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Katarína Janková,* Department of Applied Mathematics and Statistics, Faculty of Mathematics, Physics and Informatics, Comenius University, Bratislava, Slovakia.

ON MEASURES OF CHAOS FOR DISTRIBUTIONALLY CHAOTIC MAPS

Abstract

Let f be a distributionally chaotic map of the interval such that the endpoints of the minimal periodic portions of any basic set are periodic. Then the principal measure of chaos, $\mu_p(f)$, is not greater than twice the spectral measure of chaos $\mu_s(f)$. This proves an assertion of Schweizer et al. in a special case.

1 Introduction.

The notion of distributional chaos was introduced by Schweizer and Smítal in [5] for continuous maps of the interval and later studied by many authors, not only on the interval. To express the size of chaos, the principal measure μ_p and the spectral measure μ_s of chaos which are based on differences between upper and lower distribution functions are useful. In [5], the following theorem is stated without a proof.

Theorem 1 ([5], Thm. 6.13)**.** *For any f in $C(I,I)$, $\mu_s(f) \le \mu_p(f) \le 2\mu_s(f)$.*

The aim of this paper is to establish this inequality in a special case, cf. our theorem below. Before stating the results, we recall some notation. For any $x, z \in I$, let

$$\xi(x,z,f,t,n) = \frac{1}{n}\#\{0 \le i < n : |f^i(x) - f^i(z)| < t\},$$

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and define *lower* and *upper distribution functions* by

$$F_{xz}(t) = \liminf_{n\to\infty} \xi(x,z,f,t,n) \text{ and } F^*_{xz}(t) = \limsup_{n\to\infty} \xi(x,z,f,t,n),$$

respectively. The *principal measure* $\mu_p(f)$ *of chaos* generated by f is given by

$$\mu_p(f) = \sup_{x,z\in I} \mu(x,z,f), \text{ where } \mu(x,z,f) = \int_0^1 (F^*_{xz}(t) - F_{xz}(t))\,dt.$$

The measure of chaos is closely related to the structure of ω-limit sets of f. By an ω-limit set, we mean the set of limit points of some trajectory $\{f^n(x)\}$, $x \in I$. These sets were originally studied by A. N. Sharkovsky [4], and later by A. Blokh [2], who gave characterizations of various types of ω-limit sets. According to Sharkovsky, a maximal (with respect to inclusion) ω-limit set $\tilde\omega$ is of the *second type* if it is infinite and contains a periodic point. (It is known that then the periodic points are dense in it.) In recent works [5] [6], this set has been called (similarly as in Blokh [2]) a basic set, and we shall use this terminology. Otherwise, $\tilde\omega$ is of the *first type*. In this case, $\tilde\omega$ either is a periodic orbit or it is infinite and has a periodic decomposition of arbitrarily high periods. We say that an ω-limit set $\tilde\omega$ has a periodic decomposition of period k if there is a *minimal* compact periodic interval $J \subset I$ of period k such that $\bigcup_{i=0}^k f^i(J) \supset \tilde\omega$. Since J is minimal, the convex hulls of the *periodic portions* $\omega_i = f^i(J)\cap\tilde\omega$ of $\tilde\omega$ are nonoverlapping, but may have endpoints in common. Recall that a basic set has a maximal decomposition into periodic portions (i.e., these portions are indecomposable). It is also known that any basic set $\tilde\omega$ is either a nowhere dense perfect set or a finite periodic collection of compact intervals which are such that $f \mid \tilde\omega$ is transitive. For other properties of maximal ω-limit sets, see, e.g., Theorem 3.7 of [6].

Two points $x, z \in I$ are *isotectic* if, for every integer $n > 0$, the ω-limit sets $\omega_{f^n}(x)$ and $\omega_{f^n}(z)$ are contained in the same maximal ω-limit set of f^n. By $Iso(f)$ we shall denote the set $\{(x,y)\in I\times I : x, y \text{ are isotectic}\}$. The *spectrum* $\Sigma(f)$ of f is the set of all minimal elements of the set $\{F_{xz}; x, z \in Iso(f)\}$ which is a finite set. The *spectral measure of chaos* is given by

$$\mu_s(f) = \max\Big\{\int_0^1 (1 - F(t))\,dt; F \in \Sigma(f)\Big\}.$$

Since, for any $F \in \Sigma$, there are points $x, y \in I$ with $F_{xy} = F$ and $F^*_{xy} = 1$ (cf. [5] or [6]), we get $\mu_s(f) \le \mu_p(f)$. By [6], for any two maximal ω-limit sets $\tilde\omega_1$ and $\tilde\omega_2$ there exists $\max_{x\in\tilde\omega_1, z\in\tilde\omega_2} \mu(x,z,f) =: \mu(\tilde\omega_1,\tilde\omega_2)$. In [3] it was shown that if the principal measure of chaos is positive, then it is generated by a pair of points such that at least one of them belongs to a basic set. We extend this result by showing that, in certain cases, $\mu(\tilde{\omega}_1, \tilde{\omega}_2)$ is generated by a pair of points such that one is periodic (see Lemmas 2 and 3 below). In our next proof we shall use fundamental Lemma 3.3 from [1] which we restate as follows.

Lemma 1 ([1]). *Let $\tilde{\omega}$ be a nowhere dense, indecomposable basic set and $\delta > 0, \lambda > 0$. Then, there is a compact portion W of $\tilde{\omega}$ and an integer $K = K(\delta, \lambda)$ with the following properties: for any $n > K$ and any $x \in \tilde{\omega}$, there is a compact portion U of $\tilde{\omega}$ contained in W such that:*

(i) $W \subset f^n(U)$,

(ii) $\#\{i \leq n; \mid f^i(u) - f^i(x) \mid \geq \delta\} < \lambda n$ *for any* $u \in U$.

2 Main Results.

Lemma 2. *Let $\tilde{\omega}$ be a basic set and p, q be periodic points belonging to the same minimal periodic portion ω_1 of $\tilde{\omega}$. Then, for any $\varepsilon > 0$, there is an integer $K = K(\varepsilon, p, q)$ such that if $\{m_i\}_{i=1}^{\infty}$ is a sequence of nonnegative integers with $m_{i+1} - m_i > K$, then there is a $u \in \omega_1$ such that, for any $i \geq 1$:*

(i) $\frac{1}{m_{2i}-m_{2i-1}} \#\{m_{2i-1} \leq j < m_{2i} : |f^j(u) - f^j(p)| > \varepsilon\} < \varepsilon$,

(ii) $\frac{1}{m_{2i+1}-m_{2i}} \#\{m_{2i} \leq j < m_{2i+1} : |f^j(u) - f^j(q)| > \varepsilon\} < \varepsilon$.

Proof. We may assume that p and q belong to the same periodic portion of f, say ω_1. First, assume that ω is nowhere dense. Let $\omega_1, \omega_2, \ldots, \omega_m$ be minimal periodic portions of ω. Consider $g = f^m$. Choose $\varepsilon_1 > 0$ such that $|y_1 - y_2| < \varepsilon_1$ implies $|f^i(y_1) - f^i(y_2)| < \varepsilon$ for any $y_1, y_2 \in I$ and any $i = 1, 2, \ldots, m$. By Lemma 1, there exists a compact portion W of ω_1 and $K_1 = K(\varepsilon, p, q)$ such that for any $n > K_1$, there are $U(p, n)$ and $U(q, n)$ contained in W such that $g^n(U) \supset W$,

$$\frac{1}{n}\#\{0 \leq i < n : |g^i(u) - g^i(p)| > \varepsilon_1\} < \varepsilon \tag{1}$$

for $u \in U(p, n)$, and

$$\frac{1}{n}\#\{0 \leq i < n : |g^i(u) - g^i(q)| > \varepsilon_1\} < \varepsilon \tag{2}$$

for $u \in U(q, n)$. Let $\{m_i\}$ be such that $|m_i - m_{i+1}| > mK_1$ and $r_i = [\frac{m_i}{m}]$, where $[z]$ denotes the integer part of z. For any i let $U_{2i-1} = U(p, r_{2i} - r_{2i-1})$

and $U_{2i} = U(q, r_{2i+1} - r_{2i})$. Now we construct a sequence of sets $V_1 \supset V_2 \supset \dots$ such that $V_1 = U_1$, and for $i > 1$, $V_{i+1} \subset V_i$ is such that $g^{r_{i+1}-r_i}(V_{i+1}) = U_{i+1}$. If we take $K = mK_1$ and $u \in \bigcap_{i=1}^{\infty} V_i$, then u and K have the desired properties. In the other case when $\tilde{\omega}$ is the union of finitely many compact intervals, the proof is easy and we omit it. □

Lemma 3. *Let $\tilde{\omega}$ be basic with minimal periodic portions $\omega_0, \dots, \omega_{n-1}$ such that the endpoints of all ω_i form one or two periodic orbits (of periods $2n$ or n, respectively). Let p be any of these endpoints, and let d_i be the length of ω_i. Then, there is some $u \in \tilde{\omega}$, such that u and p are isotectic and $\mu(u, p, f) \geq \frac{1}{n}\sum_{i=0}^{n-1} d_i$.*

PROOF. Assume, as we may, that p and q are the endpoints of ω_0. Let $\varepsilon > 0$ and $\{m_i\}$ an increasing sequence of positive integers divisible by n such that

$$\lim_{k\to\infty} (m_1 + \dots + m_k)/m_{k+1} = 0. \tag{3}$$

Then, by Lemma 1, there is a $u \in \omega_0$ such that (1) and (2) are satisfied for any $i \geq i(\varepsilon)$. Obviously, (1) implies $F^*_{up}(\varepsilon) > 1 - \varepsilon$. By (2),

$$\frac{1}{m_{2i+1} - m_{2i}} \#\{m_{2i} \leq j < m_{2i+1} : |f^j(u) - f^j(p)| < d_{j(\mathrm{mod} n)} - \varepsilon\} < \varepsilon,$$

whence

$$\xi(f^{m_{2i}+j}(u), f^j(p), f^n, t, (m_{2i+1} - m_{2i})/n) =: \xi_j(t) < \varepsilon \text{ if } t \leq d_{j(\mathrm{mod} n)} - \varepsilon, \tag{4}$$

and $\xi_j(t) = 1$ if $t \geq d_{j(\mathrm{mod} n)}$. Let $\nu(t) = \#\{0 \leq i < n; t \leq d_i\}$; i.e., $\nu(t)$ is the number of periodic portions of diameter not less than t. Then, by (4),

$$\xi(f^{m_{2i}}(u), p, f, t, m_{2i+1} - m_{2i}) \leq \frac{1}{n}(\varepsilon\nu(t+\varepsilon) + n - \nu(t+\varepsilon)). \tag{5}$$

Letting $\varepsilon \to 0$, since ν is right continuous by (3) and (5), we get $F_{up}(t) \leq 1 - \nu(t)/n$, while $F^*_{up}(\varepsilon) > 1 - \varepsilon$ yields $F^*_{up} \equiv 1$. Thus, $\mu(u, p, f) \geq \frac{1}{n}\int_0^1 \nu(t)\,dt$. It is easy to verify that $\frac{1}{n}\int_0^1 \nu(t)\,dt = \frac{1}{n}\sum_{i=0}^{n-1} d_i$. □

The following lemma shows that the principal measure of chaos generated by points u and p in the preceding proof is the greatest possible in the sense that any two points x and y lying in the same portion of ω generate a measure of chaos less than $\frac{1}{n}\sum_{k=0}^{n} d_k$.

Lemma 4. *Let ω^1, ω^2 be basic sets with minimal periodic portions of periods m, r and lengths $d_0^1, \dots, d_{m-1}^1$ and $d_0^2, \dots, d_{r-1}^2$, respectively. Then, for any $x \in \omega^1$ and $y \in \omega^2$,*

$$\mu(x,y,f) \le \frac{1}{m}\sum_{k=0}^{m-1} d_k^1 + \frac{1}{r}\sum_{k=0}^{r-1} d_k^2.$$

Proof. We may assume that $f^i(x)$ belongs to the periodic portion ω_i^1 of ω^1 of length d_i^1, $0 \le i < m$. Similarly, $f^j(y) \in \omega_j^2 \subset \omega_2$, where ω_j^2 is a periodic portion of the length d_j^2 if $0 \le j < r$. Thus, for any $k \ge 0$,

$$l_k \le |f^k(x) - f^k(y)| \le d^1_{k(\mathrm{mod}\ m)} + d^2_{k(\mathrm{mod}\ r)} + l_k,$$

where l_k is the distance between the portions $\omega^1_{k(\mathrm{mod}\ m)}$ and $\omega^2_{k(\mathrm{mod}\ r)}$. Then, obviously, $\xi(f^k(x), f^k(y), f^{mr}, t) = 0$ if $t < l_k$, and $\xi(f^k(x), f^k(y), f^{mr}, t) = 1$ if $t > l_k + d^1_{k(\mathrm{mod}\ m)} + d^2_{k(\mathrm{mod}\ r)}$. This implies

$$\mu(f^k(x), f^k(y), f^{mr}) \le d^1_{k(\mathrm{mod}\ m)} + d^2_{k(\mathrm{mod}\ r)},$$

and hence,

$$\mu(x,y,f) \le \frac{1}{mr}\sum_{k=0}^{mr-1} \mu(f^k(x), f^k(y), f^{mr}) \le \frac{1}{m}\sum_{k=0}^{m} d_k^1 + \frac{1}{r}\sum_{k=0}^{r} d_k^2. \qquad \square$$

Lemma 5. *If u, p, f are as in Lemma 3, then $\mu(u,p,f) = \frac{1}{n}\sum_{i=0}^{n-1} d_i$.*

Proof. As in the preceding proof, we obtain the inequality

$$\mu(f^k(u), f^k(p), f^m) \le d_{k(\mathrm{mod} m)},$$

where d_k denotes the length of the periodic portion ω_k, and consequently

$$\mu(u,p,f) \le \frac{1}{mr}\sum_{k=0}^{mr-1} \mu(f^k(u), f^k(p), f^{mr}) \le \frac{1}{m}\sum_{k=0}^{m} d_k,$$

which, together with Lemma 3, gives the equality. $\square$

Lemma 6. *Let ω-limit sets ω^1, ω^2 of f be maximal and such that if ω^i ($i = 1, 2$) is basic. Then the set of endpoints of its minimal periodic portions consist of one or two periodic orbits. If $\mu_p(f) = \mu(x,y,f)$, for some $x \in \omega^1$ and $y \in \omega^2$, then $\mu_p(f) \le 2\mu_s(f)$.*

PROOF. The assertion is trivial if $\mu(f) = 0$. So we may assume that ω^1 is a basic set, with minimal periodic portions of period m and lengths $d_0^1, \dots d_{m-1}^1$. Consider three cases.

(i) Let ω^2 be basic, with minimal periodic portions of period r and lengths $d_0^2, \dots, d_{r-1}^2$. Applying Lemmas 3 and 4, we obtain $u_1, v_1 \in \omega_1^1$ and $u_2, v_2 \in \omega_1^2$ such that

$$\mu(x,y,f) \leq \frac{1}{m}\sum_{j=0}^{m-1} d_j^1 + \frac{1}{r}\sum_{j=0}^{r-1} d_j^2 = \mu(u_1,v_1,p) + \mu(u_2,v_2,f) \leq 2\mu_s(f). \quad (6)$$

The second inequality follows since the pairs (u_1, v_1) and (u_2, v_2) are isotectic.

(ii) Let ω^2 be the orbit of a periodic point of period $r \geq 1$. Then, the inequality follows by (6), where we have $d_j^2 = 0$, $0 \leq j < r$, and $u_2 = v_2$.

(iii) If ω^2 is an infinite maximal ω limit set of the first type, then for any $\varepsilon > 0$ and any $y \in \omega^2$, there is a periodic point p of sufficiently large period r such that

$$\frac{1}{r}\#\{0 \leq j < r; |f^j(y) - f^j(p)| > \varepsilon\} < \varepsilon$$

which for $\varepsilon \to 0$, reduces this case to case (ii). $\square$

Our main result, the next theorem, follows from Lemma 6.

Theorem 2. *Let f be a continuous map of the interval such that endpoints of minimal periodic portions of any basic set of f form periodic orbits. Then, $\mu_p(f) \leq 2\mu_s(f)$.*

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