RESEARCH

Real Analysis Exchange
ISSN:0147-1937
Vol. 32(1), 2006/2007, pp. 3–18

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ON THE MAXIMAL FAMILIES FOR SOME CLASSES OF STRONGLY QUASICONTINUOUS FUNCTIONS ON $\mathbb{R}^m$

Abstract

Some generalizations of the notions of approximate quasicontinuity on $\mathbb{R}^m$ and the maximal families (additive, multiplicative, lattice and with respect to the composition) for these classes of functions are investigated.

1 Preliminaries.

Let $\mathbb{R}$, $\mathbb{Q}$, $\mathbb{Z}$ and $\mathbb{N}$ denote, respectively, the set of all real numbers, of all rationals, of all integers and of all positive integers.

Throughout the present paper we shall use the following differentiation basis $\mathcal{P}$ in the product space $\mathbb{R}^m$ for $m \in \mathbb{N}$. For every $n \in \mathbb{N}$ and for each system of integers $k_1, \ldots, k_m$ we define the m-dimensional cube

$$P^n_{k_1,\ldots,k_m} = \left[\frac{k_1-1}{2^n}, \frac{k_1}{2^n}\right) \times \left[\frac{k_2-1}{2^n}, \frac{k_2}{2^n}\right) \times \cdots \times \left[\frac{k_m-1}{2^n}, \frac{k_m}{2^n}\right).$$

Moreover, let

$$\mathcal{P}_n = \{P^n_{k_1,\ldots,k_m}; k_1, \ldots, k_m \in \mathbb{Z}\} \text{ and } \mathcal{P} = \bigcup_{n=1}^{\infty} \mathcal{P}_n.$$

Observe that:

(1) if $(k_1, \ldots, k_m) \neq (l_1, \ldots, l_m)$, then $P^n_{k_1,\ldots,k_m} \cap P^n_{l_1,\ldots l_m} = \emptyset$,

Key Words: density topology, strong quasicontinuity, maximal families, additive, multiplicative, lattice.
Mathematical Reviews subject classification: 26A05, 26A15
Received by the editors September 3, 2004
Communicated by: B. S. Thomson

(2) $\mathbb{R}^m = \bigcup_{k_1,\ldots,k_m\in\mathbb{Z}} P^n_{k_1,\ldots,k_m}$,

(3) if $n_1 > n_2$, then for each system $(k_1, \ldots, k_m)$ there is a unique system $(l_1, \ldots l_m)$ such that $P^{n_1}_{k_1,\ldots,k_m} \subset P^{n_2}_{l_1,\ldots,l_m}$,

(4) for each point $\mathbf{x} \in \mathbb{R}^m$ and for each $n \in \mathbb{N}$ there is a unique system $(k_1(\mathbf{x}), \ldots, k_m(\mathbf{x}))$ such that $\mathbf{x} \in P^n_{k_{1(\mathbf{x})},\ldots,k_{m(\mathbf{x})}} = P^n(\mathbf{x})$.

Evidently, for each index $k \in \mathbb{N}$ and each point $\mathbf{x} \in \mathbb{R}^m$, we have

$$P^{k+1}(\mathbf{x}) \subset P^k(\mathbf{x}),\ \{\mathbf{x}\} = \bigcap_{k=1}^{\infty} P^k(\mathbf{x}) \text{ and } \lim_{k\to\infty} \operatorname{diam}(P^k(\mathbf{x})) = 0,$$

where $\operatorname{diam}(P)$ denotes the diameter of the cube P.

Let $\lambda^*_m,\ (\lambda_m)$ denote outer Lebesgue measure in $\mathbb{R}^m$, (Lebesgue measure in $\mathbb{R}^m$ respectively), let $\mathcal{L}_m$ denote the family of all λ_m-measurable sets (i.e., the sets measurable in the Lebesgue sense) in $\mathbb{R}^m$ and let $A \subset \mathbb{R}^m$ be an arbitrary set.

For $\mathbf{x} \in \mathbb{R}^m$ we define *the upper outer density* (*the lower density*) of the set A at the point $\mathbf{x}$ by

$$d_u(A,\ \mathbf{x}) = \lim_{n\to\infty} \sup \frac{\lambda^*_m(A \cap P^n(\mathbf{x}))}{\lambda_m(P^n(\mathbf{x}))}, \Big(d_l(A,\ \mathbf{x}) = \lim_{n\to\infty} \inf \frac{\lambda^*_m(A \cap P^n(\mathbf{x}))}{\lambda_m(P^n(\mathbf{x}))}\Big).$$

A point $\mathbf{x} \in \mathbb{R}^m$ is called *an outer density point* (with respect to the basis $\mathcal{P}$) of the set $A \subset \mathbb{R}^m$ iff $d_l(A,\ \mathbf{x}) = 1$. A point $\mathbf{x} \in \mathbb{R}^m$ is called *a density point* (with respect to the basis $\mathcal{P}$) of the set $A \subset \mathbb{R}^m$ iff there exists a λ_m-measurable set $B \subset A$ such that $d_l(B,\ \mathbf{x}) = 1$. Let

$$\phi(A) = \{\mathbf{x} \in \mathbb{R}^m; \mathbf{x} \text{ is a density point of A with respect to } \mathcal{P}\}$$

and put

$$\mathcal{T}_d = \{A \in \mathcal{L}_m; A \subset \phi(A)\}.$$

The family $\mathcal{T}_d$ is a topology called *the density topology* ([1], [2] and [15]). Denote by $\mathcal{T}_e$ the Euclidean topology in $\mathbb{R}^m$. Observe that $\mathcal{T}_e \subset \mathcal{T}_d$ and $\mathcal{T}_e \neq \mathcal{T}_d$. If $A \in \mathcal{T}_e$, then we will say that A is an open set.

If $\mathbf{x} \in \mathbb{R}^m$ is a continuity point of the mapping $f : (\mathbb{R}^m,\ \mathcal{T}_e) \to (\mathbb{R},\ \mathcal{T}_e)$, then we say simply that $\mathbf{x}$ is continuity point of the function $f : \mathbb{R}^m \to \mathbb{R}$.

A point $\mathbf{x} \in \mathbb{R}^m$ is called *an approximate continuity point* of the function $f : \mathbb{R}^m \to \mathbb{R}$ if $\mathbf{x}$ is a continuity point of the mapping $f : (\mathbb{R}^m,\ \mathcal{T}_d) \to (\mathbb{R},\ \mathcal{T}_e)$.

We will denote by $C(f)$ (by $A(f)$) the set of all continuity points (approximate continuity points respectively) of the function $f : \mathbb{R}^m \to \mathbb{R}$. The set

$D(f) = \mathbb{R}^m \setminus C(f)$ denotes the set of all discontinuity points of the function f.

Moreover, denote by $\mathcal{C}$, (by $\mathcal{A}$), [by $\mathcal{C}_{ae}$] the class of all continuous functions $f : \mathbb{R} \to \mathbb{R}$ (approximately continuous functions $f : \mathbb{R}^m \to \mathbb{R}$), [the class of all functions $f : \mathbb{R}^m \to \mathbb{R}$ which are λ_m-almost everywhere continuous ; i.e., for which $\lambda_m(D(f)) = 0$, respectively].

Let $\mathcal{T}$ be any topology of subsets of the space $\mathbb{R}^m$ and let $\mathbf{x} \in \mathbb{R}^m$ be a point.

Definition 1. The function $f : \mathbb{R}^m \to \mathbb{R}$ is $\mathcal{T}$- quasicontinuous at the point $\mathbf{x}$ if for every $\varepsilon > 0$ and for every set $U \in \mathcal{T}$ containing $\mathbf{x}$ there is a nonempty set $V \in \mathcal{T}$ such that $V \subset U$ and $f(V) \subset (f(\mathbf{x}) - \varepsilon, f(\mathbf{x}) + \varepsilon)$.

If $\mathcal{T} = \mathcal{T}_e$, then we say simply that f is quasicontinuous at $\mathbf{x}$ ([10], [11]). If $\mathcal{T} = \mathcal{T}_d$, then f is called *approximately quasicontinuous* (with respect to $\mathcal{P}$) *at the point* $\mathbf{x}$ and we write $f \in Q_{ap}(\mathbf{x})$. If for every $\mathbf{x} \in \mathbb{R}^m,\ f \in Q_{ap}(\mathbf{x})$, then we say that f is *approximately quasicontinuous* (with respect to $\mathcal{P}$). The class of all approximately quasicontinuous functions $f : \mathbb{R}^m \to \mathbb{R}$ we denote by Q_{ap} ([4], [5]).

Let $A \subset \mathbb{R}$ be an arbitrary set. For $x \in \mathbb{R}$ we define the lower bilateral density of the set A at x by

$$D_l(A, x) = \lim_{h \to 0} \frac{\lambda_1([x - h, x + h] \cap A)}{2h}.$$

A point $x \in \mathbb{R}$ is called a bilateral density point of the set $A \subset \mathbb{R}$ iff there is a λ_1-measurable set $B \subset A$ such that $D_l(B, x) = 1$. Let

$$\Phi(A) = \{x \in \mathbb{R} :\ x \text{ is a bilateral density point of } A\}.$$

The family $\tau_d = \{A \in \mathcal{L}_1; A \subset \Phi(A)\}$ is a topology called the density topology ([1], [15]).

Similarly as above, a point $x \in \mathbb{R}$ is called an approximate continuity point of the function $f : \mathbb{R} \to \mathbb{R}$ if x is a continuity point of the mapping $f : (\mathbb{R}, \tau_d) \to (\mathbb{R}, \mathcal{T}_e)$. If $\mathcal{T} = \tau_d$, then a function $f : \mathbb{R} \to \mathbb{R}$ which is τ_d-quasicontinuous is called approximately quasicontinuous ([4], [5]).

Definition 2. [(Grande [7])]. A function $f : \mathbb{R} \to \mathbb{R}$ is said to be strongly τ_d-quasicontinuous at a point $x \in \mathbb{R}$ if for every $\eta > 0$ and for every set $U \in \tau_d$ containing x there is an open interval I such that $U \cap I \neq \emptyset$ and $|f(t) - f(x)| < \eta$ for every $t \in I \cap U$.

Denote by $\mathrm{int}(A)$ the interior (Euclidean) of the set A. The family

$$\mathcal{T}_{ae} = \{A \in \mathcal{T}_d; \lambda_m(A \setminus \mathrm{int}(A)) = 0\}$$

is also a topology ([12]). If a point $\mathbf{x} \in \mathbb{R}^m$ is a continuity point of the mapping $f : (\mathbb{R}^m,\ \mathcal{T}_{ae}) \to (\mathbb{R},\ \mathcal{T}_e)$, then we say that the function $f : \mathbb{R}^m \to \mathbb{R}$ is $\mathcal{T}_{ae}$- *continuous at a point* $\mathbf{x}$. A function $f : \mathbb{R}^m \to \mathbb{R}$ is $\mathcal{T}_{ae}$-*continuous* (everywhere) iff $f \in \mathcal{A} \cap \mathcal{C}_{ae}$ ([12], [3]). The class of all $\mathcal{T}_{ae}$-continuous functions $f : \mathbb{R}^m \to \mathbb{R}$ we denote by $\mathcal{C}(\mathcal{T}_{ae})$.

2 New Definitions and Notions.

Now we define some classes of strongly quasicontinuous functions $f : \mathbb{R}^m \to \mathbb{R}$, which we will investigate in this paper. By analogy, classes of such functions for the case $m = 1$ were introduced by Z. Grande ([9]) with respect to the bilateral density.

Definition 3. Let $f : \mathbb{R}^m \to \mathbb{R}$ be a function and let $\mathbf{x} \in \mathbb{R}^m$ be a point. Then

- $f \in Q_s(\mathbf{x})$; i.e., f is called *strongly quasicontinuous at a point* $\mathbf{x}$ if for every real $\varepsilon > 0$ and for each set $A \in \mathcal{T}_d$ containing $\mathbf{x}$, there is a nonempty open set O such that $A \cap O \neq \emptyset$ and $f(O \cap A) \subset (f(\mathbf{x}) - \varepsilon, f(\mathbf{x}) + \varepsilon)$.

  If for every $\mathbf{x} \in \mathbb{R}^m$, $f \in Q_s(\mathbf{x})$, then we say that f is *strongly quasicontinuous*. Denote by Q_s the class of all strongly quasicontinuous functions $f : \mathbb{R}^m \to \mathbb{R}$.

- $f \in Q_{s_1}(\mathbf{x})$ ($f \in Q_{s_2}(\mathbf{x})$); i.e., f is called s_1-*strongly quasicontinuous* (f is called s_2- *strongly quasicontinuous* respectively) *at a point* $\mathbf{x}$ if for each real $\varepsilon > 0$ and for each set $A \in \mathcal{T}_d$ containing $\mathbf{x}$ there exists a nonempty open set O such that $O \cap A \neq \emptyset$, $O \cap A \subset C(f)$ ($O \cap A \subset A(f)$ respectively) and $f(O \cap A) \subset (f(\mathbf{x}) - \varepsilon,\ f(\mathbf{x}) + \varepsilon)$.

  If for each $\mathbf{x} \in \mathbb{R}^m$, $f \in Q_{s_1}(\mathbf{x})$ ($f \in Q_{s_2}(\mathbf{x})$), then we say that f is s_1-*strongly quasicontinuous* (f is s_2-*strongly quasicontinuous* respectively). Denote by Q_{s_1}, by Q_{s_2}) the class of all functions $f : \mathbb{R}^m \to \mathbb{R}$ which are s_1-strongly quasicontinuous (s_2-strongly quasicontinuous respectively).

The notion of strong quasicontinuity (for the bilateral density topology in $\mathbb{R}$) introduced by Z. Grande in [7] is more general than that above (for $m = 1$). For example, if $f : \mathbb{R} \to \mathbb{R}$ is defined by

$$f(x) = \begin{cases} 0 & \text{for } x \leq 0 \\ 1 & \text{for } x > 0, \end{cases}$$

then the function f is strongly quasicontinuous at 0 in the sense of Grande, but $f \notin Q_s(0)$. If $f : \mathbb{R} \to \mathbb{R}$ is strongly quasicontinuous at x in the above sense ($f \in Q_s(x)$), then f is strongly quasicontinuous at x in the sense of Grande.

From the definitions above it follows that $Q_{s_1} \subset Q_{s_2} \subset Q_s \subset Q_{ap}$. The inclusions above are proper ([13]); moreover, $Q_s \subset \mathcal{C}_{ae}$, ([6]).

Let $\xi(\mathbf{x})$ be a property of a function $f : \mathbb{R}^m \to \mathbb{R}$ at a point $\mathbf{x}$ (we will write $f \in \xi(\mathbf{x})$) such that the following are true.

- If f is continuous at $\mathbf{x}$, then $f \in \xi(\mathbf{x})$;
- if $f \in \xi(\mathbf{x})$, then $-f \in \xi(\mathbf{x})$;
- if $f \in \xi(\mathbf{x})$ and the restricted function $g|_O = f|_O$ for some open set O containing $\mathbf{x}$, then $g \in \xi(\mathbf{x})$.

Denote by S the family of all functions $f : \mathbb{R}^m \to \mathbb{R}$ such that for every real $\varepsilon > 0$, for every point $\mathbf{x}$ and for every set $A \in \mathcal{T}_d$ containing $\mathbf{x}$ there is a nonempty open set O such that $O \cap A \neq \emptyset$, $f(O \cap A) \subset (f(\mathbf{x}) - \varepsilon, f(\mathbf{x}) + \varepsilon)$ and $f \in \xi(\mathbf{t})$ for every $\mathbf{t} \in O \cap A$.

For a set $H \subset \mathbb{R}^m$ and for a real $\eta > 0$, let

$$\mathcal{O}(H, \eta) = \bigcup_{\mathbf{x} \in H} K(\mathbf{x}, \eta), \text{ where } K(\mathbf{x}, \eta) = \{\mathbf{u} \in \mathbb{R}^m; |\mathbf{x} - \mathbf{u}| < \eta\}.$$

The following lemma will be used in the proofs of the next results.

Lemma 1. *Let $\mathbf{x} \in \mathbb{R}^m$ and let $H \subset \mathbb{R}^m$ be a nonempty set such that the upper density $d_u(\operatorname{int}(H), \mathbf{x}) = c > 0$. Then, there exists a sequence of pairwise disjoint sets $H_n \subset \operatorname{int}(H)$, $(n = 1, 2, \ldots)$ such that*

(1) each set H_n, $n = 1, 2, \ldots$, is the union of a finite family of cubes from $\mathcal{P}$ whose closures are pairwise disjoint;

(2) $\mathbf{x} \notin H_n$ for each $n = 1, 2, \ldots$;

(3) the family $(H_n)_n$ converges to the point $\mathbf{x}$ in the sense of the Hausdorff metric;

(4) the upper density $d_u\left(\bigcup_{n \in \mathbb{N}} \operatorname{int}(H_n), \mathbf{x}\right) = c$.

PROOF. Let $U = \mathcal{O}(H, 1)$. There is the first positive integer $n(1)$ such that the cube $P^{n(1)}(\mathbf{x}) \in \mathcal{P}_{n(1)}$ is contained in U and

$$\frac{\lambda_m\left((\operatorname{int}(H)) \cap P^{n(1)}(\mathbf{x})\right)}{\lambda_m\left(P^{n(1)}(\mathbf{x})\right)} > \frac{1}{2} \cdot c.$$

There is also a finite family of cubes

$$Q_{1,n(1)},\ Q_{2,n(1)},\ \ldots,\ Q_{i(n(1)),n(1)} \in \mathcal{P}$$

whose closures are pairwise disjoint and contained in $\operatorname{int}(P^{n(1)}(\mathbf{x}) \cap H) \setminus \{\mathbf{x}\}$ and

$$\frac{\lambda_m\left(\bigcup_{i=1}^{i(n(1))} Q_{i,n(1)}\right)}{\lambda_m(P^{n(1)}(\mathbf{x}))} \geq \left(1 - \frac{1}{2}\right) \cdot c.$$

Let $H_1 = \bigcup_{i \leq i(n(1))} Q_{i,n(1)}$ and observe that $\operatorname{cl}(H_1) = \bigcup_{i \leq i(n(1))} cl\left(Q_{i,n(1)}\right)$.

In general, for $j > 1$ we find the first positive integer $n(j)$ such that the cube $P^{n(j)}(\mathbf{x}) \in \mathcal{P}_{n(j)}$, $P^{n(j)}(\mathbf{x}) \subset P^{n(j-1)}(\mathbf{x}) \setminus \operatorname{cl}(H_{j-1})$ with $\operatorname{diam}\left(P^{n(j)}(\mathbf{x})\right) < \frac{1}{2} \cdot \operatorname{diam}\left(P^{n(j-1)}(\mathbf{x})\right)$ and

$$\frac{\lambda_m\left(\operatorname{int}(H) \cap P^{n(j)}(\mathbf{x})\right)}{\lambda\left(P^{n(j)}(\mathbf{x})\right)} > \left(1 - \frac{1}{2^j}\right) \cdot c.$$

For such an integer $n(j)$ there is a finite family of cubes

$$Q_{1,n(j)},\ Q_{2,n(j)},\ \ldots,\ Q_{i(n(j)),n(j)} \in \mathcal{P}$$

whose closures are pairwise disjoint and contained in the set $\int\left(P^{n(j)}(\mathbf{x}) \cap H\right) \setminus \{\mathbf{x}\}$ and such that

$$\frac{\lambda_m\left(\bigcup_{i=1}^{i(n(j))} Q_{i,n(j)}\right)}{\lambda_m(P^{n(j)}(\mathbf{x}))} \geq \left(1 - \frac{1}{2^j}\right) \cdot c.$$

Let $H_j = \bigcup_{i \leq i(n(j))} Q_{i,n(j)}$ and observe that

$$\operatorname{cl}(H_j) = \bigcup_{i \leq i(n(j))} \operatorname{cl}\left(Q_{i,n(j)}\right).$$

The sequence $(H_j)_j$ satisfies the conditions (1)–(4) of our lemma. $\square$

3 The Maximal Families.

In this paper the main results are the m-dimensional analogs of the results from [8, 14]. Now, let

- $Max_{add}(S) = \{f : \mathbb{R}^m \to \mathbb{R}; f + g \in S \text{ for every } g \in S\}$;
- $Max_{mult}(S) = \{f : \mathbb{R}^m \to \mathbb{R}; f \cdot g \in S \text{ for every } g \in S\}$;

- $Max_{max}(S) = \{f : \mathbb{R}^m \to \mathbb{R}; \max(f, g) \in S \text{ for every } g \in S\}$;
- $Max_{min}(S) = \{f : \mathbb{R}^m \to \mathbb{R}; \min(f, g) \in S \text{ for every } g \in S\}$;
- $Max_{comp}(S) = \{f : \mathbb{R} \to \mathbb{R}; f \circ g \in S \text{ for every } g \in S\}$.

Remark 1. *Evidently, $\mathcal{C} \subset S \cup \mathcal{C}(\mathcal{T}_{ae}) \subset Q_s$. So, every function $f \in S$ is λ_m-almost everywhere continuous ($f \in \mathcal{C}_{ae}$) ([6],[7]).*

Remark 2. *The inclusion*

$$Max_{add}(S) \cup Max_{mult}(S) \cup Max_{max}(S) \cup Max_{min}(S) \subset S$$

is true.

PROOF. Since the constant functions $g_1 = 0$ and $g_2 = 1$ belong to S, for all functions $f_1 \in Max_{add}(S)$, $f_2 \in Max_{mult}(S)$ we obtain that $f_1 = f_1 + g_1 \in S$ and $f_2 = f_2 \cdot g_2 \in S$. So, $Max_{add}(S) \cup Max_{mult}(S) \subset S$.

If $f \notin S$, then there are a real $\varepsilon > 0$, a point $\mathbf{x}$ and a set $A \in \mathcal{T}_d$ containing $\mathbf{x}$ such that for every nonempty open set O with $O \cap A \neq \emptyset$ there is a point $\mathbf{t} \in O \cap A$ such that $|f(\mathbf{t}) - f(\mathbf{x})| \geq \varepsilon$ or $f \notin \xi(\mathbf{t})$. Then the functions $\max(f, f(\mathbf{x}) - \varepsilon)$ and $\min(f, f(\mathbf{x}) + \varepsilon)$ are not in $\xi(\mathbf{x})$. So, $f \notin Max_{max}(S) \cup Max_{min}(S)$, and the proof is completed. □

3.1 The Family $Max_{add}(S)$.

In this part we suppose that the property $\xi(\mathbf{x})$, $\mathbf{x} \in \mathbb{R}^m$, is such that if $f, g \in \xi(\mathbf{x})$, then $f + g \in \xi(\mathbf{x})$; i.e., that $\xi(\cdot)$ has the additive property.

Theorem 1. *Assume that $\xi(\mathbf{x})$, $\mathbf{x} \in \mathbb{R}^m$, has the additive property. Then $\mathcal{C}(\mathcal{T}_{ae}) \cap S = Max_{add}(S)$.*

PROOF. Let $f \in \mathcal{C}(\mathcal{T}_{ae}) \cap S$ and $g \in S$. Fix a real $\varepsilon > 0$, a point $\mathbf{x} \in \mathbb{R}^m$ and a set $A \in \mathcal{T}_d$ containing $\mathbf{x}$. Since $f \in \mathcal{C}(\mathcal{T}_{ae})$, the point $\mathbf{x}$ is a density point of the set

$$B = \operatorname{int}\left(\left\{\mathbf{t} \in \mathbb{R}^m; |f(\mathbf{t}) - f(\mathbf{x})| < \frac{\varepsilon}{2}\right\}\right).$$

Consequently, $\mathbf{x}$ is a density point of the set $B \cap A$. Since $g \in S$, there is a nonempty open set $O \subset B$ such that $O \cap A \neq \emptyset$, $|g(\mathbf{t}) - g(\mathbf{x})| < \frac{\varepsilon}{2}$ and $g \in \xi(\mathbf{t})$ for every $\mathbf{t} \in O \cap A$. From the relation $f \in S$ it follows that there is a nonempty open set $O' \subset O$ such that $O' \cap A \neq \emptyset$ and $f \in \xi(\mathbf{t})$ for each point $\mathbf{t} \in O' \cap A$. Consequently, $O' \cap A \neq \emptyset$, $f + g \in \xi(\mathbf{t})$ and

$$|(f(\mathbf{t}) + g(\mathbf{t})) - (f(\mathbf{x}) + g(\mathbf{x}))| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

for each point $\mathbf{t} \in O \cap A$. So, $f \in Max_{add}(S)$ and the inclusion $\mathcal{C}(\mathcal{T}_{ae}) \cap S \subset Max_{add}(S)$ is proved.

For the proof of the inclusion $Max_{add}(S) \subset \mathcal{C}(\mathcal{T}_{ae}) \cap S$, fix a function $f \in Max_{add}(S)$. By Remark 2, the function $f \in S$. If $f \notin \mathcal{C}(\mathcal{T}_{ae})$, there are a point $\mathbf{x} \in \mathbb{R}^m$ and a real $\varepsilon > 0$ such that the set $\mathrm{cl}(\{\mathbf{t} \in \mathbb{R}^m; |f(\mathbf{t}) - f(\mathbf{x})| > \varepsilon\})$ has a positive upper density at a point $\mathbf{x}$. Without loss of generality, we can assume that

$$d_u(\mathrm{cl}(\{\mathbf{t} \in \mathbb{R}^m; f(\mathbf{t}) > f(\mathbf{x}) + \varepsilon\}),\ \mathbf{x}) > 0.$$

Since $f \in S \subset Q_s$ is λ_m-almost everywhere continuous ([6]), we obtain

$$\lambda_m(\mathrm{cl}(\{\mathbf{t}; f(\mathbf{t}) > f(\mathbf{x}) + \varepsilon\}) \setminus \{\mathbf{t}; f(\mathbf{t}) \geq f(\mathbf{x}) + \varepsilon\}) = 0,$$

and consequently,

$$d_u\left(\mathrm{int}\left(\left\{\mathbf{t} \in \mathbb{R}^m; f(\mathbf{t}) > f(\mathbf{x}) + \frac{\varepsilon}{2}\right\}\right),\ \mathbf{x}\right) > 0.$$

For $H = \left\{\mathbf{t} \in \mathbb{R}^m; f(\mathbf{t}) > f(\mathbf{x}) + \frac{\varepsilon}{2}\right\}$, there exists a sequence of pairwise disjoint sets $H_n \subset \mathrm{int}(H),\ n = 1,\ 2,\ \ldots$ which satisfies conditions (1)–(4) of Lemma 1.

Now, put

$$g(\mathbf{t}) = \begin{cases} -f(\mathbf{x}) + \frac{\varepsilon}{2} & \text{if} (\mathbf{t} = \mathbf{x}) \vee (\mathbf{t} \in H_n,\ n = 1, 2, \ldots) \\ -f(\mathbf{t}) & \text{otherwise on } \mathbb{R}^m. \end{cases}$$

The function $g \in S$. Indeed, fix a real $\eta > 0$, a point $\mathbf{u} \in \mathbb{R}^m$ and a set $A \in \mathcal{T}_d$ containing $\mathbf{u}$. If $\mathbf{u} \in H_n$ for some $n \in \mathbb{N}$, then there is a nonempty open set $O \subset H_n$ with $O \cap A \neq \emptyset$ and $g(O \cap A) \subset (g(\mathbf{u}) - \eta, g(\mathbf{u}) + \eta)$. Moreover, $g \in \xi(\mathbf{u})$ for each point $\mathbf{u} \in O \cap A$ (in this case the function $g|_O$ is constant and equals $-f(\mathbf{x}) + \frac{\varepsilon}{2}$ on the set O). Note, if $\mathbf{u} = \mathbf{x}$, then by (4) of Lemma 1 there is an index $n \in \mathbb{N}$ with $A \cap \mathrm{int}(H_n) \neq \emptyset$. So, it is enough to suppose that $O = \mathrm{int}(H_n)$ in this case. If $\mathbf{u} \notin \bigcup_{n=1}^{\infty} H_n \cup \{\mathbf{x}\}$, then there is an open set O such that $O \cap (\bigcup_{n=1}^{\infty} H_n \cup \{\mathbf{x}\}) = \emptyset$ and $O \cap A \neq \emptyset$. Since $g|_O = -f|_O$, $f(O \cap A) \subset (f(\mathbf{u}) - \eta, f(\mathbf{u}) + \eta)$ and $f \in \xi(\mathbf{u})$ for every $\mathbf{u} \in O \cap A$, we obtain

$$g(O \cap A) = -f(O \cap A) \subset (-f(\mathbf{u}) - \eta, -f(\mathbf{u}) + \eta) = (g(\mathbf{u}) - \eta, g(\mathbf{u}) + \eta)$$

and $g \in \xi(\mathbf{u})$ for each point $\mathbf{u} \in O \cap A$.

But, observe that $f(\mathbf{x}) + g(\mathbf{x}) = \frac{\varepsilon}{2}$, $f(\mathbf{t}) + g(\mathbf{t}) > \varepsilon$ for $\mathbf{t} \in H_n$, $(n = 1, 2, \ldots)$ and $f(\mathbf{t}) + g(\mathbf{t}) = 0$ otherwise on $\mathbb{R}^m$. So, $f + g \notin S$ and consequently $f \notin Max_{add}(S)$. This contradiction finishes the proof. $\square$

Corollary 1. *If the property* $\xi(\mathbf{x}),\ \mathbf{x} \in \mathbb{R}^m$, *denotes that*

- $f(\mathbf{x}) \in \mathbb{R}^m$, *then* $S = Q_s$ *and* $Max_{add}(Q_s) = \mathcal{C}(\mathcal{T}_{ae}) \cap Q_s$;
- $\mathbf{x} \in C(f)$, *then* $S = Q_{s_1}$ *and* $Max_{add}(Q_{s_1}) = \mathcal{C}(\mathcal{T}_{ae}) \cap Q_{s_1}$;
- $\mathbf{x} \in A(f)$, *then* $S = Q_{s_2}$ *and* $Max_{add}(Q_{s_2}) = \mathcal{C}(\mathcal{T}_{ae}) \cap Q_{s_2}$.

3.2 The Families $Max_{max}(S)$ and $Max_{min}(S)$.

In this part we suppose that if $f, g \in \xi(\mathbf{x}),\ \mathbf{x} \in \mathbb{R}^m$, then $\max(f,g),\ \min(f,g) \in \xi(\mathbf{x})$. Then, we say that $\xi(\cdot)$ has the lattice property.

Theorem 2. *Let* $\xi(\mathbf{x}),\ \mathbf{x} \in \mathbb{R}^m$, *has the lattice property. Then,*

$$Max_{max}(S) = Max_{min}(S) = \mathcal{C}(\mathcal{T}_{ae}) \cap S.$$

PROOF. For the proof of the inclusion

$$\mathcal{C}(\mathcal{T}_{ae}) \cap S \ \subset\ Max_{max}(S) \cap Max_{min}(S),$$

we take a function $f \in \mathcal{C}(\mathcal{T}_{ae}) \cap S$ and a function $g \in S$. Fix a real $\varepsilon > 0$, a point $\mathbf{x} \in \mathbb{R}^m$ and a set $A \in \mathcal{T}_d$ containing $\mathbf{x}$. Let $h = \max(f,g)$. Consider the following cases.

(1) $f(\mathbf{x}) > g(\mathbf{x})$. Let $a = f(\mathbf{x}) - g(\mathbf{x})$ and let $b = \min\left(\frac{a}{2}, \varepsilon\right)$. Since $f \in \mathcal{C}(\mathcal{T}_{ae})$, $\mathbf{x}$ is a density point of the set $B = \operatorname{int}(\{\mathbf{t}; |f(\mathbf{t}) - f(\mathbf{x})| < b\})$. By the relation $g \in S$ being applied to the point $\mathbf{x}$ and the set $B \cap A \in \mathcal{T}_d$, it follows that there is an open set O such that $O \cap (A \cap B) \neq \emptyset,\ g \in \xi(\mathbf{t})$ and $|g(\mathbf{t}) - g(\mathbf{x})| < b$ for each point $\mathbf{t} \in O \cap (A \cap B)$.

Since $f \in S$, there is an open set $O' \subset O \cap B$ with $O' \cap (A \cap B) \neq \emptyset$ and $f \in \xi(\mathbf{t})$ for each point $\mathbf{t} \in O' \cap (A \cap B)$. Observe that for $\mathbf{u} \in O' \cap (A \cap B)$, we have

$$f(\mathbf{u}) > f(\mathbf{x}) - b \geq g(\mathbf{x}) + 2b - b = g(\mathbf{x}) + b > g(\mathbf{u}),$$

so $h(\mathbf{u}) = f(\mathbf{u})$. Moreover, $h(\mathbf{x}) = f(\mathbf{x})$, and for each point $\mathbf{u} \in O \cap (A \cap B)$ we have $h \in \xi(\mathbf{u})$ and $|h(\mathbf{u}) - h(\mathbf{x})| = |f(\mathbf{u}) - f(\mathbf{x})| < b \leq \varepsilon$.

(2) $f(\mathbf{x}) < g(\mathbf{x})$. In this case the proof is analogous as above.

(3) $f(\mathbf{x}) = g(\mathbf{x})$. Let $b = \varepsilon$ and choose an open set O' as above in case (1). Then, $O' \cap (A \cap B) \neq \emptyset$ and for $\mathbf{u} \in O' \cap (A \cap B)$ we obtain $h \in \xi(\mathbf{u})$ and

$$|h(\mathbf{u}) - h(\mathbf{x})| \leq \max(|f(\mathbf{u}) - f(\mathbf{x})|, |g(\mathbf{u}) - g(\mathbf{x})|) < b = \varepsilon.$$

So, $h = \max(f,g) \in S$. The proof $\min(f,g) \in S$ is analogous.

Finally, since by Remark 2 the inclusion $Max_{max}(S) \cup Max_{min}(S) \subset S$ is true, we shall show the inclusion

$$Max_{max}(S) \cup Max_{min}(S) \ \subset \ \mathcal{C}(\mathcal{T}_{ae}).$$

Let $f \in Max_{max}(S)$ be a function. By Remark 2, $f \in S$. If $f \notin \mathcal{C}(\mathcal{T}_{ae})$, then there are a point $\mathbf{x} \in \mathbb{R}^m$ and a real $\varepsilon > 0$ such that

$$d_u(\mathrm{cl}(\{\mathbf{t} \in \mathbb{R}^m; |f(\mathbf{t}) - f(\mathbf{x})| > \varepsilon\}),\ \mathbf{x}) > 0.$$

If $d_u(\mathrm{cl}(\{\mathbf{t} \in \mathbb{R}^m; f(\mathbf{t}) > f(\mathbf{x}) + \varepsilon\}), \mathbf{x}) > 0$, then, as before in the proof of Theorem 1, for $H = \left\{\mathbf{t} \in \mathbb{R}^m; f(\mathbf{t}) > f(\mathbf{x}) + \frac{\varepsilon}{2}\right\}$, there exists a sequence of pairwise disjoint sets $H_n \subset \mathrm{int}(H)$, $n = 1,\ 2,\ \ldots$ such that conditions (1)–(4) of Lemma 1 are satisfied. Let the function $g_1 : \mathbb{R}^m \to \mathbb{R}$ be defined by

$$g_1(\mathbf{t}) = \begin{cases} f(\mathbf{x}) - \varepsilon & \text{if } (\mathbf{t} = \mathbf{x}) \vee (\mathbf{t} \in H_n,\ n = 1, 2, \ldots) \\ f(\mathbf{x}) + \varepsilon & \text{otherwise on } \mathbb{R}^m. \end{cases}$$

Note that $g_1 \in S$. Moreover, $\max(f(\mathbf{x}), g_1(\mathbf{x})) = f(\mathbf{x})$ and $\max(f(\mathbf{t}), g_1(\mathbf{t})) > f(\mathbf{x}) + \frac{\varepsilon}{2}$ for $\mathbf{t} \neq \mathbf{x}$. So, $\max(f, g_1) \notin S$ and consequently $f \notin Max_{max}(S)$, yielding a contradiction.

Now, consider the case $d_u(\mathrm{cl}(\{\mathbf{t} \in \mathbb{R}^m; f(\mathbf{t}) < f(\mathbf{x}) - \varepsilon\}),\ \mathbf{x}) > 0$. Then, as before in this proof, there are disjoint sets

$$K_n \subset \mathrm{int}\left(\left\{\mathbf{t} \in \mathbb{R}^m; f(\mathbf{t}) < f(\mathbf{x}) - \frac{\varepsilon}{2}\right\}\right),\ n = 1,\ 2,\ \ldots$$

which satisfy conditions (1)–(4) of Lemma 1. Let the function $g_2 : \mathbb{R}^m \to \mathbb{R}$ be defined as g_1 before, but for the sets K_n, $n = 1,\ 2,\ \ldots$. Then, $g_2 \in S$ and $\max(f(\mathbf{x}), g_2(\mathbf{x})) = f(\mathbf{x})$, $\max(f(\mathbf{t}), g_2(\mathbf{t})) < f(\mathbf{x}) - \frac{\varepsilon}{2}$ for $\mathbf{t} \in K_n$, $(n = 1, 2, \ldots)$ and $\max(f(\mathbf{t}), g_2(\mathbf{t})) \geq f(\mathbf{x}) + \varepsilon$ otherwise on $\mathbb{R}^m$. So, in this case also, $\max(f, g_2) \notin S$ and consequently $f \notin Max_{max}(S)$, yielding a contradiction.

We can prove the inclusion $Max_{min}(S) \subset \mathcal{C}(\mathcal{T}_{ae})$ analogously. $\square$

Corollary 2. *If the property $\xi(\mathbf{x})$, $\mathbf{x} \in \mathbb{R}^m$, denotes that*

- $f(\mathbf{x}) \in \mathbb{R}^m$, *then* $S = Q_s$ *and* $Max_{max}(Q_s) = Max_{min}(Q_s) = \mathcal{C}(\mathcal{T}_{ae}) \cap Q_s$;
- $\mathbf{x} \in C(f)$, *then* $S = Q_{s_1}$ *and* $Max_{max}(Q_{s_1}) = Max_{min}(Q_{s_1}) = \mathcal{C}(\mathcal{T}_{ae}) \cap Q_{s_1}$;
- $\mathbf{x} \in A(f)$, *then* $S = Q_{s_2}$ *and* $Max_{max}(Q_{s_2}) = Max_{min}(Q_{s_2}) = \mathcal{C}(\mathcal{T}_{ae}) \cap Q_{s_2}$.

3.3 The Family $Max_{comp}(S)$.

Suppose that for every functions $f : \mathbb{R} \to \mathbb{R}$ belonging to $\mathcal{C}$ and for every function $g \in \xi(\mathbf{x})$, $\mathbf{x} \in \mathbb{R}^m$, we have $f \circ g \in \xi(\mathbf{x})$, i.e., $\xi(\cdot)$ is invariant with respect to the composition with the continuous functions from $\mathbb{R}$ to $\mathbb{R}$.

Theorem 3. *Assume that $\xi(\mathbf{x})$, $\mathbf{x} \in \mathbb{R}^m$, is invariant with respect to the composition with the continuous functions from $\mathcal{C}$. Then, $Max_{comp}(S) = \mathcal{C}$.*

Proof. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function and let $g \in S$ be a function. Fix a real $\varepsilon > 0$, a point $\mathbf{x}$ and a set $A \in \mathcal{T}_d$ containing $\mathbf{x}$. Since f is continuous at $g(\mathbf{x})$, there is a real $\delta > 0$ such that if $|\mathbf{u} - g(\mathbf{x})| < \delta$, then $|f(\mathbf{u}) - f(g(\mathbf{x}))| < \varepsilon$. Since $g \in S$, there is a nonempty open set O such that $O \cap A \neq \emptyset, g \in \xi(\mathbf{t})$ and $|g(\mathbf{t}) - g(\mathbf{x})| < \delta$ for each point $\mathbf{t} \in O \cap A$. Observe that for every point $\mathbf{t} \in O \cap A$ we obtain $f \circ g \in \xi(\mathbf{t})$ and $|f(g(\mathbf{t})) - f(g(\mathbf{x}))| < \varepsilon$. So, $f \circ g \in S$, and consequently $\mathcal{C} \subset Max_{comp}(S)$.

Suppose that $f : \mathbb{R} \to \mathbb{R}$ is not continuous at a point $y \in \mathbb{R}$. Then there is a sequence of points $y_n \neq y, n = 1, 2, \ldots$, such that $\lim_{n\to\infty} y_n = y$ and $\lim_{n\to\infty} f(y_n) \neq f(y)$. Let $P^1(\mathbf{0}) \in \mathcal{P}_1$ be a cube containing a point $\mathbf{x} = \mathbf{0}$. For $\mathbf{x} = \mathbf{0}$ and $H = P^1(\mathbf{0})$ there exists a family of sets $H_j \subset \operatorname{int}(P^1(\mathbf{0})), j = 1, 2, \ldots$ which satisfies conditions (1)–(4) of Lemma 1. Put

$$g(\mathbf{x}) = \begin{cases} y_n & \text{if } \mathbf{x} \in H_n,\ n = 1, 2, \ldots \\ y & \text{if } \mathbf{x} = \mathbf{0} \\ y_1 & \text{otherwise on } \mathbb{R}^m. \end{cases}$$

The function $g \in S$. Indeed, fix a real $\varepsilon > 0$, a point $\mathbf{x} \in \mathbb{R}^m$ and a set $A \in \mathcal{T}_d$ containing $\mathbf{x}$. If $\mathbf{x} \neq \mathbf{0}$, then there exists a cube $P(\mathbf{x}) \in \mathcal{P}$ containing $\mathbf{x}$ such that the restricted function $g|_{\mathrm{cl}(P(\mathbf{x}))}$ is constant and there exists an open set $O \subset P(\mathbf{x})$ such that $O \cap A \neq \emptyset, g(O \cap A) \subset (g(\mathbf{x}) - \varepsilon, g(\mathbf{x}) + \varepsilon)$ and $g \in \xi(\mathbf{u})$ for each point $\mathbf{u} \in U \cap A$. If $\mathbf{x} = \mathbf{0}$, then there exists an index $n \in \mathbb{N}$ such that $|y_n - y| < \varepsilon$ and there is a nonempty open set $O \subset H_n$ such that $O \cap A \neq \emptyset$. Obviously, $g|_{O\cap A}$ is constant. So, $g \in \xi(\mathbf{u})$ for each $\mathbf{u} \in O \cap A$ and since $|g(\mathbf{u}) - g(\mathbf{0})| = |y_n - y|$ for each $\mathbf{u} \in O$, we obtain $g(O \cap A) \subset (g(\mathbf{0}) - \varepsilon, g(\mathbf{0}) + \varepsilon)$. But observe, $f \circ g \notin Q_s(\mathbf{0})$ and thus $f \circ g \notin S$. This contradiction shows that for every function $g \in S$ if $f \circ g \in S$, then $f \in \mathcal{C}$ and the proof is completed. □

Corollary 3. *If the property $\xi(\mathbf{x})$, $\mathbf{x} \in \mathbb{R}^m$, denotes that*

- $f(\mathbf{x}) \in \mathbb{R}^m$*, then* $S = Q_s$ *and* $Max_{comp}(Q_s) = \mathcal{C}$;
- $\mathbf{x} \in C(f)$*, then* $S = Q_{s_1}$ *and* $Max_{comp}(Q_{s_1}) = \mathcal{C}$;

- $\mathbf{x} \in A(f)$, *then* $S = Q_{s_2}$ *and* $Max_{comp}(Q_{s_2}) = \mathcal{C}$.

3.4 The Family $Max_{mult}(S)$.

Suppose that the property $\xi(\mathbf{x})$, $\mathbf{x} \in \mathbb{R}^m$, is such that

- if $f, g \in \xi(\mathbf{x})$, then $f \cdot g \in \xi(\mathbf{x})$;
- if $f \in \xi(\mathbf{x})$ and there is an open set O such that $d_u(O, \mathbf{x}) = 1$ and $f(\mathbf{x}) \neq 0 \notin f(O)$, then every extension of the function $h(\mathbf{t}) = \frac{1}{f(\mathbf{t})}$ for $\mathbf{t} \in O \cup \{\mathbf{x}\}$ belongs to $\xi(\mathbf{x})$.

Lemma 2. *If a function* $f \in S$ *is not* $\mathcal{T}_{ae}$*-continuous at a point* $\mathbf{x} \in \mathbb{R}^m$ *at which* $f(\mathbf{x}) \neq 0$*, then there is a function* $g \in S$ *such that the product* $f \cdot g \notin S$.

PROOF. Arguing as in the proof of Theorem 1, we can show that there is a real $\varepsilon > 0$ and a family of sets $H_n \subset \operatorname{int}\left(\left\{\mathbf{t} \in \mathbb{R}^m; f(\mathbf{t}) > f(\mathbf{x}) + \frac{\varepsilon}{2}\right\}\right)$, $n = 1, 2, \ldots$ which satisfy conditions (1)–(4) of Lemma 1.

Put

$$g(\mathbf{t}) = \begin{cases} 1 & \text{if } (\mathbf{t} = \mathbf{x}) \vee (\mathbf{t} \in H_n, n = 1, 2, \ldots), \\ 0 & \text{otherwise on } \mathbb{R}^m, \end{cases}$$

and observe that $g \in S$. But $f(\mathbf{x}) \cdot g(\mathbf{x}) = f(\mathbf{x}) \neq 0$ and for every point $\mathbf{t} \neq \mathbf{x}$ we have $f(\mathbf{t}) \cdot g(\mathbf{t}) = 0$ or $|f(\mathbf{t}) \cdot g(\mathbf{t}) - f(\mathbf{x}) \cdot g(\mathbf{x})| = |f(\mathbf{t}) - f(\mathbf{x})| > \frac{\varepsilon}{2}$. So, $f \cdot g \notin Q_s(\mathbf{x})$, and thus $f \cdot g \notin S$. This completes the proof. $\square$

Lemma 3. *Let* $f \in S$ *be a function and let* $\mathbf{x} \in \mathbb{R}^m$ *be a point such that* $f(\mathbf{x}) = 0$*. If* $d_u(\{\mathbf{t} \in \mathbb{R}^m; f(\mathbf{t}) = 0\}, \mathbf{x}) > 0$*, then for every function* $g \in S$*, for every real* $\varepsilon > 0$ *and for every set* $A \in \mathcal{T}_d$ *containing* $\mathbf{x}$ *there is an open set* O *such that* $O \cap A \neq \emptyset$*, the product* $f \cdot g \in \xi(\mathbf{t})$ *and* $|f(\mathbf{t}) \cdot g(\mathbf{t})| < \varepsilon$ *for each point* $\mathbf{t} \in O \cap A$.

PROOF. Fix a function $g \in S$, a real $\varepsilon > 0$ and a set $A \in \mathcal{T}_d$ containing $\mathbf{x}$. Since $f, g \in S$, they are λ_m-almost everywhere continuous. Observe that the set

$$B = \{\mathbf{t} \in A; f(\mathbf{t}) = 0 \text{ and } f \text{ is continuous at } \mathbf{t}\}$$

is of positive λ_m-measure. Find a point $\mathbf{u} \in B$ such that $f(\mathbf{u}) = 0$ and the function g is continuous at $\mathbf{u}$. Let O be an open set containing $\mathbf{u}$ such that there is a real $r > 0$ with $|g(\mathbf{t})| < r$ for each point $\mathbf{t} \in O$. Observe that $\mathbf{u} \in O \cap A \in \mathcal{T}_d$. Since $f \in S$ and $f(\mathbf{u}) = 0$, there is an open set $O' \subset O$ such that $O' \cap A \neq \emptyset$, $f \in \xi(\mathbf{t})$ and $|f(\mathbf{t})| < \frac{\varepsilon}{r}$ for each point $\mathbf{t} \in O' \cap A$. But $g \in S$ and $\emptyset \neq O' \cap A \in \mathcal{T}_d$, so there is an open set $O'' \subset O'$ such that $O'' \cap A \neq \emptyset$

and $g \in \xi(\mathbf{t})$ for each point $\mathbf{t} \in O'' \cap A$. Finally, observe that for $\mathbf{t} \in O'' \cap A$, we have

$$f \cdot g \in \xi(\mathbf{t}) \text{ and } |f(\mathbf{t}) \cdot g(\mathbf{t}) - f(\mathbf{x}) \cdot g(\mathbf{x})| = |f(\mathbf{t}) \cdot g(\mathbf{t})| < \frac{\varepsilon}{r} \cdot r = \varepsilon.$$

This completes the proof. □

Lemma 4. *Suppose that the function $f \in S$ is not $\mathcal{T}_{ae}$- continuous at a point $\mathbf{x}$ at which $f(\mathbf{x}) = 0$. If*

$$d_u(\{\mathbf{t} \in \mathbb{R}^m; f(\mathbf{t}) = 0\},\ \mathbf{x}) = 0,$$

there is a function $g \in S$ such that $f \cdot g \notin S$.

Proof. Since f is λ_m-almost everywhere continuous, we obtain

$$\lambda_m\left(\mathrm{cl}\left(\{\mathbf{t} \in \mathbb{R}^m; f(\mathbf{t}) = 0\}\right) \setminus \{\mathbf{t} \in \mathbb{R}^m; f(\mathbf{t}) = 0\}\right) = 0$$

$$\text{and } d_u\left(\mathrm{cl}\left(\{\mathbf{t} \in \mathbb{R}^m; f(\mathbf{t}) = 0\}\right),\ \mathbf{x}\right) = 0.$$

Since f is not $\mathcal{T}_{ae}$-continuous at $\mathbf{x}$, there is a real $\varepsilon > 0$ such that the set $\mathrm{cl}(\{\mathbf{t} \in \mathbb{R}^m; |f(\mathbf{t})| > \varepsilon\})$ has positive upper density at a point $\mathbf{x}$. Moreover, since $\{\mathbf{t} \in \mathbb{R}^m; |f(\mathbf{t})| > \varepsilon\} = \{\mathbf{t} \in \mathbb{R}^m; f(\mathbf{t}) > \varepsilon\} \cup \{\mathbf{t} \in \mathbb{R}^m; f(\mathbf{t}) < -\varepsilon\}$, we obtain

$$d_u(\{\mathbf{t} \in \mathbb{R}^m; f(\mathbf{t}) > \varepsilon\},\ \mathbf{x}) > 0 \text{ or } d_u(\{\mathbf{t} \in \mathbb{R}^m; f(\mathbf{t}) < -\varepsilon\},\ \mathbf{x}) > 0. \tag{3.1}$$

Without loss of generality, we can assume that the first of the inequalities (3.1) is true. Since f is λ_m-almost everywhere continuous, we have $d_u(\mathrm{int}(H),\ \mathbf{x}) > 0$ for $H = \left\{\mathbf{t} \in \mathbb{R}^m; f(\mathbf{t}) > \frac{\varepsilon}{2}\right\} \cap P^{n(1)}(\mathbf{x})$, where $n(1)$ is the first positive integer such that $P^{n(1)}(\mathbf{x}) \in \mathcal{P}_{n(1)}$ and $P^{n(1)}(\mathbf{x}) \subset \mathcal{O}\left(\left\{\mathbf{t} \in \mathbb{R}^m; f(\mathbf{t}) > \frac{\varepsilon}{2}\right\},\ 1\right)$. By Lemma 1 applied to the set H and the point $\mathbf{x}$, there exists a sequence $(H_n)_n$ of subsets of $\mathrm{int}(H)$ such that conditions (1)–(4) of Lemma 1 are satisfied.
Let $K = \{\mathbf{t} \in P^{n(1)}(\mathbf{x}); f(\mathbf{t}) = 0\}$. The upper density $d_u(\mathrm{cl}(K),\ \mathbf{x}) = 0$. We will prove that there is an open (in $P^{n(1)}(\mathbf{x})$) set $V \supset \mathrm{cl}(K) \setminus \{\mathbf{x}\}$ contained in $P^{n(1)}(\mathbf{x}) \setminus \bigcup_{n=1}^{\infty} H_n \setminus \{\mathbf{x}\}$ such that

$$d_u(V,\ \mathbf{x}) = 0 \text{ and } \lambda_m(\mathrm{cl}(V) \setminus V) = 0.$$

Let $(s_n)_n$ be a sequence of positive numbers such that

$$\lim_{n\to\infty} \frac{s_n}{\lambda_m\left(P^{n+2}(\mathbf{x})\right)} = 0.$$

Since the set

$$T = \mathrm{cl}\left(P^n(\mathbf{x}) \setminus P^{n+1}(\mathbf{x})\right) \cap \mathrm{cl}(K)$$

is compact for each $n \geq n(1)$, there exists a finite family of open balls

$$B_1^n,\ B_2^n,\ \ldots,\ B_{i(n)}^n \subset P^n(\mathbf{x}) \setminus \mathrm{cl}\left(P^{n+2}(\mathbf{x})\right) \setminus \mathrm{cl}\left(\bigcup_{n=1}^{\infty} H_n\right)$$

such that

$$\bigcup_{i=1}^{i(n)} B_i^n \supset T \text{ and } \lambda_m\left(\bigcup_{i=1}^{i(n)} B_i^n \setminus T\right) < \frac{s_n}{4^n}.$$

Observe that the set $V = \bigcup_{n \geq n(1)} \bigcup_{i=1}^{i(n)} B_i^n$ is open and satisfies all requirements. Let

$$B = P^{n(1)}(\mathbf{x}) \setminus \left(V \cup \bigcup_{n=1}^{\infty} H_n \cup \{\mathbf{x}\}\right)$$

and put

$$g(\mathbf{t}) = \begin{cases} \varepsilon & \text{if } (\mathbf{t} = \mathbf{x}) \vee (\mathbf{t} \in H_n,\ n = 1, 2, \ldots), \\ 0 & \text{if } (\mathbf{t} \in V) \vee (\mathbf{t} \in B \text{ and } d_u(V,\ \mathbf{t}) > 0), \\ \frac{1}{f(\mathbf{t})} & \text{if } \mathbf{t} \in B \text{ and } d_u(V,\ \mathbf{t}) = 0, \\ f(\mathbf{t}) & \text{if } \mathbf{t} \in \mathbb{R}^m \setminus P^{n(1)}(\mathbf{x}). \end{cases}$$

We can prove that $g \in S$ by methods used above. But the product $f \cdot g \notin Q_s(\mathbf{x})$. Indeed, observe that on $P^{n(1)}(\mathbf{x})$ we have

$$f(\mathbf{x}) \cdot g(\mathbf{x}) = 0,$$

$$f(\mathbf{t}) \cdot g(\mathbf{t}) > \tfrac{\varepsilon^2}{2} \operatorname{int}(\text{ for } \mathbf{t} \in H_n,\ n \in \mathbb{N},$$

$$f(\mathbf{t}) \cdot g(\mathbf{t}) = 0 \text{ if } \mathbf{t} \in P^{n(1)}(\mathbf{x}) \setminus \left(\bigcup_{n=1}^{\infty} H_n \cup \{\mathbf{x}\}\right) \text{ and } d_u(V,\ \mathbf{t}) > 0,$$

$$f(\mathbf{t}) \cdot g(\mathbf{t}) = 1 \text{ if } \mathbf{t} \in B \text{ and } d_u(V,\ \mathbf{t}) = 0,$$

and for each $\mathbf{t} \in \mathbb{R}^m \setminus P^{n(1)}(\mathbf{x})$ we have $g(\mathbf{t}) \cdot f(\mathbf{t}) = (f(\mathbf{t}))^2$. If A is the set of all density points of the set $B \cup \bigcup_{n=1}^{\infty} H_n$ and $\eta = \frac{1}{2} \cdot \min\left\{1, \frac{\varepsilon^2}{2}\right\}$, then $\mathbf{x} \in A$ and for each open set O with $O \cap A \neq \emptyset$ the image $f(O \cap A)$ is not contained in $(f(\mathbf{x}) - \eta,\ f(\mathbf{x}) + \eta) = (-\eta,\ \eta)$. So, $f \cdot g \notin S$. □

Lemma 5. *If a function $f \in S$ is $\mathcal{T}_{ae}$-continuous at a point $\mathbf{x} \in \mathbb{R}^m$, then for every function $g \in S$, for every set $A \in \mathcal{T}_d$ containing $\mathbf{x}$ and for every real $\varepsilon > 0$ there is a nonempty open set O such that $O \cap A \neq \emptyset$, $f \cdot g \in \xi(\mathbf{t})$ and $|f(\mathbf{t}) \cdot g(\mathbf{t}) - f(\mathbf{x}) \cdot g(\mathbf{x})| < \varepsilon$ for each point $\mathbf{t} \in O \cap A$.*

PROOF. Fix a real $\varepsilon > 0$, a set $A \in \mathcal{T}_d$ containing $\mathbf{x}$. Since f is $\mathcal{T}_{ae}$-continuous at $\mathbf{x}$, the point $\mathbf{x}$ is a density point of the set

$$B = \operatorname{int}\left\{\mathbf{t} \in \mathbb{R}^m; |f(\mathbf{t}) - f(\mathbf{x})| < \frac{\varepsilon}{2 \cdot \max(|g(\mathbf{x})|, 1)}\right\},$$

Consequently, $\mathbf{x}$ is a density point of the set $B \cap A$. Since $f \in S$, there is a nonempty open set $O \subset B$ such that $O \cap A \neq \emptyset$ and $f \in \xi(\mathbf{t})$ for each point $\mathbf{t} \in O \cap A$. Since $g \in S$, there is a nonempty open set $O' \subset O$ such that $O' \cap A \neq \emptyset$,

$$|g(\mathbf{t}) - g(\mathbf{x})| < \frac{\varepsilon}{2 \cdot \max(\sup_{\mathbf{t} \in O' \cap A} |f(\mathbf{t})|,\ 1)}$$

and $g \in \xi(\mathbf{t})$ for each $\mathbf{t} \in O' \cap A$. Consequently, we obtain that $f \cdot g \in S(\mathbf{t})$ and

$$|f(\mathbf{t}) \cdot g(\mathbf{t}) - f(\mathbf{x}) \cdot g(\mathbf{x})| \leq |f(\mathbf{t})| \cdot |g(\mathbf{t}) - g(\mathbf{x})| + |g(\mathbf{x})| \cdot |f(\mathbf{t}) - f(\mathbf{x})| <$$

$$\sup_{\mathbf{t} \in O' \cap A} |f(\mathbf{t})| \cdot \frac{\varepsilon}{2 \cdot \max(\sup_{\mathbf{t} \in O' \cap A} |f(\mathbf{t})|,\ 1)} + |g(\mathbf{x})| \cdot \frac{\varepsilon}{2 \cdot \max(|g(\mathbf{x})|,\ 1)} \leq \varepsilon.$$

So, $f \cdot g \in S$ and the proof is completed. □

From Lemmas 2, 3, 4 and 5 we immediately obtain the following theorem.

Theorem 4. *A function $f \in Max_{mult}(S)$ if and only if $f \in S$ and satisfies the following condition.*

(m) if f is not $\mathcal{T}_{ae}$-continuous at a point $\mathbf{x} \in \mathbb{R}^m$, then $f(\mathbf{x}) = 0$ and $d_u(\{\mathbf{t} \in \mathbb{R}^m; f(\mathbf{t}) = 0\}, \mathbf{x}) > 0$.

Corollary 4. *If the property $\xi(\mathbf{x}), \mathbf{x} \in \mathbb{R}^m$, denotes that*

- *$f(\mathbf{x}) \in \mathbb{R}$, then $S = Q_s$ and $f \in Max_{mult}(Q_s)$ if and only if $f \in Q_s$ and satisfies the condition (m);*
- *$\mathbf{x} \in C(f)$, then $S = Q_{s_1}$ and $f \in Max_{mult}(Q_{s_1})$ if and only if $f \in Q_{s_1}$ and satisfies the condition (m);*
- *$\mathbf{x} \in A(f)$, then $S = Q_{s_2}$ and $f \in Max_{mult}(Q_{s_2})$ if and only if $f \in Q_{s_2}$ and satisfies the condition (m).*

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