_{i=1}^p [a_i,b_i] \subset [a,b]$. Then $\{D_i\}_{i=1,2,\ldots,p}$ is said to be a *$\delta^k$-fine partial division of* $[a,b]$ if each $D_i$ is a $\delta^k$-fine division of $[a_i,b_i]$. Its corresponding partial Riemann sum is given by $\displaystyle\sum_{i=1}^p s(f,g;D_i)$.

With this notion of partial division an analogue of the Saks-Henstock Lemma was proved (see [3]).

**Theorem 2.3.** *(Saks-Henstock Lemma) If $(f,g) \in GR_k[a,b]$ and $J(g;c)$ exists for all $c \in (a,b)$, then for every $\epsilon > 0$ there exists a positive function $\delta$ on $[a,b]$ such that for any $\delta^k$-fine division $D$ of $[a,b]$ and for any $\delta^k$-fine partial division $\{D_i\}_{i=1,2,\ldots,p}$ of $[a,b]$*

$$\left| s(f,g;D) - \int_a^b f\,dg \right| < \epsilon \text{ and } \left| \sum_{i=1}^p \left( s(f,g;D_i) - \int_{a_i}^{b_i} f\,dg \right) \right| < (k+1)\epsilon$$

*where $D_i$ is a $\delta^k$-fine division of $[a_i,b_i]$.*

**Remark 2.4.** In view of the above definition of partial division and Saks-Henstock lemma we observe that unlike the usual partial division in Henstock's theory we have to consider "point-division" pairs and also we see that these are the building blocks of the $GR_k$-integral. In other words, "a point" and "an interval around that point" are not helpful for our purpose; rather "a point" and "a $\delta^k$-fine division in a neighbourhood of the point" are what we need.

Keeping this in mind we introduce the concept of *local tagging* which for $k = 1$, reduces to usual "point-interval" tagging.

**Definition 2.5.** Given a function $\delta : [a,b] \to \mathbb{R}_+$ and a point $x \in [a,b]$, then a $\delta^k$-fine division $D$ of $[u,v] \subseteq [a,b]$ is said to be locally tagged at $x$ if $[u,v] \subset (x - \delta(x), x + \delta(x))$ with either $u = x$ or $v = x$.

It may be noted here that for local tagging at $x$ we need $\delta$ to be defined in a neighbourhood of $x$. But for simple presentation we considered $\delta$ to be defined on $[a, b]$.

**Definition 2.6.** A family of triplets $\{x_i, D_i, [a_i, b_i]\}_{i=1}^p$ is called a regulated $\delta^k$-fine division of $[a, b]$ if each $D_i$ is a $\delta^k$-fine division of $[a_i, b_i]$ locally tagged at $x_i$ where $[a_i, b_i], i = 1, 2, \ldots, p$ are non-overlapping with $\cup[a_i, b_i] = [a, b]$. Further, $\{x_i, D_i, [a_i, b_i]\}_{i=1}^p$ is called a regulated $\delta^k$-fine partial division of $[a, b]$ if $\cup[a_i, b_i] \subseteq [a, b]$.

## 3 $GR_k^*$-integral

We now introduce a $GR_k$-type integral that admits the fundamental theorem of calculus among other results. Also we will show that if $J(g; c)$ exists for all $c \in (a, b)$, then the new integral contains the $GR_k$ integral. We denote the new integral by $GR_k^*$ integral which is defined as follows.

**Definition 3.1.** Let $f : [a, b] \to \mathbb{R}$ and $g : [a, b]^{k+1} \to \mathbb{R}$ such that $J(g; c)$ exists for all $c \in (a, b)$. We say that $f$ is $GR_k^*$ integrable to $A$ with respect to $g$ on $[a, b]$ if for all $\epsilon > 0$ there exists $\delta : [a, b] \to \mathbb{R}_+$ such that for any regulated $\delta^k$-fine division $\{x_i, D_i, [a_i, b_i]\}_{i=1}^p$ of $[a, b]$ we have

$$\left| \sum_{i=1}^{p} s(f, g; D_i) + \sum_{i=1}^{p-1} (k-1) f(b_i) J(g; b_i) - A \right| < \epsilon$$

We can easily verify that $GR_k^*$ integral is well defined.

If $f$ is $GR_k^*$ integrable with respect to $g$, we write $(f, g) \in GR_k^*[a, b]$ and denote the integral also by $\int_a^b f \, dg$.
In what follows we always assume that $J(g; x)$ exists for all $x \in (a, b)$.

**Theorem 3.2.** *Let $a < c < b$. If $(f, g) \in GR_k^*[a, c]$ and $(f, g) \in GR_k^*[c, b]$, then $(f, g) \in GR_k^*[a, b]$ and*

$$\int_a^b f \, dg = \int_a^c f \, dg + \int_c^b f \, dg + (k-1) f(c) J(g; c).$$

PROOF. Since $(f, g) \in GR_k^*[a, c] \cap GR_k^*[c, b]$, for $\epsilon > 0$, there exists $\delta_1(x) > 0$ and $\delta_2(x) > 0$ defined on $[a, c]$ and $[c, b]$ respectively such that

$$\left| \sum_{i=1}^{p} s(f, g; D_i) + \sum_{i=1}^{p-1} (k-1) f(b_i) J(g; b_i) - \int_a^c f \, dg \right| < \epsilon,$$

and

$$\left|\sum_{j=1}^{q} s(f,g;P_j) + \sum_{j=1}^{q-1}(k-1)f(d_j)J(g;d_j) - \int_c^b f\,dg\right| < \epsilon,$$

for every regulated $\delta_1^k$-fine division $\{x_i, D_i, [a_i, b_i]\}_{i=1}^p$ of $[a,c]$ and regulated $\delta_2^k$-fine division $\{y_j, P_j, [c_j, d_j]\}_{j=1}^q$ of $[c,b]$ respectively. We define $\delta(x) = \min\{\delta_1(x), c-x\}$ when $x \in [a,c)$, $\min\{\delta_2(x), x-c\}$ when $x \in (c,b]$, and min $\{\delta_1(c), \delta_2(c)\}$ when $x = c$. For any $\delta$-fine division $D$ of $[a,b], c$ is always an associated point of $D$. Let $\{z_l, Q_l, [u_l, v_l]\}_{l=1}^r$ be a regulated $\delta^k$-fine division of $[a,b]$. So, $c$ is one of $u_l$ or $v_l$ for some $l = 1, 2, \ldots, r$. Then,

$$\begin{aligned}
&\Bigg|\sum_{l=1}^{r} s(f,g;Q_l) + \sum_{l=1}^{r-1}(k-1)f(v_l)J(g;v_l) \\
&\qquad\qquad - \Big\{\int_a^c f\,dg + \int_c^b f\,dg + (k-1)f(c)J(g;c)\Big\}\Bigg| \\
&\qquad \le \left|\sum_1 s(f,g;Q_l) + \sum_1 (k-1)f(v_l)J(g;v_l) - \int_a^c f\,dg\right| \\
&\qquad\quad + \left|\sum_2 s(f,g;Q_l) + \sum_2 (k-1)f(v_l)J(g;v_l) - \int_c^b f\,dg\right| < 2\epsilon,
\end{aligned}$$

where $\sum_1, \sum_2$ denote respectively the partial sum and jumps over $[a,c]$ and $[c,b]$ respectively. So, $(f,g) \in GR_k^*[a,b]$ and the result follows. □

It is to be noted that Remark 2.2 still holds for $GR_k^*$ integral. Similar to Theorem 2.3 [3] we can prove the following Cauchy condition for $GR_k^*$integral.

**Theorem 3.3.** *$(f,g) \in GR_k^*[a,b]$ if and only if for every $\epsilon > 0$ there exists a positive function $\delta$ on $[a,b]$ such that for all regulated $\delta^k$-fine divisions $\{x_i, D_i, [a_i, b_i]\}_{i=1}^p$ and $\{y_j, P_j, [c_j, d_j]\}_{j=1}^q$ of $[a,b]$ we have*

$$\begin{aligned}
&\Bigg|\Big(\sum_{i=1}^{p} s(f,g;D_i) + \sum_{i=1}^{p-1}(k-1)f(b_i)J(g;b_i)\Big) \\
&\qquad - \Big(\sum_{j=1}^{q} s(f,g;P_j) + \sum_{j=1}^{q-1}(k-1)f(d_j)J(g;d_j)\Big)\Bigg| < \epsilon.
\end{aligned}$$

Also similar to Theorem 2.4 of [3], we have the following result.

**Theorem 3.4.** *If $(f,g) \in GR_k^*[a,b]$ and $a \le c < d \le b$, then $(f,g) \in GR_k^*[c,d]$.*

Next we come to the Saks-Henstock Lemma for the $GR_k^*$ integral.

**Theorem 3.5.** *(Saks-Henstock Lemma) $(f,g) \in GR_k^*[a,b]$ if and only if there exists a function $F$, $g$-nearly additive with respect to $f$, satisfying the condition that for all $\epsilon > 0$ there exists $\delta : [a,b] \to \mathbb{R}_+$ such that for all regulated $\delta^k$-fine partial division $\{x_i, D_i, [a_i,b_i]\}_{i=1}^p$ of $[a,b]$ we have*

$$\left|\sum_{i=1}^{p}\{s(f,g;D_i) - F(a_i,b_i)\}\right| < \epsilon.$$

PROOF. Let $(f,g) \in GR_k^*[a,b]$. Since $(f,g) \in GR_k^*[a,b]$, for $\epsilon > 0$ there exists $\delta(x) > 0$ , $x \in [a,b]$ such that for all regulated $\delta^k$-fine division $\{z_r, Q_r, [u_r,v_r]\}_{r=1}^t$ of $[a,b]$ we have

$$\left|\sum_{r=1}^{t} s(f,g;Q_r) + \sum_{r=1}^{t-1}(k-1)f(v_r)J(g;v_r) - F(a,b)\right| < \epsilon,$$

where $F(u,v) = \int_u^v f dg$. We define $F(u,v) = (k-1)f(u)J(g;u)$ when $u = v$. Let $\{x_i, D_i, [a_i,b_i]\}_{i=1}^p$ be a regulated $\delta^k$-fine partial division of $[a,b]$, and $\cup_{j=1}^q [c_j,d_j]$ be the closure of the complements of $\cup_{i=1}^p [a_i,b_i]$ in $[a,b]$. By Theorem-3.4, $(f,g) \in GR_k^*[c_j,d_j], j = 1,2,\ldots,q$ and so we can find $\delta_j(x) > 0, j = 1,2,\ldots,q$ defined on $[c_j,d_j]$ such that for all regulated $\delta_j^k$-fine division $\{y_{js}, D_{js}, [c_{js},d_{js}]\}_{s=1}^{m_j}$ of $[c_j,d_j], j = 1,2,\ldots,q$ we have

$$\left|\sum_{s=1}^{m_j} s(f,g;D_{js}) + (k-1)\sum_{s=1}^{m_j-1} f(d_{js})J(g;d_{js}) - F(c_{js},d_{js})\right| < \frac{\epsilon}{q}.$$

We may assume that $\delta_j(x) \le \delta(x)$ for $x \in [c_j,d_j], j = 1,2,\ldots,q$.

Now $\{y_{js}, D_{js}, [c_{js},d_{js}]\}_{s=1}^{m_j}; j = 1,2,\ldots,q$ and $\{x_i, D_i, [a_i,b_i]\}_{i=1}^p$ is a regulated $\delta^k$-fine division of $[a,b]$.
Let $\Lambda$ be the set of common end points of $[a_i,b_i]$ and $[c_j,d_j]$.
Now,

$$\left|\sum_{j=1}^{q}\sum_{s=1}^{m_j} s(f,g;D_{js}) + \sum_{i=1}^{p} s(f,g;D_i) + (k-1)\sum_{j=1}^{q}\sum_{s=1}^{m_j-1} f(d_{js})J(g;d_{js}) \right.$$
$$\left. + (k-1)\sum_{x\in\Lambda} f(x)J(g;x) - F(a,b)\right| < \epsilon.$$

Also by Theorem 3.2,

$$\begin{aligned}
F(a,b) &= \sum_{i=1}^{p} F(a_i,b_i) + (k-1)\sum_{x\in\Lambda} f(x)J(g;x) \\
&\quad + \sum_{j=1}^{q} F(c_j,d_j)\Big|\sum_{i=1}^{p} s(f,g;D_i) - \sum_{i=1}^{p} F(a_i,b_i)\Big| \\
&\leq \Big|\sum_{j=1}^{q}\sum_{s=1}^{m_j} s(f,g;D_{js}) \\
&\quad + \sum_{i=1}^{p} s(f,g;D_i) + (k-1)\sum_{j=1}^{q}\sum_{s=1}^{m_j-1} f(d_{js})J(g;d_{js}) \\
&\quad + (k-1)\sum_{x\in\Lambda} f(x)J(g;x) - F(a,b)\Big| \\
&\quad + \Big|(k-1)\sum_{j=1}^{q}\sum_{s=1}^{m_j-1} f(d_{js})J(g;d_{js}) \\
&\quad + \sum_{j=1}^{q}\sum_{s=1}^{m_j} s(f,g;D_{js}) - \sum_{j=1}^{q} F(c_j,d_j)\Big| < 2\epsilon.
\end{aligned}$$

Conversely, let there exist $\delta(x) > 0$, for $x \in [a,b]$ and a function $F$, $g$-nearly additive with respect to $f$ so that the given condition holds for any regulated $\delta^k$-fine division in $[a,b]$. Let $\{x_i, D_i, [a_i,b_i]\}_{i=1}^{p}$ be any regulated $\delta^k$-fine division of $[a,b]$ such that $\left|\sum_{i=1}^{p} s(f,g;D_i) - \sum_{i=1}^{p} F(a_i,b_i)\right| < \epsilon$. Now $F$ being $g$-nearly additive, $F(a,b) = \sum_{i=1}^{p} F(a_i,b_i) + (k-1)\sum_{i=1}^{p-1} f(b_i)J(g;b_i)$. So, $\left|\sum_{i=1}^{p} s(f,g;D_i) + (k-1)\sum_{i=1}^{p-1} f(b_i)J(g;b_i) - F(a,b)\right| < \epsilon$. Hence , $(f,g) \in GR_k^*[a,b]$. $\square$

**Remark 3.6.** In view of the above lemma, it follows that if $J(g;c)$ exists for all $c \in (a,b)$, then $(f,g) \in GR_k[a,b]$ implies that $(f,g) \in GR_k^*[a,b]$. Also,when $(f,g)$ is $GR_k[a,b] \cap GR_k^*[a,b]$ , the respective integrals are equal.

In [3] we used the concept of bounded variation of $k$th order of $g$. We now introduce the notion of local bounded variation of $k$th order.

For $X \subset [a,b]$, we define

$$LV_g^k(X) = \inf_\delta \sup \Big\{ \sum_{i=1}^{p} |s(1,g;D_i)| \Big\},$$

where the sup is taken over all regulated $\delta^k$-fine partial division $\{x_i, D_i, [a_i,b_i]\}_{i=1}^p$ of $[a,b]$ such that $x_i \in X$. Then $X \subset [a,b]$ is said to be of *$Lg^k$-variation zero* if $LV_g^k(X) = 0$.

For example, let $X$ be countable. $LV_g^k(X) = 0$ if and only if $J(g;x) = 0$ at every $x \in X$. A function $g$ is said to be $LBV^k(X)$ if $LV_g^k(X)$ is finite.

Also $g$ is said to be $LBV^kG(X)$ if $X = \cup_{j=1}^{\infty} X_j$ such that $g$ is $LBV^k(X_j)$ for each $j$.

We can easily show that, $g \in BV^k[a,b]$ implies that $g \in LBV^k[a,b]$

A property is said *to hold $Lg^k$ a.e.* if it holds everywhere in $[a,b]$ except on a set of $Lg^k$-variation zero. It is easy to verify the following result.

**Theorem 3.7.** *If $f_1$ or $f_2$ is $GR_k^*$ integrable with respect to $g$ on $[a,b]$ and $f_1 = f_2$, $Lg^k$ a.e. in $[a,b]$, then the other is also integrable and $\int_a^b f_1\, dg = \int_a^b f_2\, dg$.*

## 4 Some Results

**Definition 4.1.** Let $F$ be a function $g$-nearly additive with respect to $f$ on $[a,b]$. $F$ is said to be $g$-regular with respect to $f$ at $x \in [a,b]$ if for all $\epsilon > 0$, there exists a function $\delta(x) > 0$, defined on $[a,b]$ such that for all $\delta^k$-fine divisions $D = \{[x_i, x_{i+k}]; \xi_i\}_{i=0,1,2\ldots,n-k}$ of $[u,v] \subset [a,b]$ locally tagged at $x$ we have

$$|(D)\sum f(\xi_i) g(x_i, \ldots, x_{i+k}) - F(u,v)| < \epsilon (D) \sum |g(x_i, \ldots, x_{i+k})|.$$

**Definition 4.2.** Let $f : [a,b] \to \mathbb{R}$, $g : [a,b]^{k+1} \to R$ and $F$ be $g$-nearly additive with respect to $f$ in $[a,b]$. Given $\epsilon > 0$ and a function $\delta : [a,b] \to R_+$, we define $\Gamma_{\epsilon,\delta} = \big\{(x,D) : D = \{[x_i, x_{i+k}]; \xi_i\}_{i=0}^{n-k}$ is a $\delta^k$-fine division of $[u,v]$ locally tagged at $x \in [a,b]$ such that

$$|(D)\sum f(\xi_i) g(x_i, \ldots, x_{i+k}) - F(u,v)| \geq \epsilon (D) \sum |g(x_i, \ldots, x_{i+k})|\big\}.$$

**Theorem 4.3.** *Let $F$ be $g$-nearly additive function defined on $[a,b]$ and $g \in LBV^kG[a,b]$. If $F$ is $g$-regular at all $x \in [a,b]$, then $(f,g) \in GR_k^*[a,b]$ with primitive $F$.*

PROOF. Since $g \in LBV^kG[a,b]$, there exists non-overlapping $E_j, j = 1,2,\ldots$ such that $[a,b] = \cup E_j$ and $g \in LBV^k(E_j)$ for all $j$. So, there exists $\delta_{1j}(x) > 0, j = 1,2,\ldots$ defined on $[a,b]$ and $M_j > 0$ such that $\sum_{i=1}^{p_j} |s(1,g;D_{ij})| < M_j$ for all regulated $\delta_{1j}^k$-fine division $\{x_{ij}, D_{ij}, [a_{ij}, b_{ij}]\}_{i=1}^{p_j}$, $x_{ij} \in E_j$. $F$ being $g$-regular at all $x \in [a,b]$, for $\epsilon > 0$ and $j = 1,2,\ldots$ there exists $\delta_{2j}(x) > 0$, for $x \in [a,b]$ such that for all $\delta_{2j}^k$-fine division $D = \{[x_i, x_{i+k}]; \xi_i\}$ of $[u,v]$ locally tagged at $x \in [a,b]$ we have

$$|F(u,v) - (D)\sum f(\xi_i)g(x_i,\ldots,x_{i+k})| < \frac{\epsilon}{2^j M_j}(D)\sum |g(x_i,\ldots,x_{i+k}|.$$

We define $\delta(x) = \min\{\delta_{1j}(x), \delta_{2j}(x)\}, x \in E_j$. Let $\{x_i, D_i, [a_i,b_i]\}_{i=1}^p$ be any regulated $\delta^k$-fine partial division of $[a,b]$.

$$\begin{aligned}
\left|\sum_{i=1}^p \{s(f,g;D_i) - F(a_i,b_i)\}\right| &\le \sum_{i=1}^p \Big|s(f,g;D_i) - F(a_i,b_i)\Big| \\
&= \sum_{j=1}^\infty \sum_{x_i \in E_j} \Big|s(f,g;D_i) - F(a_i,b_i)\Big| \\
&\le \sum_{j=1}^\infty \frac{\epsilon}{2^j M_j} \sum_{x \in E_j} \Big|s(1,g;D_i)\Big| < \epsilon.
\end{aligned}$$

So $(f,g) \in GR_k^*[a,b]$ with primitive $F$. □

**Theorem 4.4.** *Let $F$ be a function $g$-nearly additive with respect to $f$ on $[a,b]$ and $g \in LBV^kG[a,b]$. Then $(f,g) \in GR_k^*[a,b]$ with primitive $F$ if for all $\epsilon > 0$ there exists $\delta : [a,b] \to \mathbb{R}_+$ such that for all regulated $\delta^k$-fine partial division $\{x_i, D_i, [a_i,b_i]\}_{i=1}^p$ of $[a,b]$, where $(x_i, D_i) \in \Gamma_{\epsilon,\delta}, i = 1,2,\ldots,p$ we have*

$$\sum_{i=1}^p |s(f,g;D_i)| < \epsilon \text{ and } \sum_{i=1}^p |F(a_i,b_i)| < \epsilon.$$

*The converse also holds if $[a,b] = \cup_{l=1}^\infty X_l$ is such that for each $l$ there exists a $\delta_l : [a,b] \to \mathbb{R}_+$ and $M_l > 0$ so that for any $\delta^k$-fine division $D = \{[x_i, x_{i+k}]; \xi_i\}_{i=0}^{n-k}$ of $[u,v]$ locally tagged at $x \in X_l$, we have*

$$\left|\sum_{i=0}^{n-k} f(\xi_i)g(x_i,\ldots,x_{i+k})\right| \le M_l \sum_{i=0}^{n-k} \Big|g(x_i,\ldots,x_{i+k})\Big|.$$

PROOF. Let the given conditions hold. Since, $g \in LBV^kG[a,b]$, there exists non-overlapping $E_j, j = 1,2,\ldots$ with $[a,b] = \cup_{j=1}^\infty E_j$ such that $g \in LBV^k(E_j)$.

So, we can find $\delta_{1j}(x) > 0, x \in [a,b]$ and $M_j > 0$ such that for all regulated $\delta_{1j}^k$-fine partial division $\{x_i, D_i, [a_i,b_i]\}_{i=1}^p$ of $[a,b], x_i \in E_j$ we have $\sum_{i=1}^p |s(1,g;D_i)| < M_j$. For $x \in X'(\epsilon) \cap E_j$, there exists $\delta_{2j}(x) > 0$ defined on $[a,b]$ such that for all $\delta_{2j}^k$-fine division $D = \{[x_i, x_{i+k}]; \xi_i\}$ of $[u,v]$ locally tagged at $x$, we have

$$|(D)\sum_{i=1}^{n-k} f(\xi_i)g(x_i,\ldots,x_{i+k}) - F(u,v)| < \frac{\epsilon}{2^j M_j}\sum_{i=1}^{n-k}|g(x_i,\ldots,x_{i+k})|.$$

Let $\delta(x)$ =min$\{\delta_{1j}(x), \delta_{2j}(x)\}, x \in E_j$ and $\{x_i, D_i, [a_i,b_i]\}_{i=1}^p$ be a regulated $\delta^k$-fine partial division of $[a,b]$. Then

$$\begin{aligned}
|\sum_{i=1}^{p}\{s(f,g;D_i) - F(a_i,b_i)\}| \le & \sum_{x_i\in X(\epsilon)}|s(f,g;D_i)| + \sum_{x_i\in X(\epsilon)}|F(a_i,b_i)| \\
& + \sum_{x_i\in X'(\epsilon)}|s(f,g;D_i) - F(a_i,b_i)| \\
<& 2\epsilon + \sum_{j=1}^{\infty}\sum_{x_i\in X'(\epsilon)\cap E_j}|s(f,g;D_i) - F(a_i,b_i)| \\
<& 2\epsilon + \sum_{j=1}^{\infty}\sum_{x_i\in X'(\epsilon)\cap E_j}\frac{\epsilon}{2^j M_j}|s(1,g;D_i)| \\
=& 2\epsilon + \sum_{j=1}^{\infty}\frac{\epsilon}{2^j M_j}M_j = 3\epsilon.
\end{aligned}$$

So,$(f,g) \in GR_k^*[a,b]$.

Conversely, let $(f,g) \in GR_k^*[a,b]$ with primitive $F$ and $f$ satisfies the given condition. We may assume that $X_l$ are disjoint. So, there exists $M_l > 1$ and $\delta_{1l}(x) > 0$, for $x \in [a,b]$ such that

$$|(D)\sum f(\xi_i)g(x_i,\ldots,x_{i+k})| \le M_l\sum|g(x_i,\ldots,x_{i+k})|,$$

for any $\delta_{1l}^k$-fine division $D = \{[x_i, x_{i+k}]; \xi_i\}$ of $[u,v]$ locally tagged at $x \in X_l$. By the Henstock Lemma, there exists $\delta_{2l}(x) > 0$ on $[a,b]$ such that for every regulated $\delta_{2l}^k$-fine partial division $\{x_i, D_i, [a_i,b_i]\}_{i=1}^p$ of $[a,b]$ we have

$$\Big|\sum_{i=1}^{p}\{s(f,g;D_i) - F(a_i,b_i)\}\Big| < \frac{\epsilon^2}{2^{l+1}M_l}.$$

We define $\delta(x) = \min\{\delta_{1l}(x), \delta_{2l}(x)\}, x \in X_l$. Let $\{x_i, D_i, [a_i, b_i]\}_{i=1}^p$ be a regulated $\delta^k$-fine partial division of $[a,b]$, where $(x_i, D_i) \in \Gamma_{\epsilon,\delta}$. Then,

$$\sum_{i=1}^{p} |s(f,g;D_i)| \le \sum_{l=1}^{\infty} \sum_{x_i \in X_l} |s(f,g;D_i) \le \sum_{l=1}^{\infty} M_l \sum_{x_i \in X_l} |s(1,g;D_i)|$$
$$\le \sum_{l=1}^{\infty} \frac{M_l}{\epsilon} \sum_{x_i \in X_l} |s(f,g;Di) - F(a_i,b_i)| \le \sum_{l=1}^{\infty} \frac{M_l 2\epsilon^2}{\epsilon 2^{l+1} M_l} = \epsilon.$$

Furthermore, $\delta$ may be chosen such that

$$\sum_{i=1}^{p} \Big|F(a_i,b_i)\Big| \le \sum_{i=1}^{p} \Big|s(f,g;D_i) - F(a_i,b_i)\Big| + \sum_{i=1}^{p} \Big|s(f,g;D_i)\Big| < \epsilon.$$

We note that in this part we do not need $g \in LBV^kG[a,b]$. □

**Corollary 4.5.** *If $f$ satisfies the condition of the second part of the above Theorem, and $F$ be a function $g$-nearly additive with respect to $f$ on $[a,b]$ and $g \in LBV^kG[a,b]$, then $(f,g) \in GR_k^*[a,b]$ with primitive $F$ if and only if for all $\epsilon > 0$ there exists $\delta : [a,b] \to R_+$ such that for all regulated $\delta^k$-fine partial division $\{x_i, D_i, [a_i,b_i]\}_{i=1}^p$ of $[a,b]$, where $(x_i, D_i) \in \Gamma_{\epsilon,\delta}$ we have*

$$\sum_{i=1}^{p} |s(1,g;D_i)| < \epsilon \text{ and } \sum_{i=1}^{p} |F(a_i,b_i)| < \epsilon.$$

Proof. If $(f,g) \in GR_k^*[a,b]$ with primitive $F$, then the above inequalities are satisfied which can be easily seen from the proof of the first part of Theorem-4.4.

Conversely, let the above conditions hold. We define $X_l$ as in the above theorem. Let $\epsilon > 0$. Then for every $\epsilon_l = \dfrac{\epsilon}{M_l 2^{l+1}}$, there exists $\delta_l(x) > 0$ such that $\sum_{i=1}^p |s(1,g;D_i)| < \epsilon_l$ and $\sum_{i=1}^p |F(a_i,b_i)| < \epsilon_l$, whenever $\{x_i, D_i, [a_i,b_i]\}$ is a regulated $\delta_l^k$-fine partial division of $[a,b]$, where $(x_i, D_i) \in \Gamma_{\epsilon_l,\delta_l}$. Now let $\delta$ be so that $\delta(x) \le \delta_l(x), x \in X_l$. Then for any regulated $\delta^k$-fine partial division $\{x_i, D_i, [a_i,b_i]\}_{i=1}^p$ of $[a,b]$, where $(x_i, D_i) \in \Gamma_{\epsilon,\delta} \subset \cap_{l=1}^{\infty} \Gamma_{\epsilon_l,\delta_l}$,

$$\sum_{i=1}^{p} |s(f,g;D_i)| \le \sum_{l=1}^{\infty} \sum_{x_i \in X_l} |s(1,g;D_i)| < \epsilon.$$

The proof is complete by Theorem 4.4. □

## 5 Convergence Theorem

**Definition 5.1.** Let $(f_n, g) \in GR_k^*[a,b]$. $\{(f_n, g)\}$ is said to be equi-$GR_k^*$-integrable on $[a,b]$ if for all $\epsilon > 0$ there exists $\delta(x) > 0$, $x \in [a,b]$, independent of $n$, such that

$$\Big|\sum_{i=1}^{p} s(f_n, g; D_i) + (k-1)\sum_{i=1}^{p-1} f_n(b_i)J(g; b_i) - \int_a^b f_n dg\Big| < \epsilon,$$

for all $n$, whenever $\{x_i, D_i, [a_i, b_i]\}_{i=1}^{p}$ is a regulated $\delta^k$-fine division of $[a,b]$.

**Theorem 5.2.** *Let $g \in LBV^k[a,b]$. If (i) $\{(f_n, g)\}$is equi-$GR_k^*$-integrable, (ii)$f_n(x) \to f(x)$ as $n \to \infty$ for all $x \in [a,b]$, then $(f,g) \in GR_k^*[a,b]$ and*

$$\int_a^b f dg = \lim_{n\to\infty} \int_a^b f_n dg.$$

Proof. Since $\{(f_n, g)\}$ is equi-$GR_k^*$-integrable, for every $\epsilon > 0$, there exists $\delta_0(x) > 0$, for $x \in [a,b]$ independent of $n$ such that

$$\left|\sum_{i=1}^{p} s(f_n, g; D_i) + (k-1)\sum_{i=1}^{p-1} f_n(b_i)J(g; b_i) - A_n\right| < \epsilon,$$

for all regulated $\delta^k$-fine division $\{(x_i, D_i, [a_i, b_i])\}$ of $[a,b]$ where $A_n = \int_a^b f_n dg$. Now $g \in LBV^k[a,b]$ implies that there exists $\delta_1(x) > 0$, $x \in [a,b]$ and $M > 0$ such that $\sum_{i=1}^{p} |s(1, g; D_i)| < M$ for all regulated $\delta_1^k$-fine division $\{(x_i, D_i, [a_i, b_i])\}$ of $[a,b]$. Let $\delta(x) = \min\,\{\delta_0(x), \delta_1(x)\}, x \in [a,b]$. Since $f_n(x) \to f(x)$ as $n \to \infty$, for all $x \in [a,b]$ and since there are finite number of associated points in regulated $\delta^k$-fine partial division $\{(x_i, D_i, [a_i, b_i])\}_{i=1}^{p}$ we can find $N_1$ such that $|f_n(\xi) - f(\xi)| < \frac{\epsilon}{M}$ for all $n > N_1$ and for all $\xi \in \Lambda =$ the set of associated points of $D_i$. So, for all $n > N_1$ we have $\sum_{i=1}^{p} |s(f_n, g; D_i) - s(f, g; D_i)| < \epsilon$. Also there exists $N_2 > N_1$ such that for all $n > N_2$,

$$\left|(k-1)\sum_{i=1}^{p-1} f_n(b_i)J(g; b_i) - (k-1)\sum_{i=1}^{p-1} f(b_i)J(g; b_i)\right| < \epsilon.$$

Now, for $n, m > N_2$

$$\begin{aligned}|A_n - A_m| \leq & \left| \sum_{i=1}^{p} s(f_m, g; D_i) + (k-1) \sum_{i=1}^{p-1} f_m(b_i) J(g; b_i) - A_m \right| \\ & + \left| \sum_{i=1}^{p} s(f_n, g; D_i) + (k-1) \sum_{i=1}^{p-1} f_n(b_i) J(g; b_i) - A_n \right| \\ & + \left| \sum_{i=1}^{p} s(f_m, g; D_i) - \sum_{i=1}^{p} s(f_n, g; D_i) \right| \\ & + \left| (k-1) \sum_{i=1}^{p-1} f_n(b_i) J(g; b_i) - (k-1) \sum_{i=1}^{p-1} f_m(b_i) J(g; b_i) \right| < 4\epsilon.\end{aligned}$$

So $\{A_n\}$ is a Cauchy sequence. Let $A = \lim_{n\to\infty} A_n$ and for $n > N > N_2$ let $\left| \int_a^b f_n dg - A \right| < \epsilon$. For fixed $n > N$, we have

$$\begin{aligned}& \left| \sum_{i=1}^{p} s(f, g; D_i) + (k-1) \sum_{i=1}^{p-1} f(b_i) J(g; b_i) - A \right| \\ & \quad \leq \left| \sum_{i=1}^{p} s(f_n, g; D_i) + (k-1) \sum_{i=1}^{p-1} f_n(b_i) J(g; b_i) - A_n \right| \\ & \quad + \left| (k-1) \sum_{i=1}^{p-1} f_n(b_i) J(g; b_i) - (k-1) \sum_{i=1}^{p-1} f(b_i) J(g; b_i) \right| \\ & \quad + \left| \sum_{i=1}^{p} s(f_n, g; D_i) - \sum_{i=1}^{p} s(f, g; D_i) \right| + \left| A_n - A \right| < 4\epsilon.\end{aligned}$$

So, $(f, g) \in GR_k^*[a, b]$ and $\int_a^b f dg = \lim_{n\to\infty} \int_a^b f_n dg$. □

**Acknowledgement**: The authors are really grateful to the referee for his valuable observations which played a key role in the development of the paper.

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