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A NOTE ON ALGEBRAIC SUMS OF SUBSETS OF THE REAL LINE

Abstract

We investigate the algebraic sums of sets for a large class of invariant σ-ideals and σ-fields of subsets of the real line. We give a simple example of two Borel subsets of the real line such that its algebraic sum is not a Borel set. Next we show a similar result to Proposition 2 from A. Kharazishvili paper [4]. Our results are obtained for ideals with coanalytical bases.

1 Introduction

We shall work in ZFC set theory. By ω we denote natural numbers. By $\triangle$ we denote the symmetric difference of sets. The cardinality of a set X we denote by $|X|$. By $\mathbb{R}$ we denote the real line and by $\mathbb{Q}$ we denote rational numbers. If A and B are subsets of $\mathbb{R}^n$ and $b \in \mathbb{R}^n$, then $A + B = \{a + b : a \in A \wedge b \in B\}$ and $A + b = A + \{b\}$. Similarly, if $A \subseteq \mathbb{R}$, $B \subseteq \mathbb{R}^n$ and $a \in \mathbb{R}$, then $A \cdot B = \{a \cdot b : a \in A \wedge b \in B\}$ and $a \cdot B = \{a\} \cdot B$.

We say that a family $\mathcal{F}$ of subsets of $\mathbb{R}$ is invariant if for each $A \in \mathcal{F}$, $a \in \mathbb{Q}$ and $b \in \mathbb{R}$ we have $a \cdot A + b \in \mathcal{F}$ (see [3]).

Let E be a polish space. If $x \in E$ and $\varepsilon > 0$, then by $B(x, \varepsilon)$ we denote the ball with center x and radius ε. The family of Borel subsets of E we denote by $\mathrm{Bor}\,(E)$. For each $\alpha < \omega_1$ such that $\alpha > 0$ we denote by $\Sigma^0_\alpha(E)$ the α-th additive class of Borel subsets of E and by $\Pi^0_\alpha(E)$ the α-th multiplicative class of Borel subsets of E; i.e., $\Sigma^0_1(E)$ coincides with the family of all open subsets

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*The research of the first author is partially supported by KBN grant 3-P03A-051-20 of E, $\Pi^0_1(E)$ is the family of all closed subsets of E, $\Sigma^0_2(E)$ is the family of all F_σ subsets of E, $\Pi^0_2(E)$ is the family of all G_δ subsets of E and so on. By $\Sigma^1_1(E)$ and $\Pi^1_1(E)$ we denote the family of all analytic and coanalytic subsets of the space E.

Let I be an ideal of subsets of a set X, $\mathcal{S}$ any family of subsets of X. We say that I is an ideal with $\mathcal{S}$-base, or that I has $\mathcal{S}$-base, if for every $A \in I$ there exists $B \in I \cap \mathcal{S}$ such that $A \subseteq B$. In particular, if E is a polish space and I is an ideal of subsets of E, then we say that the ideal I has a Borel base if it has $\mathrm{Bor}\,(E)$-base.

Notice that the σ-ideal $\mathbb{L}$ of Lebesgue measure zero subsets of $\mathbb{R}$ has a G_δ-base and that the σ-ideal $\mathbb{K}$ of first category subsets of $\mathbb{R}$ has a F_σ-base. Notice that it is quite easy to construct invariant σ-ideals with Π^1_1-base which has no Σ^1_1-base. Namely, let P be a non-empty perfect subset of the real line consisting of linearly independent over $\mathbb{Q}$ elements (see [5]). Let us fix a subset $A \in \Pi^1_1(\mathbb{R}) \setminus \Sigma^1_1(\mathbb{R})$ of P. Let I be the σ-ideal of subsets of the real line generated by the family of all translations of A. Then I is invariant over translations, has $\Pi^1_1(\mathbb{R})$-base and has no $\Sigma^1_1(\mathbb{R})$-base.

Suppose that $\mathcal{S}$ is a σ-field of subsets of a set X and that I is a σ-ideal of subsets of the set X. Then by $S[I]$ we denote the smallest σ-field containing $S \cup I$. It is easy to check that $S[I] = \{A \triangle B : A \in \mathcal{S} \wedge B \in I\}$.

We consider the field $\mathbb{R}$ as a linear space over the field $\mathbb{Q}$ of rational numbers. Let us recall that any base of the space $\mathbb{R}$ over $\mathbb{Q}$ is called a Hamel base. For each $X \subseteq \mathbb{R}$ we denote by $\mathrm{Span}(X)$ the linear subspace of $\mathbb{R}$ generated by the set X. If $X \subseteq \mathbb{R}$ and $n \in \omega$, then we put

$$\mathrm{Span}(X, n) = \underbrace{\mathbb{Q} \cdot X + \ldots + \mathbb{Q} \cdot X}_{n}.$$

Therefore for each set $X \subseteq \mathbb{R}$ we have $\mathrm{Span}(X) = \bigcup\{\mathrm{Span}(X, n) : n \in \omega\}$.

Let us recall that a boolean algebra $\mathcal{B}$ satisfies the countable chain condition if each family of pairwise disjoint elements from $\mathcal{B}$ is countable. In [3] and [4] A. Kharazishvili proved the following theorem.

Theorem 1.1. *Let I be an invariant σ-ideal of subsets of $\mathbb{R}$, let $\mathcal{S}$ be an invariant σ-algebra of subsets of $\mathbb{R}$, containing I and let the quotient boolean algebra $\mathcal{S}/I$ satisfy the countable chain condition. Then the next four sentences are equivalent:*

1. $(\exists X, Y \in I)(X + Y \notin I)$,
2. $(\exists X \in I)(X + X \notin I)$,

3. *there exists a linearly independent (over rational numbers $\mathbb{Q}$) set $X \in I$ such that* $\mathrm{Span}(X) \notin \mathcal{S}$,

4. $(\exists X \in I)(\mathrm{Span}(X) \notin \mathcal{S})$.

2 On Algebraic Sums of Borel Sets

In this section we recall some probably well - known facts about algebraic sums of Borel subsets of the space $\mathbb{R}^n$. It is clear that if A is a nonempty open subset of $\mathbb{R}^n$ and B is an arbitrary subset of $\mathbb{R}^n$, then $A+B$ is an open set. If A is compact and B is closed, then $A+B$ is closed. If A and B are closed, then $A+B$ is a F_σ set. But it may not be a closed set. Indeed, let $A=\sqrt{2}\cdot\mathbb{Z}$. Then $A+\mathbb{Z}$ is a dense subset of $\mathbb{R}$. The following result is a generalization of this observation.

Theorem 2.1. *For each ordinal number $\alpha<\omega_1$ there exists a set $A \in \Pi^0_\alpha(\mathbb{R})$ such that $A+\mathbb{Z} \in \Sigma^0_{\alpha+1}(\mathbb{R}) \setminus \Pi^0_\alpha(\mathbb{R})$.*

Proof. Let $B \subseteq (0,1)$ be such set that $B \in \Sigma^0_{\alpha+1}(\mathbb{R}) \setminus \Pi^0_\alpha(\mathbb{R})$. Therefore $B=\bigcup_{n\in\omega} A_n$, where $A_n \in \Pi^0_\alpha(\mathbb{R})$ for each natural number n. Let $A=\bigcup_{n\in\omega}(A_n+n)$. Then $A \in \Pi^0_\alpha(\mathbb{R})$ and $(A+\mathbb{Z})\cap(0,1)=B$. □

Suppose that $A,B \subseteq \mathbb{R}^n$ are Borel sets. Then $A+B=\{f(x,y): x \in A \wedge y \in B\}$, where $f(x,y)=x+y$. Therefore $A+B$ is a continuous image of the Borel set $A\times B$; so $A+B$ is an analytic set.

It is quite easy to give an example of two Borel subsets $A,B \subseteq \mathbb{R}^2$ such that $A+B$ is not a Borel set. Namely, let us consider a G_δ subset $A \subseteq [0,1]^2$ such that $\pi_2(A)$ is not a Borel set, where π_2 is the projection defined by $\pi_2(x,y)=y$. Finally let us put $B=[0,1]\times\{0\}$. Note that $(A\setminus B)\cap(\{0\}\times\mathbb{R})=\{0\}\times\pi_2(A)$ and therefore $A\setminus B$ is not a Borel set. In fact we showed that for each natural number $n\geq 2$ there exist a G_δ subset A of $\mathbb{R}^n$ and a compact subset B of $\mathbb{R}^n$ such that $A\setminus B$ is not a Borel set.

In 1954 B. Sodnomow (see [7]), P. Erdős and A. H. Stone in 1969 (see [2]) and C. A. Rogers in 1969 (see [6]) showed that there exists a G_δ subset A of the real line and a compact subset B of the real line such that $A+B$ is not a Borel set. We shall present now a simple proof of this fact.

Let N be a perfect and compact subset of the real line of algebraically independent elements. Let P^* and Q^* be two disjoint nonempty perfect subsets of N. Then $\mathrm{Span}(P^*)\cap\mathrm{Span}(Q^*)=\{0\}$. Let $G^* \subseteq P^*\times Q^*$ be such a G_δ set such that $\pi_2(G^*)$ is not a Borel set. Let us fix an element $r \in Q^*$. Using similar arguments as above we show that $G^*\setminus(P^*\times\{r\})$ is not a Borel subset of $\mathbb{R}^2$. Let $P=P^*\setminus P^*$ and $Q=Q^*\setminus Q^*$. Then $G^*\setminus(P^*\times\{r\})\subseteq P\times Q$ and

$P \times Q$ is a compact set. Let $\varphi(x,y) = x + y$. Then φ is a continuous one-to-one function on $P \times Q$. Therefore φ is a homeomorphism between $P \times Q$ and $\varphi(P \times Q)$. Let $G = \varphi(G^*)$ and $P = -(\varphi(P^*) + r)$. Then G is a G_δ subset of $\mathbb{R}$ and P is a compact set. But $G + P = \varphi(G^* \setminus (P^* \times \{r\})$; so $G + P$ is not a Borel set.

3 On Summable Families of Sets

Let $\mathcal{A}$ and $\mathcal{S}$ be two families of sets. We say that $\mathcal{A}$ is $\mathcal{S}$-summable if for every family $\mathcal{B} \subseteq \mathcal{A}$ we have $\bigcup \mathcal{B} \in \mathcal{S}$. A family $\mathcal{A}$ is point-finite if $\{X \in \mathcal{A} : x \in X\}$ is finite for each x. The following result is a slightly stronger formulation of the main result from [1].

Theorem 3.1. *Suppose that E is a polish space and that $I \subseteq P(E)$ is a σ-ideal with a $\Pi^1_1(E)$-base. Suppose also that $\mathcal{A}$ is a point-finite and $\Sigma^1_1(E)[I]$-summable family. Then there exists a countable subfamily $\mathcal{B} \subseteq \mathcal{A}$ such that $\bigcup \mathcal{A} \setminus \bigcup \mathcal{B} \in I$.*

PROOF. Without loss of generality we may assume that the cardinality of the family $\mathcal{A}$ is less or equal to that of the continuum. Let T be an arbitrary subset of $\mathbb{R}$ which contain no perfect subset and $|T| = |\mathcal{A}|$. Let $\mathcal{A} = \{A_t : t \in T\}$. Let us put $R = \{(x,t) \in E \times \mathbb{R} : x \in A_t\}$. Let D be a countable dense subset of $\mathbb{R}$. Since the family $\mathcal{A}$ is point-finite, we get

$$R = \bigcap_{n\in\omega} \bigcup_{d\in D} (R^{-1}(B(d, \frac{1}{n})) \times B(d, \frac{1}{n})).$$

For any $n \in \omega$ and $d \in D$ let $S_{n,d} \in \Sigma^1_1(E)$ and $A_{n,d} \in I$ be such that $R^{-1}(B(d, \frac{1}{n})) = S_{n,d} \triangle A_{n,d}$. Let us put $B = \bigcup_n \bigcup_d A_{n,d}$ and let us fix some $C \in \Pi^1_1(E)$ such that $C \in I$ and $B \subseteq C$. Then the set $Q = R \cap ((E \setminus C) \times \mathbb{R})$ is an analytic subset of the product polish space $E \times \mathbb{R}$. Hence the set $S = \{t \in T : (\exists x)((x,t) \in Q\}$ is a subset of T which is an analytic subset of the real numbers, so is countable. Thus we have $\bigcup_{t\in S} A_t \supseteq \bigcup \mathcal{A} \setminus C$. $\square$

It is possible to apply Theorem directly 3.1 to the family of all analytic subsets of a polish space (put $I = \{\emptyset\}$), to the family of all Lebesgue measurable subsets of $\mathbb{R}$, to the σ-field of all subsets with the Baire property of a fixed polish space and so on. We shall now formulate some other applications of this theorem.

Corollary 3.2. *Let E be a polish space and let $\mathcal{A}$ be an uncountable family of pairwise disjoint $\Sigma^1_1(E)$ sets. Then there exists a subfamily $\mathcal{B} \subseteq \mathcal{A}$ such that $\bigcup \mathcal{B}$ is not in the class $\Sigma^1_1(E)$.*

Corollary 3.3. *Let E be a polish space and let $I \subseteq P(E)$ be a σ-ideal with a $\Pi^1_1(E)$-base and let $\mathcal{A}$ be an uncountable pairwise disjoint family of subsets of* $\mathrm{Bor}\,(E)[I] \setminus I$. *Then there exists a subfamily $\mathcal{B} \subseteq \mathcal{A}$ such that $\bigcup \mathcal{B}$ is not in the class* $\mathrm{Bor}\,(E)[I]$.

Corollary 3.4. *(see [1]) Suppose that E is a polish space and that $I \subseteq P(E)$ is a σ-ideal with a $\Pi^1_1(E)$-base. Suppose also that $\mathcal{A} \subseteq I$ is a point-finite family of sets such that $\bigcup \mathcal{A} \notin I$. Then there exists a subfamily $\mathcal{B}$ of $\mathcal{A}$ such that* $\bigcup \mathcal{B} \notin \mathrm{Bor}\,(E)[I]$.

For a given family $\mathcal{S}$ of sets let us denote by $\mathcal{S}^-$ the family of sets which are hederatively in $\mathcal{S}$; i.e., $\mathcal{S}^- = \{A : (\forall B \subseteq A)(B \in \mathcal{S})\}$. Note that if *Leb* denotes the family of all Lebesgue measurable subsets of the real line, then *Leb*$^-$ is the family of all Lebesgue measure zero subsets of the reals. A similar remark is holds for the Baire property.

Corollary 3.5. *Suppose that E is a polish space and that $I \subseteq P(E)$ is a σ-ideal with a $\Pi^1_1(E)$-base. Then* $(\mathrm{Bor}\,(E)[I])^- = I$.

Proof. The inclusion $I \subseteq (\mathrm{Bor}\,(E)[I])^-$ is obvious. Suppose that $A \in (\mathrm{Bor}\,(E)[I])^-$. Let $\mathcal{A} = \{\{x\} : x \in A\}$. Then $\mathcal{A}$ is point-finite and $\mathrm{Bor}\,(E)[I]$-summable. Therefore there exists a countable family $\mathcal{B} \subseteq \mathcal{A}$ such that $\bigcup \mathcal{A} \setminus \bigcup \mathcal{B} \in I$. But then $A = \bigcup \mathcal{B} \cup (\bigcup \mathcal{A} \setminus \bigcup \mathcal{B})$, so $A \in I$. □

4 Main Result

In this section we prove the main result of this paper. We begin with one auxiliary result, whose easy proof is omitted.

Lemma 4.1. *Let C be a linearly independent over $\mathbb{Q}$ subset of the real line and let A, B be disjoint subsets of the set C. Then* $\mathrm{Span}(A) + b_1 \cap \mathrm{Span}(A) + b_2 = \emptyset$ *for different $b_1, b_2 \in B$.*

Theorem 4.2. *Let $I \subseteq P(\mathbb{R})$ be an invariant σ-ideal with a $\Pi^1_1(\mathbb{R})$-base. Then the following are equivalent:*

1. $(\exists A, B \in I)(A + B \notin I)$,
2. $(\exists A, B \in I)(A + B \notin \mathrm{Bor}\,(\mathbb{R})[I])$.

Proof. The implication (2) $\Rightarrow$ (1) is obvious. We prove the implication (1) $\Rightarrow$ (2). Suppose that A, $B \in I$ and $A + B \notin I$. Observe that the set $D = A \cup B$ is such that $D \in I$ and $D + D \notin I$. Let A^* be a maximal linearly independent subset of D over $\mathbb{Q}$. Then $D + D \subseteq \mathrm{Span}(A^*)$ and hence

$\mathrm{Span}(A^*) \notin I$. Let $\{A_n : n < \omega\}$ be an increasing family of subsets of the set A^* such that:

1. $A^* = \bigcup_{n\in\omega} A_n$,
2. $(\forall n \in \omega)(|A^* \setminus A_n| > \omega)$.

Then $\mathrm{Span}(A^*) = \bigcup_{n\in\omega} \mathrm{Span}(A_n)$. Hence there exists $m \in \omega$ such that $\mathrm{Span}(A_m) \notin I$. But $\mathrm{Span}(A_m) = \bigcup_{k\in\omega} \mathrm{Span}(A_m, k)$. Therefore we find natural number k such that $\mathrm{Span}(A_m, k-1) \in I$ and $\mathrm{Span}(A_m, k) \notin I$. We may assume that $\mathrm{Span}(A_m, k) \in \mathrm{Bor}(\mathbb{R})[I]$; otherwise the theorem is proved. Let $C = \mathrm{Span}(A_m, k)$ and let $T = A^* \setminus A_m$. Lemma 4.1 implies that $\mathcal{A} = \{C + t : t \in T\}$ is uncountable family of pairwise disjoint subsets of $\mathrm{Bor}(\mathbb{R})[I] \setminus I$. Let us apply Corollary 3.3 to the family $\mathcal{A}$. Then we obtain a set $Z \subseteq T$ such that $\bigcup_{t\in Z}(C + t) = C + Z \notin \mathrm{Bor}(\mathbb{R})[I]$. Hence we have

$$\underbrace{\mathbb{Q} \cdot A_m + \cdots + \mathbb{Q} \cdot A_m}_{k-1} + \mathbb{Q} \cdot A_m + Z \notin \mathrm{Bor}(\mathbb{R})[I].$$

Note that if $\mathbb{Q} \cdot A_m + Z \in I$ or $\mathbb{Q} \cdot A_m + Z \notin \mathrm{Bor}(\mathbb{R})[I]$, then the proof of the theorem is done. Suppose that $\mathbb{Q} \cdot A_m + Z \in \mathrm{Bor}(\mathbb{R})[I] \setminus I$. Then $\mathcal{A} = \{\mathbb{Q} \cdot A_m + z : z \in Z\}$ is uncountable pairwise disjoint family of subsets of ideal I such that $\bigcup \mathcal{A} \notin I$. So by Corollary 3.4 there exists a set $S \subseteq Z$ such that $\mathbb{Q} \cdot A_m + S \notin \mathrm{Bor}(\mathbb{R})[I]$. This finishes the proof of the theorem. $\square$

It is easy to see that a similar result also holds for an invariant σ-ideal of subsets of the Euclidean space $\mathbb{R}^n$ with a $\Pi^1_1(\mathbb{R}^n)$-base, where n is an arbitrary natural number. We can apply our theorem to some natural σ-ideals I such that the quotient boolean algebra does not satisfy the countable chain condition. Hence we may apply our theorem in such cases when Kharazishvili's theorem mentioned in the introduction does not apply. For example, let us consider the σ-ideal of subsets of $\mathbb{R}^2$

$$\mathcal{M} = \{A \subseteq \mathbb{R}^2 : (\exists B \in \mathrm{Bor}(\mathbb{R}^2))(A \subseteq B \wedge (\forall x)(B_x \in \mathbb{L})\},$$

where $B_x = \{y \in \mathbb{R} : (x, y) \in B\}$. Then $\mathcal{M}$ is an invariant σ-ideal of subsets of $\mathbb{R}^2$ with a Borel base. It is easy to check that the boolean algebra $\mathrm{Bor}(\mathbb{R}^2)/\mathcal{M}$ does not satisfy the countable chain condition.

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