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WEAK TYPE AND RESTRICTED WEAK TYPE (p,p) OPERATORS IN ORLICZ SPACES

Abstract

Let (Ω,μ) be a finite measure space, $\Phi(t)=\int_0^t a(s)\,ds$ and $\Psi(t)=\int_0^t b(s)\,ds$, where a and b are positive continuous functions defined on $[0,\infty)$. Consider the associated Orlicz spaces $L^\Phi(\Omega)$ and $L^\Psi(\Omega)$. In this paper we find a relationship between a and b to assure that T, a sublinear and positive homogeneous operator of restricted weak type (p,p) and of type (∞,∞), maps $L^\Psi(\Omega)$ into $L^\Phi(\Omega)$. If the two Orlicz spaces are normable, our results imply the continuity of T. This relation between a and b is sharp since it is shown to be necessary for operators like the one side maximal operators related to the Cesàro averages.

1 Introduction

Let (Ω,μ) be a finite measure space and $\mathfrak{M}(\Omega)$ be the space of measurable functions from Ω into $\overline{\mathbb{R}}$. Let Ψ be a nondecreasing continuous function such that $\Psi(0)=0$ and $\lim_{t\to\infty}\Psi(t)=\infty$. The family of functions

$$L^\Psi(\Omega)=\{f\in\mathfrak{M}(\Omega):\int_\Omega \Psi(\epsilon|f|)\,d\mu<\infty \text{ for some } \epsilon>0\ \}$$

is called an *Orlicz Space*. For more details see Rao and Ren [6].

If f is a measurable function, we define $\mu_f:(0,\infty)\to[0,\infty]$, the distribution function of f, as $\mu_f(s)=\mu(\{x\in\Omega:\ |f(x)|>s\})$ for all $s>0$.

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Let T be a sublinear and positive homogeneous operator defined on a subspace $\mathfrak{D} \subset \mathfrak{M}(\Omega)$ and taking values on $\mathfrak{M}(\Omega)$. We assume that $\mathfrak{D}$ contains all the characteristic functions of sets of finite measure and has the property that whenever $f \in \mathfrak{D}$ and g is a truncation of f, then $g \in \mathfrak{D}$. Such an operator T is of *weak type* (p,p) if there exists a constant A such that for any measurable function $f \in \mathfrak{D}$, $\mu_{Tf}(s) \le \left(\frac{A}{s}\|f\|_p\right)^p$ for all $s > 0$.

T is of *restricted weak type* (p,p) if there exists a constant A such that for any measurable function $f \in \mathfrak{D}$, $\mu_{Tf}(s) \le \left(\frac{A}{s}\int_0^\infty \mu_f^{1/p}\right)^p$ for all $s > 0$. Finally, T is of *type* (∞,∞) if there exists a constant B such that for any measurable function $f \in \mathfrak{D}$, $\|Tf\|_\infty \le B\|f\|_\infty$.

Remark 1. In terms of Lorentz spaces $L^{p,q}$, an operator T is of restricted weak type (p,p) if there exists a constant C such that $\|Tf\|_{p,\infty} \le C\|f\|_{p,1}$ for all $f \in \mathfrak{D}$. If a sublinear and positive homogeneous operator satisfies the weak type (p,p) inequality for all characteristic functions of sets of finite measure, then the operator is of restricted weak type (p,p) (see [7]).

In the sequel we will work with functions Φ and Ψ given by $\Phi(t) = \int_0^t a(s)\,ds$ and $\Psi(t) = \int_0^t b(s)\,ds$ for all $t \ge 0$, where a and b be positive continuous functions defined on $[0,\infty)$.

2 Statement of the Theorems

In [4] the Hardy-Littlewood maximal function is studied in the torus $\mathbb{T}$ and, under some assumptions on a and b, it is found that

$$\int_{\mathbb{T}} \Phi(|Mf|) \le C' + C' \int_{\mathbb{T}} \Psi(C'|f|) \text{ for all } f \in \mathfrak{M}(\mathbb{T}) \tag{1}$$

if and only if

$$\int_1^t \frac{a(s)}{s}\,ds \le Cb(Ct) \text{ for all } t > 1 \; . \tag{2}$$

In [2] the authors consider the maximal function in the context of spaces of homogeneous type solving the problem under somehow more general assumptions on Φ and Ψ. Also, in [3] a similar problem is solved for the fractional maximal function of order $0 < \alpha \le 1$, being the Hardy-Littlewood maximal function a particular case.

The properties of M used to prove (2) implies (1), are only the weak type $(1,1)$ and type (∞,∞). In consequence it is easy to extend these results to operators of weak type (p,p) with $p > 1$ and type (∞,∞) as follows.

Theorem 2.1. *Let T be of weak type (p,p) with $p \geq 1$, and of type (∞,∞). If for some constant C, a and b satisfy*

$$t^{p-1}\int_1^t \frac{a(s)}{s^p}\,ds \leq Cb(Ct) \text{ for all } t > 1 \tag{3}$$

then, there exists a constant C' such that $\int_\Omega \Phi(|Tf|)\,d\mu \leq C' + C'\int_\Omega \Psi(C'f)\,d\mu$ for all $f \in \mathfrak{D}$.

Remark 2. We notice that our assumptions on a and b allow us to obtain Kolmogorov type inequalities.

A model operator which plays the role of M in this case is the maximal $\mathcal{M}_p$, acting on Lebesgue measurable functions on $[0,1]$ given by

$$\mathcal{M}_p f(x) = \sup_{I\in\mathcal{I}, x\in I}\left(\frac{1}{|I|}\int_I f^p\right)^{1/p} \tag{4}$$

with $\mathcal{I}$ the family of all intervals contained on $[0,1]$.

For this operator we have the following theorem analogous to the results on [4]. In particular, it says that condition (3) is sharp.

Theorem 2.2. *Let $p \geq 1$ and b monotone on $[1,\infty)$. There exists a constant C' such that*

$$\int_{[0,1]} \Phi(\mathcal{M}_p f) \leq C' + C'\int_{[0,1]} \Psi(C'|f|) \text{ for all } f \in \mathfrak{M}([0,1]) \tag{5}$$

if and only if (3) holds.

For $p > 1$ there exist operators which are of restricted weak type (p,p) but not of weak type (p,p). Examples of these are the maximal operators associated to Cesàro averages of order α with $0 < \alpha < 1$, defined for $f \in \mathfrak{M}([0,1])$ by

$$M_\alpha^+ f(x) = \sup_{x<c<1}\frac{1}{(c-x)^\alpha}\int_x^c |f(s)|(c-s)^{\alpha-1}\,ds \text{ for } x \in [0,1] \tag{6}$$

and

$$M_\alpha^- f(x) = \sup_{0<c<x}\frac{1}{(x-c)^\alpha}\int_c^x |f(s)|(s-c)^{\alpha-1}\,ds \text{ for } x \in [0,1]. \tag{7}$$

which are known to be of restricted weak type $(1/\alpha, 1/\alpha)$ but not of weak type $(1/\alpha, 1/\alpha)$.

It is well known that if we start with an operator of weak type (p, p) or restricted weak type (p, p), the Marcinkiewicz Interpolation Theorem gives the same boundedness results for the intermediate spaces L^q with $p < q < \infty$.

One of the mainstay of this paper is to see what occurs when we are dealing with the more general class of Orlicz spaces.

The results of the following theorems show that the properties of boundedness differ if we start with an operator of weak type (p, p) or with one of restricted weak type (p, p), $p > 1$.

Theorem 2.3. *Let T be of restricted weak type (p, p) with $p > 1$, and of type (∞, ∞). If for some constant C, a and b satisfy*

$$\sup_{t>1} \left(\int_1^t \frac{a(s)}{s^p} \, ds \right)^{1/p} \left(\int_t^\infty b(Cs)^{-p'/p} \, ds \right)^{1/p'} < \infty, \tag{8}$$

then there exists a constant C' such that $\int_\Omega \Phi(|Tf|) \, d\mu \leq C' + C' \int_\Omega \Psi(C'|f|) \, d\mu$ for all $f \in \mathfrak{D}$.

We now introduce the linear operator $\mathcal{H}_p$ with $p > 1$ defined for $f \in \mathfrak{M}([0,1])$ by $\mathcal{H}_p f(x) = \frac{1}{x^{1/p}} \int_0^x f(s) s^{1/p-1} \, ds$ for $x \in [0,1]$. It is easy to see that the $\mathcal{H}_p$ operator is of restricted weak type (p, p) and of type (∞, ∞).

Remark 3. Note that if f is decreasing then $\mathcal{H}_p f$ is also decreasing; in fact, if x and y are in $[0,1]$ and $x < y$, using that f is decreasing, we have

$$\begin{aligned} \mathcal{H}_p f(x) &= \frac{1}{x^{1/p}} \int_0^x f(s) s^{1/p-1} \, ds \; = \frac{1}{x^{1/p}} \int_0^y f(\frac{x}{y} t) \left(\frac{x}{y} t \right)^{1/p-1} \frac{x}{y} \, dt \\ &= \frac{1}{y^{1/p}} \int_0^y f(\frac{x}{y} t) t^{1/p-1} \, dt \geq \frac{1}{y^{1/p}} \int_0^y f(t) t^{1/p-1} \, dt = \mathcal{H}_p f(y). \end{aligned}$$

Also, as it is easy to realize from its form, $\mathcal{H}_{1/\alpha}$ is related to M_α^+ and M_α^-. The next theorem tells us that, as in the case of weak type, condition (8) for a and b is sharp.

Theorem 2.4. *Let $p > 1$ and b monotone on $[1, \infty)$. There exists a constant C' shuch that*

$$\int_{[0,1]} \Phi(\mathcal{H}_p f) \leq C' + C' \int_{[0,1]} \Psi(C'|f|) \text{ for all } f \in \mathfrak{M}([0,1]) \tag{9}$$

if and only if condition (8) holds.

We should mention that similar results to Theorem 2.4 were obtained in [1] in terms of norm inequalities and under more restricted assumptions on Φ and Ψ. From Theorem 2.4 we can derive the following consequence.

Corolary 2.5. *Let $0 < \alpha < 1$ and b monotone on $[1,\infty)$. There exists a constant C' such that*

$$\int_{[0,1]} \Phi(M_\alpha^- f) \le C' + C' \int_{[0,1]} \Psi(C'|f|) \text{ for all } f \in \mathfrak{M}([0,1]) \tag{10}$$

if and only if condition (8) *holds with $p = 1/\alpha$.*

Proof. The operator M_α^- is of restricted weak type $(1/\alpha, 1/\alpha)$ and of type (∞,∞). Thus the sufficiency of condition (8) follows from Theorem 2.3. The necessity follows from the fact that for $f \in \mathfrak{M}([0,1])$ we have $M_\alpha^- f(x) \ge \mathcal{H}_p f(x)$ for almost all $x \in [0,1]$, and the result is a consequence of Theorem 2.4. □

The same is true for M_α^+ since $M_\alpha^+ f(x) = M_\alpha^- g(-x)$ with $g(x) = f(1/2-x)$ for all $x \in [0,1]$, and these two functions have the same distribution function.

Remark 4. It is not hard to find the largest spaces that are mapped into L^p. In fact, if we thake for $a(t) = t^{p-1}$ the best possible function b satisfying either (3) or (8) we get $\mathcal{M}_p$ maps the space $L^p \log L$ into L^p, whereas M_α^- and M_α^+ map the space $L^p(\log L)^p$ into L^p.

Remark 5. In particular, Remark 4 implies that condition (8) is strictly stronger than (3). In fact, the pair $a(t) = t^{p-1}$ and $b(t) = t^{p-1}\log(t+1)$ satisfies (8) but not (3).

3 Proofs of the Theorems

The proof of Theorem 2.1 requires the following lemma which tells us how to control the size of the distribution of Tf in terms of the distribution of f.

Lemma 3.1. *Let T be an operator of weak type (p,p), $p > 1$, and of type (∞,∞) with constants A and B respectively. Then, for every function f in the domain of T,*

$$\mu_{Tf}(t) \le \frac{(4Ap)^p}{t^p} \int_{t/4B}^\infty s^{p-1} \mu_f(s)\, ds \text{ for all } t > 0$$

Proof. Let $f \in \mathfrak{D}$ be given. For $t > 0$ let us define

$$f^t(x) = \begin{cases} f(x) & \text{if } |f(x)| > t/2B \\ 0 & \text{otherwise} \end{cases}$$

and $f_t = f - f^t$. Since T is sublinear

$$\mu(\{|Tf| > t\}) \le \mu(\{|Tf^t| > t/2\}) + \mu(\{|Tf_t| > t/2\}). \tag{11}$$

From the boundedness of T in L^∞ we have $|Tf_t(x)| \le t/2$, which implies $\mu(\{|Tf_t| > t/2\}) = 0$. On the other hand, using the weak type (p,p) of T, we get

$$\mu(\{|Tf^t| > t/2\}) \le \left(\frac{2A}{t}\|f^t\|_p\right)^p. \tag{12}$$

Since $\mu_{f^t}(s) = \mu_f(t/2B)$ for $s \in (0, t/2B)$, $\mu_{f^t} \le \mu_f$ and the fact that μ_f is decreasing, we have

$$\begin{aligned}
\|f^t\|_p^p &= p\int_0^\infty s^{p-1}\mu_{f^t}(s)\,ds = p\left(\int_0^{t/4B} + \int_{t/4B}^\infty\right) s^{p-1}\mu_{f^t}(s)\,ds \\
&= p\mu_f(t/2B)\int_0^{t/4B} s^{p-1}\,ds + p\int_{t/4B}^\infty s^{p-1}\mu_{f^t}(s)\,ds \\
&\le p\mu_f(t/2B)\int_{t/4B}^{t/2B} s^{p-1}\,ds + p\int_{t/4B}^\infty s^{p-1}\mu_f(s)\,ds \\
&\le p\int_{t/4B}^{t/2B} \mu_f(s)s^{p-1}\,ds + p\int_{t/4B}^\infty s^{p-1}\mu_f(s)\,ds \\
&\le 2p\int_{t/4B}^\infty s^{p-1}\mu_f(s)\,ds
\end{aligned} \tag{13}$$

and this completes the proof. $\square$

Proof of Theorem 2.1. Let f be a function in $\mathfrak{D}$

$$\begin{aligned}
\int_\Omega \Phi(|Tf|)\,d\mu = \int_0^\infty a(t)\mu_{Tf}(t)\,dt &= \left(\int_0^1 + \int_1^\infty\right) a(t)\mu_{Tf}(t)\,dt \\
&\le \mu(\Omega)\Phi(1) + \int_1^\infty a(t)\mu_{Tf}(t)\,dt.
\end{aligned}$$

From Lemma 3.1, Fubini's Theorem and inequality (3) we get

$$\begin{aligned}
\int_1^\infty a(t)\mu_{Tf}(t)\,dt &\le \int_1^\infty a(t)\left(\frac{(4Ap)^p}{t^p}\int_{t/4B}^\infty s^{p-1}\mu_f(s)\,ds\right)dt \\
&= (4Ap)^p\int_{1/4B}^\infty \mu_f(s)\left(s^{p-1}\int_1^{4Bt}\frac{a(t)}{t^p}\,dt\right)ds
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{C(4Ap)^p}{(4B)^{p-1}} \int_{1/4B}^{\infty} b(4BCs)\mu_f(s)\,ds \\
&\leq (\frac{Ap}{B})^p 4BC \int_0^{\infty} b(4BCs)\mu_f(s)\,ds \\
&= (\frac{Ap}{B})^p \int_{\Omega} \Psi(4BCf)d\mu
\end{aligned}
$$

and the proof is complete. □

Proof of Theorem 2.2. Since the operator $\mathcal{M}_p$ is simultaneously of weak type (p,p) and of type (∞,∞) the "only if" part follows from Theorem 2.1.

Suppose that (5) holds. Let $t > 1$ fixed and $f_t = t\chi_{[0,1/t^p)}$ being in $\mathfrak{M}([0,1])$.

$$
\begin{aligned}
\mathcal{M}_p f_t(x) &\geq \left[\frac{1}{x}\int_0^x [f_t(y)]^p dy\right]^{1/p} = \left[\frac{t^p}{x}\int_0^x \chi_{[0,1/t^p)} dy\right]^{1/p} \\
&\geq \begin{cases} t & \text{if } x \in [0,1/t^p) \\ \frac{1}{x^{1/p}} & \text{if } x \in (1/t^p,1) \end{cases}.
\end{aligned}
$$

Then for $1 < s < t$,

$$
\begin{aligned}
\lambda_{\mathcal{M}_p f_t}(s) &= |\{x : \mathcal{M}_p f_t(x) > s\}| \\
&\geq |\{x \in (0,1/t^p] : t > s\} \cup \{x \in (1/t^p,1] : \frac{1}{x^{1/p}} > s\}| \\
&= |(0,1/t^p] \cup \{x \in (1/t^p,1] : x < \frac{1}{s^p}\}| \\
&= |(0,1/s^p]| = \frac{1}{s^p}.
\end{aligned}
$$

Therefore it follows that

$$
\begin{aligned}
\int_0^1 \Phi(|\mathcal{M}_p f_t|) &= \int_0^{\infty} a(s)\lambda_{\mathcal{M}_p f_t}(s)\,ds \\
&\geq \int_1^t a(s)\lambda_{\mathcal{M}_p f_t}(s)\,ds \geq \int_1^t \frac{a(s)}{s^p}\,ds.
\end{aligned} \tag{14}
$$

On the other hand, let $L = \liminf_{s\to\infty} \dfrac{b(s)}{s^{p-1}}$. If this limit is zero it is easy to see that there is a function f in L^{Ψ} that $M_p f = \infty$ everywhere in $[0,1]$. In fact, since b is monotone we have also $\liminf_{s\to\infty} \dfrac{\Psi(s)}{s^p} = 0$ and then we can choose an increasing sequence of numbers t_n, $n \geq 1$, such that $t_n > 2^n$ and $\dfrac{\Psi(t_n)}{t_n^p} < \dfrac{1}{2^n}$. Then the function $f = \displaystyle\sum_{n=1}^{\infty} t_n \chi_{I_n}$, with $I_n = \left[\sum_{k=1}^{n-1} \frac{1}{t_n^p}, \sum_{k=1}^{n} \frac{1}{t_n^p}\right)$ has this property. So we only need to consider b such that $\displaystyle\liminf_{s\to\infty} \frac{b(s)}{s^{p-1}} > 0$ and since b is monotone it must be nondecresing on $[1,\infty)$. Since $\lambda_{f_t}(s) = \begin{cases} 1/t^p & \text{if } 0 < s < t \\ 0 & \text{if } s \geq t, \end{cases}$ we have

$$\begin{aligned}\int_0^1 \Psi(C'|f_t|) = C' \int_0^\infty b(C's)\lambda_{f_t}(s)\,ds = \frac{C'}{t^p}\int_0^t b(C's)\,ds \\ \leq C'\Psi(1) + \frac{C'}{t^p}\int_1^t b(C's)\,ds \leq C'\Psi(1) + \frac{C'b(C't)}{t^{p-1}}.\end{aligned} \tag{15}$$

Now since $\displaystyle\liminf_{s\to\infty} \frac{b(s)}{s^{p-1}} = L > 0$, there exists s_0 such that $M \leq \dfrac{b(s)}{s^{p-1}}$ for $s > s_0$, with $M = 1$ if $L = \infty$ and $M = \dfrac{L}{2}$ when L is finite. Then from (14) and (15) we have

$$\begin{aligned}\int_1^t \frac{a(s)}{s^p}\,ds &\leq \int_0^1 \Phi(|\mathcal{M}_p f_t|) \leq C' + C'\int_0^1 \Psi(C'|f_t|) \\ &\leq C' + C'^2\Psi(1) + \frac{C'^2 b(C't)}{t^{p-1}} \\ &\leq \frac{C' + C'^2\Psi(1)}{M s_0^{p-1}} \frac{b(s_0 t)}{t^{p-1}} + \frac{C'^2 b(C't)}{t^{p-1}} \leq C t^{1-p} b(Ct).\end{aligned}$$

with $C = \max\left\{C', s_0, C'^2 + \dfrac{C' + C'^2\Psi(1)}{M s_0^{p-1}}\right\}$. Since C is independent of t, (3) follows. $\square$

To prove Theorem 2.3 we need the analogous to Lemma 3.1.

Lemma 3.2. *Let T be an operator of restricted weak type (p,p) and of type (∞,∞) with constants A and B respectively. Then, for every function f in the domain of T,*

$$\mu_{Tf}(t) \leq \left[\frac{4A}{t}\int_{t/4B}^\infty \mu_f(s)^{1/p}\,ds\right]^p \quad \textit{for all } t > 0 \tag{16}$$

PROOF. Let $f \in \mathfrak{D}$ and $t > 0$. We define f^t and f_t as in the proof of Lemma 3.1. Then we have $\mu(\{|Tf_t| > t/2\}) = 0$. Since T is of restricted weak type, $\mu_{f^t}(s) = \mu_f(t/2B)$ for $s \in (0, t/2B)$, $\mu_f^t \leq \mu_f$ and the fact that μ_f is decreasing, we have

$$\begin{aligned}
\mu(\{|Tf^t| > t/2\}) &\leq \left[\frac{2A}{t}\int_0^\infty \mu_{f^t}(s)^{1/p}\,ds\right]^p \\
&= \left(\frac{2A}{t}\right)^p \left[\left(\int_0^{t/4B} + \int_{t/4B}^\infty\right)\mu_{f^t}(s)^{1/p}\,ds\right]^p \\
&= \left(\frac{2A}{t}\right)^p \left[\frac{t}{4B}\mu_f(t/2B)^{1/p} + \int_{t/4B}^\infty \mu_{f^t}(s)^{1/p}\,ds\right]^p \\
&\leq \left(\frac{2A}{t}\right)^p \left[\int_{t/4B}^{t/2B} \mu_f(s)^{1/p}\,ds + \int_{t/4B}^\infty \mu_f(s)^{1/p}\,ds\right]^p \\
&\leq \left[\frac{4A}{t}\int_{t/4B}^\infty \mu_f(s)^{1/p}\,ds\right]^p \qquad \square
\end{aligned}$$

PROOF OF THEOREM 2.3. This proof resembles that of Theorem 1 in [5]. Suppose (8) holds, that is, there exists a constant D such that for all $t > 1$,

$$\left(\int_1^t \frac{a(s)}{s^p}\,ds\right)^{1/p}\left(\int_t^\infty b(Cs)^{-p'/p}\,ds\right)^{1/p'} \leq D. \tag{17}$$

Let f be a function in the domain of T. From Lemma 3.2,

$$\begin{aligned}
\int_\Omega \Phi(|Tf|)\,d\mu = \int_0^\infty a(s)\mu_{Tf}(s)\,ds &\leq \left(\int_0^1 + \int_1^\infty\right) a(s)\mu_{Tf}(s)\,ds \\
&\leq \Phi(1)\mu(\Omega) + A^p\int_1^\infty a(s)\left[\frac{4}{s}\int_{s/4B}^\infty \mu_f(t)^{1/p}\,dt\right]^p ds \\
&= \Phi(1)\mu(\Omega) + \left(\frac{A}{B}\right)^p\int_1^\infty a(s)\left[\frac{1}{s}\int_s^\infty \mu_f(t/4B)^{1/p}\,dt\right]^p ds.
\end{aligned}$$

Now, if we call $h(t) = \left[\int_t^\infty b(Cr)^{-p'/p}dr\right]^{1/pp'}$ and $g(t) = \mu_f(t/4B)^{1/p}$ we have by Hölder's inequality and Fubini's Theorem

$$\int_1^\infty a(s)\left[\frac{1}{s}\int_s^\infty g(t)\,dt\right]^p ds$$

$$
\begin{aligned}
&= \int_1^\infty a(s) \left[\frac{1}{s}\int_s^\infty g(t)h(t)b(Ct)^{1/p}\frac{1}{h(t)b(Ct)^{1/p}}\,dt\right]^p ds \\
&\le \int_1^\infty \frac{a(s)}{s^p}\left[\int_s^\infty (g(t)h(t))^p b(Ct)\,dt\right]\left[\int_s^\infty b(Cr)^{-p'/p}h(r)^{-p'}dr\right]^{p/p'} ds \\
&= \int_1^\infty [g(t)h(t)]^p b(Ct)\left\{\int_1^t \frac{a(s)}{s^p}\left[\int_s^\infty b(Cr)^{-p'/p}h(r)^{-p'}dr\right]^{p/p'} ds\right\} dt.
\end{aligned}
$$

Since integration by parts yields

$$
\int_s^\infty b(Cr)^{-p'/p}h(r)^{-p'}dr = p'\left[\int_s^\infty b(Cr)^{-p'/p}dr\right]^{1/p'}
$$

and

$$
\int_1^t \frac{a(s)}{s^p}\left[\int_1^s \frac{a(r)}{r^p}dr\right]^{-1/p'} ds = p\left[\int_1^t \frac{a(r)}{r^p}dr\right]^{1/p},
$$

using inequality (17) twice we have

$$
\begin{aligned}
&\int_1^\infty [g(t)h(t)]^p b(Ct)\left\{\int_1^t \frac{a(s)}{s^p}\left[\int_s^\infty b(Cr)^{-p'/p}h(r)^{-p'}dr\right]^{p/p'} ds\right\} dt \\
&= (p')^{p/p'}\int_1^\infty [g(t)h(t)]^p b(Ct)\left\{\int_1^t \frac{a(s)}{s^p}\left[\int_s^\infty b(Cr)^{-p'/p}dr\right]^{p/(p')^2} ds\right\} dt \\
&\le (p')^{p/p'}D^{p/p'}\int_1^\infty [g(t)h(t)]^p b(Ct)\left\{\int_1^t \frac{a(s)}{s^p}\left[\int_1^s \frac{a(r)}{r^p}dr\right]^{-1/p'} ds\right\} dt \\
&\le p(p')^{p/p'}D^{p/p'}\int_1^\infty [g(t)h(t)]^p b(Ct)\left[\int_1^t \frac{a(r)}{r^p}dr\right]^{1/p} dt \\
&\le p(p')^{p/p'}D^{p}\int_1^\infty [g(t)h(t)]^p b(Ct)\left[\int_t^\infty b(Cr)^{-p'/p}dr\right]^{-1/p'} dt \\
&\le p(p')^{p/p'}D^{p}\int_1^\infty g(t)^p b(Ct)\,dt \le \frac{p(p')^{p/p'}D^p}{C}\int_\Omega \Psi(4BCf)\,d\mu
\end{aligned}
$$

and this completes the proof. □

PROOF OF THEOREM 2.4. Since the operator $\mathcal{H}_p$ is simultaneously of restricted weak type (p,p) and of type (∞,∞), from Theorem 2.3 the "only if" part is done.

To prove the "if" part, suppose that (9) holds. We first assume that b has the property

$$\int_1^\infty b(s)^{-p'/p}\,ds < \infty. \tag{18}$$

Let $t > 1$ fixed. For $s > 0$ let

$$h_t(s) = A_t b(Cs)^{-p'}$$

with $A_t = \left[tb(Ct)^{-p'/p} + \int_t^\infty b(Cs)^{-p'/p}\,ds\right]^{-p}$ and $C > \max\{(C')^2, 1\}$ such that $\int_1^\infty b(Cs)^{-p'/p} < (C')^{-p'/p}$. Observe thatin this case the monotonicity of b and condition (18) imply that b is increasing and $\lim\limits_{s\to\infty} b(s) = \infty$. Then, h_t is decreasing, $\lim_{s\to\infty} h_t(s) = 0$ and $h_t^{-1}(s)$ is well defined for $s > 0$. Now consider $f_t \in \mathfrak{M}([0,1])$ defined by $f_t = h_t^{-1}\chi_{(0,y_t)}$, with $y_t = \min\{h_t(t), 1\}$. The distribution function of f_t is for $s > 0$,

$$\begin{aligned}
\lambda_{f_t}(s) &= |\{x \in (0,1] :\ f_t(x) > s\}| \\
&= |\{x \in (0,1] :\ h_t^{-1}(x) > s \text{ and } x < y_t\}| \\
&= |\{x \in (0,1] :\ x < h_t(s) \text{ and } x < y_t\}| \\
&= \min\{h_t(s), h_t(t), 1\}.
\end{aligned}$$

From this and the fact that b is increasing we get

$$\begin{aligned}
C' \int_{[0,1]} \Psi(C'|f_t|)d\lambda &= C'^2 \int_0^\infty b(C's)\lambda_{f_t}(s)\,ds \\
&\le C\left[h_t(t)\int_0^t b(Cs)\,ds + \int_t^\infty b(Cs)h_t(s)\,ds\right] \\
&\le C\left[tb(Ct)h_t(t) + \int_t^\infty b(Cs)h_t(s)\,ds\right] \\
&\le CA_t\left[tb(Ct)^{-p'/p} + \int_t^\infty b(Cs)^{-p'/p}\,ds\right] \\
&\le C\left[tb(Ct)^{-p'/p} + \int_t^\infty b(Cs)^{-p'/p}\,ds\right]^{-p'/p} \\
&\le C\left[\int_t^\infty b(Cr)^{-p'/p}dr\right]^{-p/p'}.
\end{aligned}$$

Thus, by the choice of C we have

$$C' + C'\int_{[0,1]} \Psi(C'|f_t|) \le 2C\left[\int_t^\infty b(Cr)^{-p'/p}dr\right]^{-p/p'}. \tag{19}$$

On the other hand we will see that

$$\lambda_{\mathcal{H}_p f_t}(s) \geq \frac{1}{s^p} \text{ for all } s \in (1,t), \tag{20}$$

and therefore

$$\int_{[0,1]} \Phi(|\mathcal{H}_p f_t|) d\lambda = \int_0^\infty a(s) \lambda_{\mathcal{H}_p f_t}(s)\, ds \geq \int_1^t \frac{a(s)}{s^p}\, ds. \tag{21}$$

Thus from (21) and (19) we have

$$\begin{aligned}\int_1^t \frac{a(s)}{s^p}\, ds &\leq \int_{[0,1]} \Phi(|\mathcal{H}_p f_t|) \leq C' + C' \int_{[0,1]} \Psi(C'|f_t|) \\ &\leq 2C \left[\int_t^\infty b(Cr)^{-p'/p} dr\right]^{-p/p'}.\end{aligned}$$

Since C does not depends on t, we get (8).

It remains to prove (20). Since $\lim_{s\to\infty} h_t(s) = 0$, we have $\lim_{x\to 0} f_t(x) = \infty$ and $\lim_{x\to 0} \mathcal{H}_p f_t(x) = \infty$ (for all g decreasing, $\mathcal{H}_p g \geq g$). Also $\mathcal{H}_p f_t$ is continuous and decreasing (see Remark 3) on $(0,1]$. Then the image of $\mathcal{H}_p f_t$ is the interval $[\mathcal{H}_p f_t(1), \infty)$. For $\mathcal{H}_p f_t(1) < s < t$, we have $\lambda_{\mathcal{H}_p f_t}(s) = |\{x : \mathcal{H}_p f_t(x) > s\}| = x_s$ with $x_s \in (0,1]$ such that $s = \mathcal{H}_p f_t(x_s) = \frac{1}{x_s^{1/p}} \int_0^{x_s} f_t(x) x^{1/p-1} dx$. Then $x_s = \frac{1}{s^p}\left[\int_0^{x_s} f_t(x) x^{1/p-1} dx\right]^p$.

We have $\mathcal{H}_p f_t(1) < t = h_t^{-1}(h_t(t)) = f_t(h_t(t)) \leq \mathcal{H}_p f_t(h_t(t))$. Since $\mathcal{H}_p f_t(1) < \mathcal{H}_p f_t(h_t(t))$ and $\mathcal{H}_p f_t$ is decreasing, $h_t(t) \leq 1$. Thus, $y_t = h_t(t)$ and $x_s = \lambda_{\mathcal{H}_p f_t}(s) \geq \lambda_{\mathcal{H}_p f_t}(t) \geq \lambda_{f_t}(t) = y_t$. Therefore

$$\begin{aligned}\int_0^{x_s} f_t(x) x^{1/p-1} dx \geq \int_0^{y_t} f_t(x) x^{1/p-1} dx &= \int_0^{h_t(t)} h_t^{-1}(x) x^{1/p-1} dx \\ &= \int_0^{(h_t(t))^{1/p}} h_t^{-1}(y^p) dy = t(h_t(t))^{1/p} + \int_t^\infty (h_t(r))^{1/p} dr \\ &= A_t^{1/p}\left[tb(Ct)^{-p'/p} + \int_t^\infty b(Cr)^{-p'/p} dr\right] = 1\end{aligned}$$

Then $x_s \geq \frac{1}{s^p}$ if $\mathcal{H}_p f_t(1) < s < t$. If $1 < s < \mathcal{H}_p f_t(1)$, obviously $\lambda_{\mathcal{H}_p f_t}(s) = 1 > \frac{1}{s^p}$ and we get (20). To finish the proof of Theorem 2.4 it remains to consider the case when

$$\int_1^\infty b(s)^{-p'/p}\, ds = \infty. \tag{22}$$

We will show that, in this situation, $\mathcal{H}_p$ does not map $L^\Psi([0,1])$ into $L^\Phi([0,1])$. We may suppose that b is increasing. If b were not increasing, we could take $b \le \tilde{b}$ increasing satisfying (22) and it is enough to do the following construction with $\tilde{b}$ instead of b since $L^{\tilde{\Psi}} \supset L^\Psi$.

Consider the function $f = h^{-1}\chi_{[0,1]}$ in $\mathfrak{M}([0,1])$, where

$$h(x) = \frac{Kb(x)^{-p'}}{\left(\int_{1/2}^x b^{-p'/p}\, ds\right)^p}$$

for $x \ge 1$ and K such that $h(1) = 1$. Note that h is decreasing and so f is well defined. First we see that f is in $L^\Psi([0,1])$. Since $\int_1^\infty b^{-p'/p} = \infty$ and $b^{-p'/p}$ is continuous and decreasing on $[1,\infty)$, there exists a sequence $\{x_n\}_{n=1}^\infty$ with $x_n > 1$, such that $\int_1^{x_n} b^{-p'/p} = n$ and $\lim_{n\to\infty} x_n = \infty$. Then, from the fact that $\lambda_f(s) = h(s)$ for $s > 1$, we have

$$\int_{[0,1]} \Psi(|f|) = \int_0^\infty b(s)\lambda_f(s)\, ds \le \int_0^1 b(s)\, ds + \int_1^\infty b(s)h(s)\, ds$$

and

$$\begin{aligned}
\frac{1}{K}\int_1^\infty b(s)h(s)\, ds &= \int_1^\infty \frac{b(s)^{-p'/p}}{\left(\int_{1/2}^s b^{-p'/p}\right)^p}\, ds \\
&\le \int_1^{x_1} \frac{b(s)^{-p'/p}}{\left(\int_{1/2}^1 b^{-p'/p}\right)^p}\, ds + \sum_{n=1}^\infty \int_{x_n}^{x_{n+1}} \frac{b(s)^{-p'/p}}{\left(\int_1^{x_n} b^{-p'/p}\right)^p}\, ds \\
&= \frac{1}{\left(\int_{1/2}^1 b^{-p'/p}\right)^p} + \sum_{n=1}^\infty \frac{\int_1^{x_{n+1}} b^{-p'/p} - \int_1^{x_n} b^{-p'/p}}{\left(\int_1^{x_n} b^{-p'/p}\right)^p} \\
&= \frac{1}{\left(\int_{1/2}^1 b^{-p'/p}\right)^p} + \sum_{n=1}^\infty \frac{1}{n^p} < \infty.
\end{aligned}$$

Now we will see that $\mathcal{H}_p f$ is not in $L^\Phi([0,1])$ by showing that $\mathcal{H}_p f(x) = \infty$ for all $x \in [0,1]$. Since $\mathcal{H}_p f$ is decreasing on $[0,1]$, it is enough to show

$\mathcal{H}_p f(1) = \infty$. In fact,

$$\begin{aligned}
\frac{1}{K^{1/p}}\mathcal{H}_p f(1) &= \frac{1}{K^{1/p}}\int_0^1 h^{-1}(r) r^{1/p-1} dr \geq \frac{1}{K^{1/p}}\int_1^\infty h(r)^{1/p} dr \\
&= \int_1^\infty \frac{b(s)^{-p'/p}}{\int_{1/2}^s b^{-p'/p}}\, ds = \sum_{n=0}^\infty \int_{x_n}^{x_{n+1}} \frac{b(s)^{-p'/p}}{\int_{1/2}^s b^{-p'/p}}\, ds \\
&\geq \sum_{n=0}^\infty \int_{x_n}^{x_{n+1}} \frac{b(s)^{-p'/p}}{\int_{1/2}^{x_{n+1}} b^{-p'/p}}\, ds = \sum_{n=0}^\infty \frac{\int_1^{x_{n+1}} b^{-p'/p} - \int_1^{x_n} b^{-p'/p}}{\int_{1/2}^1 b^{-p'/p} + \int_1^{x_{n+1}} b^{-p'/p}} \\
&= \sum_{n=0}^\infty \frac{1}{\int_{1/2}^1 b^{-p'/p} + 1 + n} = \infty.
\end{aligned}$$

□

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