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RESIDUALITY OF GAMILIES OF $\mathcal{F}_\sigma$ SETS

Abstract

We prove that two natural definitions of residuality of families of $\mathcal{F}_\sigma$ sets are equivalent. We make use of the Banach-Mazur game in the proof.

1 Introduction

Properties of a typical compact set in the Euclidean space are often discussed. Here we say that a property P is fulfilled by a typical compact set if the set of all compact sets satisfying P is residual in the space of all compact sets endowed with the Hausdorff metric. It is well-known that a typical compact set in the Euclidean space is Lebesgue null (see [3], for example). In this paper we consider what a typical $\mathcal{F}_\sigma$ set means, namely we define residuality of families of $\mathcal{F}_\sigma$ sets. To the best of the author's knowledge, there has been no definition of such residuality.

We shall work in a compact, dense-in-itself metric space (X, ρ) throughout this article. Without loss of generality, we may assume that $\rho(x, y) \leqq 1$ for any $x, y \in X$. An $\mathcal{F}_\sigma$ set means an $\mathcal{F}_\sigma$ subset of X, and $\mathcal{F}_\sigma$ stands for the set of all $\mathcal{F}_\sigma$ sets. Let $\mathcal{K}$ denote the set of all compact (or equivalently closed) subsets of X. For $x \in X$ and $r > 0$, the closed ball of centre x and radius r is denoted by $\bar{B}(x, r)$. For $K \in \mathcal{K}$ and $r > 0$, we put $K[r] = \bigcup_{x \in K} \bar{B}(x, r)$. It is well-known that the Hausdorff metric d makes $\mathcal{K}$ a compact metric space. Here we define $d(K, \emptyset) = 1$ for any nonempty set $K \in \mathcal{K}$. Then for $K, L \in \mathcal{K}$ and $r \in (0, 1)$, we have $d(K, L) \leqq r$ if and only if $K \subset L[r]$ and $L \subset K[r]$, even when either K or L is empty.

Giving $\mathcal{F}_\sigma$ a topology would suffice to define residuality of families of $\mathcal{F}_\sigma$ sets, but no good topology on $\mathcal{F}_\sigma$ has been found so far. Bearing in mind that

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each $\mathcal{F}_\sigma$ set is the union of a sequence in $\mathcal{K}$, we look at the space of sequences in $\mathcal{K}$ instead. Here we might worry whether we should restrict ourselves only to increasing sequences, but our main theorem removes this concern. Let us proceed to rigorous definitions.

Convention 1.1. *Every sequence begins with the term of subscript one and the set $\mathbb{N}$ of all positive integers does not contain zero.*

The set of all sequences of sets in $\mathcal{K}$ is denoted by $\mathcal{K}^{\mathbb{N}}$ and endowed with the product topology. The closed subset $\mathcal{K}^{\mathbb{N}}_{\nearrow}$ of $\mathcal{K}^{\mathbb{N}}$ is defined as the set of all increasing sequences:

$$\mathcal{K}^{\mathbb{N}}_{\nearrow} = \left\{ (K_n) \in \mathcal{K}^{\mathbb{N}} \;\middle|\; K_1 \subset K_2 \subset \cdots \right\}.$$

Definition 1.2. For a family $\mathcal{F}$ of $\mathcal{F}_\sigma$ sets, we put

$$\mathcal{K}^{\mathbb{N}}_{\mathcal{F}} = \left\{ (K_n) \in \mathcal{K}^{\mathbb{N}} \;\middle|\; \bigcup_{n=1}^{\infty} K_n \in \mathcal{F} \right\}.$$

We say that $\mathcal{F}$ is $\mathcal{K}^{\mathbb{N}}$*-residual* if $\mathcal{K}^{\mathbb{N}}_{\mathcal{F}}$ is residual in $\mathcal{K}^{\mathbb{N}}$ and that $\mathcal{F}$ is $\mathcal{K}^{\mathbb{N}}_{\nearrow}$*-residual* if $\mathcal{K}^{\mathbb{N}}_{\mathcal{F}} \cap K^{\mathbb{N}}_{\nearrow}$ is residual in $\mathcal{K}^{\mathbb{N}}_{\nearrow}$.

Our main theorem asserts that these two notions of residuality agree with each other:

Main Theorem 1. *A family of $\mathcal{F}_\sigma$ sets is $\mathcal{K}^{\mathbb{N}}$-residual if and only if it is $\mathcal{K}^{\mathbb{N}}_{\nearrow}$-residual.*

The equivalence seems to show the appropriateness of our definitions. Moreover our definitions match the properties of a typical compact set mentioned at the beginning. We prove a lemma before we state the precise relation.

Lemma 1.3. *Let Y be a second countable topological space and Z a nonempty Baire space. Then a subset A of Y is residual if and only if $A \times Z$ is residual in $Y \times Z$.*

PROOF. It suffices to show that a subset A of Y is meagre if and only if $A \times Z$ is meagre in $Y \times Z$.

Suppose that A is meagre. Then there exist nowhere dense sets A_1, A_2, $\ldots$ such that $A = \bigcup_{n=1}^{\infty} A_n$. It is easy to see that $A_n \times Z$ is nowhere dense in $Y \times Z$ for every $n \in \mathbb{N}$. Thus $A \times Z = \bigcup_{n=1}^{\infty} (A_n \times Z)$ is meagre.

Conversely suppose that $A \times Z$ is meagre. Then the Kuratowski-Ulam theorem shows that for every z in a residual set in Z, the set $\{ y \in Y \mid (y, z) \in A \times Z \} = A$ is meagre. Therefore A is meagre since Z is a nonempty Baire space. $\square$

Remark 1.4. *We shall use this lemma for $Y = \mathcal{K}$ and $Z = \mathcal{K}^{\mathbb{N}}$ in the next proposition. In this situation, the 'if' part can be replaced by the following lemma, which is Lemma 4.25 of [2] by Phelps:*

> *Let M be a complete metric space, Y a Hausdorff space and $f\colon M \longrightarrow Y$ a continuous open surjective mapping. If G is the intersection of countably many dense open subsets of M, then its image $f(G)$ is residual in Y.*

Indeed it suffices to substitute $\mathcal{K} \times \mathcal{K}^{\mathbb{N}}$ for M, $\mathcal{K}$ for Y, and the first projection for f. In order to prove this lemma, Phelps used the Banach-Mazur game, which we shall look at from the next section onwards.

Proposition 1.5. *Let $\mathcal{I}$ be a σ-ideal on X. Then $\mathcal{I} \cap \mathcal{K}$ is residual in $\mathcal{K}$ if and only if $\mathcal{I} \cap \mathcal{F}_\sigma$ is $\mathcal{K}^{\mathbb{N}}$-residual.*

PROOF. Since

$$\left\{ (K_n) \in \mathcal{K}^{\mathbb{N}} \;\middle|\; \bigcup_{n=1}^{\infty} K_n \in \mathcal{I} \right\} = \left\{ (K_n) \in \mathcal{K}^{\mathbb{N}} \;\middle|\; K_n \in \mathcal{I} \text{ for every } n \in \mathbb{N} \right\}$$
$$= \bigcap_{n=1}^{\infty} \big(\underbrace{\mathcal{K} \times \cdots \times \mathcal{K}}_{n-1 \text{ times}} \times (\mathcal{I} \cap \mathcal{K}) \times \mathcal{K} \times \mathcal{K} \times \cdots\big),$$

we see that $\mathcal{I} \cap \mathcal{F}_\sigma$ is $\mathcal{K}^{\mathbb{N}}$-residual if and only if $(\mathcal{I} \cap \mathcal{K}) \times \mathcal{K} \times \mathcal{K} \times \cdots$ is residual in $\mathcal{K}^{\mathbb{N}}$. Lemma 1.3 shows that this is equivalent to the condition that $\mathcal{I} \cap \mathcal{K}$ is residual in $\mathcal{K}$. $\square$

This proposition shows, for example, that a typical $\mathcal{F}_\sigma$ subset of the interval $[0, 1]$ is null.

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2 Banach-Mazur games

It is known that we can grasp residuality in terms of the Banach-Mazur game.

Definition 2.1. Let Y be a topological space, S a subset of Y, and $\mathcal{A}$ a family of subsets of Y. Suppose that every set in $\mathcal{A}$ has nonempty interior and that every nonempty open subset of Y contains a set in $\mathcal{A}$. The $(Y, S, \mathcal{A})$-*Banach-Mazur game* is described as follows. Two players, called Player I and Player II, alternately choose a set in $\mathcal{A}$ with the restriction that they must choose a subset of the set chosen in the previous turn. Player II will win if the intersection of all the sets chosen by the players is contained in S; otherwise Player I will win.

Remark 2.2. *The assumptions on $\mathcal{A}$ ensure that the players can continue to take sets.*

Fact 2.3. *The $(Y, S, \mathcal{A})$-Banach-Mazur game has a winning strategy for Player II if and only if S is residual in Y.*

For the proof of this fact, we refer the reader to Theorem 1 in [1].

In order to prove our main theorem, we look at the following Banach-Mazur games:

Definition 2.4. Let $\mathcal{F}$ be a family of $\mathcal{F}_\sigma$ sets.

Let $\mathcal{B}$ denote the family of all sets of the form

$$\bar{B}\bigl((K_n), a, r\bigr) = \left\{\, (A_n) \in \mathcal{K}^{\mathbb{N}} \;\middle|\; d(A_n, K_n) \leqq r \text{ for } n = 1, \ldots, a \,\right\},$$

where a is a positive integer, (K_n) is a sequence in $\mathcal{K}^{\mathbb{N}}$ such that $K_1, \ldots, K_a$ are pairwise disjoint finite sets, and r is a positive real number less than 1 such that any two distinct points in $\bigcup_{j=1}^{a} K_j$ have distance at least $3r$. The $(\mathcal{K}^{\mathbb{N}}, \mathcal{K}^{\mathbb{N}}_{\mathcal{F}}, \mathcal{B})$-Banach-Mazur game is called the $(\mathcal{K}^{\mathbb{N}}, \mathcal{F})$-*BM game* for ease of notation.

Let $\mathcal{B}_{\nearrow}$ denote the family of all sets of the form

$$\bar{B}_{\nearrow}\bigl((L_n), b, s\bigr) = \left\{\, (A_n) \in \mathcal{K}^{\mathbb{N}}_{\nearrow} \;\middle|\; d(A_n, L_n) \leqq s \text{ for } n = 1, \ldots, b \,\right\},$$

where b is a positive integer, (L_n) is a sequence in $\mathcal{K}^{\mathbb{N}}_{\nearrow}$ such that $L_1, \ldots, L_b$ are finite, and s is a positive real number less than 1 such that any two distinct points in L_b have distance at least $3s$. The $(\mathcal{K}^{\mathbb{N}}_{\nearrow}, \mathcal{K}^{\mathbb{N}}_{\mathcal{F}} \cap \mathcal{K}^{\mathbb{N}}_{\nearrow}, \mathcal{B}_{\nearrow})$-Banach-Mazur game is called the $(\mathcal{K}^{\mathbb{N}}_{\nearrow}, \mathcal{F})$-*BM game*.

Remark 2.5. *Notice that the families $\mathcal{B}$ and $\mathcal{B}_\nearrow$ satisfy the assumptions in Definition 2.1 since X is dense-in-itself.*

Convention 2.6. *Whenever we write $\bar{B}\big((K_n), a, r\big)$ or $\bar{B}_\nearrow\big((L_n), b, s\big)$, we assume that (K_n), a, r; (L_n), b, s satisfy the conditions in Definition 2.4.*

Remark 2.7. *A trivial observation shows that $\bar{B}\big((K_n), a, r\big) \subset \bar{B}\big((K'_n), a', r'\big)$ implies $a \geqq a'$ and $r \leqq r'$ and that $\bar{B}_\nearrow\big((L_n), b, s\big) \subset \bar{B}_\nearrow\big((L'_n), b', s'\big)$ implies $b \geqq b'$ and $s \leqq s'$.*

Fact 2.3 enables us to translate our main theorem into the following:

Theorem 2.8. *For a family $\mathcal{F}$ of $\mathcal{F}_\sigma$ sets, the $(\mathcal{K}^{\mathbb{N}}, \mathcal{F})$-BM game has a winning strategy for Player II if and only if the $(\mathcal{K}^{\mathbb{N}}_\nearrow, \mathcal{F})$-BM game does.*

3 Proof of our main theorem

In this section we shall prove Theorem 2.8, which, as we have already mentioned, implies our main theorem. Hereafter we fix a family $\mathcal{F}$ of $\mathcal{F}_\sigma$ sets and call the Banach-Mazur games without referring to $\mathcal{F}$.

3.1 Outline of the proof

This subsection is devoted to the outline of the proof that $\mathcal{K}^{\mathbb{N}}_\nearrow$-residuality implies $\mathcal{K}^{\mathbb{N}}$-residuality, or equivalently, that if the $\mathcal{K}^{\mathbb{N}}_\nearrow$-BM game has a winning strategy for Player II then so does the $\mathcal{K}^{\mathbb{N}}$-BM game. Figure 1 illustrates this, and Figure 2 allows us to guess easily the outline of the proof of the other implication.

Suppose that Player I chose $\bar{B}\big((K_n^{(1)}), a^{(1)}, r^{(1)}\big)$ in the first turn. Player II transfers it to a certain set, say $\bar{B}_\nearrow\big((\tilde{K}_n^{(1)}), \tilde{a}^{(1)}, \tilde{r}^{(1)}\big)$, in the $\mathcal{K}^{\mathbb{N}}_\nearrow$-BM game. Then the winning strategy in the $\mathcal{K}^{\mathbb{N}}_\nearrow$-BM game tells Player II to take a set $\bar{B}_\nearrow\big((L_n^{(1)}), b^{(1)}, s^{(1)}\big)$. Player II transfers it to a set $\bar{B}\big(\tilde{L}_n^{(1)}, \tilde{b}^{(1)}, \tilde{s}^{(1)}\big)$, which will be the real reply in the $\mathcal{K}^{\mathbb{N}}$-BM game. In a similar way, after Player I replies $\bar{B}\big((K_n^{(2)}), a^{(2)}, r^{(2)}\big)$, Player II obtains $\bar{B}_\nearrow\big((\tilde{K}_n^{(2)}), \tilde{a}^{(2)}, \tilde{r}^{(2)}\big)$, $\bar{B}_\nearrow\big((L_n^{(2)}), b^{(2)}, s^{(2)}\big)$, and $\bar{B}\big((\tilde{L}_n^{(2)}), \tilde{b}^{(2)}, \tilde{s}^{(2)}\big)$. Player II continues this strategy.

Since $\mathcal{K}^{\mathbb{N}}$ and $\mathcal{K}^{\mathbb{N}}_\nearrow$ are compact, the intersections of the closed sets chosen by the players are nonempty. By modifying the winning strategy for the $\mathcal{K}^{\mathbb{N}}_\nearrow$-BM game, we may assume that $\lim_{m\to\infty} s^{(m)} = 0$, so that the intersection in

$$
\begin{array}{lcccl}
 & \mathcal{K}^{\mathbb{N}}\text{-BM game} & & \mathcal{K}^{\mathbb{N}}_{\nearrow}\text{-BM game} & \\
\text{Player I:} & \bar{B}\big((K_n^{(1)}), a^{(1)}, r^{(1)}\big) & \longrightarrow & \bar{B}_{\nearrow}\big((\tilde{K}_n^{(1)}), \tilde{a}^{(1)}, \tilde{r}^{(1)}\big) & \\
 & & & & \text{strategy} \\
\text{Player II:} & \bar{B}\big((\tilde{L}_n^{(1)}), \tilde{b}^{(1)}, \tilde{s}^{(1)}\big) & \longleftarrow & \bar{B}_{\nearrow}\big((L_n^{(1)}), b^{(1)}, s^{(1)}\big) & \\
\text{Player I:} & \bar{B}\big((K_n^{(2)}), a^{(2)}, r^{(2)}\big) & \longrightarrow & \bar{B}_{\nearrow}\big((\tilde{K}_n^{(2)}), \tilde{a}^{(2)}, \tilde{r}^{(2)}\big) & \\
 & & & & \text{strategy} \\
\text{Player II:} & \bar{B}\big((\tilde{L}_n^{(2)}), \tilde{b}^{(2)}, \tilde{s}^{(2)}\big) & \longleftarrow & \bar{B}_{\nearrow}\big((L_n^{(2)}), b^{(2)}, s^{(2)}\big) & \\
\vdots & \vdots & & \vdots & \\
 & \downarrow & & \downarrow & \\
 & (P_n) & & (Q_n) &
\end{array}
$$

Figure 1: Outline of the proof that $\mathcal{K}^{\mathbb{N}}_{\nearrow}$-residuality implies $\mathcal{K}^{\mathbb{N}}$-residuality

$$
\begin{array}{llccc}
 & & \mathcal{K}^{\mathbb{N}}\text{-BM game} & & \mathcal{K}^{\mathbb{N}}_{\nearrow}\text{-BM game} \\
\text{Player I:} & & \bar{B}\big((\tilde{L}_n^{(1)}), \tilde{b}^{(1)}, \tilde{s}^{(1)}\big) & \longleftarrow & \bar{B}_{\nearrow}\big((L_n^{(1)}), b^{(1)}, s^{(1)}\big) \\
 & \text{strategy} & & & \\
\text{Player II:} & & \bar{B}\big((K_n^{(1)}), a^{(1)}, r^{(1)}\big) & \longrightarrow & \bar{B}_{\nearrow}\big((\tilde{K}_n^{(1)}), \tilde{a}^{(1)}, \tilde{r}^{(1)}\big) \\
\text{Player I:} & & \bar{B}\big((\tilde{L}_n^{(2)}), \tilde{b}^{(2)}, \tilde{s}^{(2)}\big) & \longleftarrow & \bar{B}_{\nearrow}\big((L_n^{(2)}), b^{(2)}, s^{(2)}\big) \\
 & \text{strategy} & & & \\
\text{Player II:} & & \bar{B}\big((K_n^{(2)}), a^{(2)}, r^{(2)}\big) & \longrightarrow & \bar{B}_{\nearrow}\big((\tilde{K}_n^{(2)}), \tilde{a}^{(2)}, \tilde{r}^{(2)}\big) \\
\vdots & & \vdots & & \vdots \\
 & & \downarrow & & \downarrow \\
 & & (P_n) & & (Q_n)
\end{array}
$$

Figure 2: Outline of the proof of that $\mathcal{K}^{\mathbb{N}}$-residuality implies $\mathcal{K}^{\mathbb{N}}_{\nearrow}$-residuality

this game is a singleton. Furthermore, since the transfers are executed so that $\tilde{s}^{(m)} \leqq s^{(m)}$ holds for every $m \in \mathbb{N}$ as will be stated below, the intersection in the $\mathcal{K}^{\mathbb{N}}$-BM game is also a singleton.

We write

$$\bigcap_{m=1}^{\infty} \bar{B}\big((K_n^{(m)}), a^{(m)}, r^{(m)}\big) = \bigcap_{m=1}^{\infty} \bar{B}\big((\tilde{L}_n^{(m)}), \tilde{b}^{(m)}, \tilde{s}^{(m)}\big) = \big\{(P_n)\big\} \quad \text{and}$$
$$\bigcap_{m=1}^{\infty} \bar{B}_{\nearrow}\big((\tilde{K}_n^{(m)}), \tilde{a}^{(m)}, \tilde{r}^{(m)}\big) = \bigcap_{m=1}^{\infty} \bar{B}_{\nearrow}\big((L_n^{(m)}), b^{(m)}, s^{(m)}\big) = \big\{(Q_n)\big\}.$$

Notice that

$$\lim_{m\to\infty}(K_n^{(m)}) = \lim_{m\to\infty}(\tilde{L}_n^{(m)}) = (P_n) \quad \text{and} \quad \lim_{m\to\infty}(\tilde{K}_n^{(m)}) = \lim_{m\to\infty}(L_n^{(m)}) = (Q_n).$$

Since Player II follows the winning strategy in the $\mathcal{K}^{\mathbb{N}}_{\nearrow}$-BM game, we have $(Q_n) \in \mathcal{K}^{\mathbb{N}}_{\mathcal{F}} \cap \mathcal{K}^{\mathbb{N}}_{\nearrow}$, or equivalently $\bigcup_{n=1}^{\infty} Q_n \in \mathcal{F}$. Thus all we have to show is that $(P_n) \in \mathcal{K}^{\mathbb{N}}_{\mathcal{F}}$, and to this aim it suffices to prove that $\bigcup_{n=1}^{\infty} P_n = \bigcup_{n=1}^{\infty} Q_n$.

3.2 Details of the transfers

3.2.1 Conditions and definitions

A *stage* consists of two moves (one in the $\mathcal{K}^{\mathbb{N}}$-BM game and one in the $\mathcal{K}^{\mathbb{N}}_{\nearrow}$-BM game) which lie at the same height in Figures 1 and 2. When we describe the situation at a fixed stage, we omit the integer m indicating the stage unless ambiguity may be caused: for example, we write K_n in place of $K_n^{(m)}$. This is not only for simple notation; we try to offer explanation of the transfers which will go in the proofs of both implications, and this omission solves the problem that when we describe the stage having, say, $K_n^{(m)}$, the previous stage can have $L_n^{(m-1)}$ or $L_n^{(m)}$ depending on which implication we look at.

The transfers are executed so that the following conditions, written as $(*)$ afterwards, are fulfilled:

(1) $\tilde{a} \geqq a$, $\tilde{b} \geqq b$, $\tilde{r} \leqq r/2$, and $\tilde{s} \leqq s/2$;

(2) $\bigcup_{j=1}^{n} K_j \subset \tilde{K}_n$ for $n = 1, \ldots, a$, and $\bigcup_{j=1}^{n} \tilde{L}_j \subset L_n$ for $n = 1, \ldots, b$;

(3) $\bigcup_{n=1}^{a} K_n = \tilde{K}_{\tilde{a}}$ and $\bigcup_{n=1}^{\tilde{b}} \tilde{L}_n = L_b$.

For $x \in \bigcup_{n=1}^{a} K_n = \tilde{K}_{\tilde{a}}$, its *affiliation* (n_1, n_2) is the pair of the integer $n_1 \in \{1, \ldots, a\}$ with $x \in K_{n_1}$, called the *first affiliation* of x, and the least integer $n_2 \in \{1, \dots, \tilde{a}\}$ with $x \in \tilde{K}_{n_2}$, called the *second affiliation* of x. We give a similar definition for the points in $\bigcup_{n=1}^{\tilde{b}} \tilde{L}_n = L_b$: for $x \in \bigcup_{n=1}^{\tilde{b}} \tilde{L}_n = L_b$, its *affiliation* (n_1, n_2) is the pair of the integer $n_1 \in \{1, \dots, \tilde{b}\}$ with $x \in \tilde{L}_{n_1}$, called the *first affiliation* of x, and the least integer $n_2 \in \{1, \dots, b\}$ with $x \in L_{n_2}$, called the *second affiliation* of x. Strictly speaking, we should specify the stage at which the affiliations are defined, because, for instance, it may be that $L_{b^{(m)}}^{(m)} \cap L_{b^{(m')}}^{(m')} \neq \emptyset$ for distinct m and m'. However, since we can easily guess the stage from the context, we choose not to specify it in order to avoid complexity.

Remark 3.1. *Condition (2) in $(*)$ is equivalent to the condition that the first affiliation is always greater than or equal to the second affiliation.*

Let us look at $\bar{B}\bigl((K_n), a, r\bigr) \in \mathcal{B}$ and $\bar{B}_{\nearrow}\bigl((\tilde{K}_n), \tilde{a}, \tilde{r}\bigr) \in \mathcal{B}_{\nearrow}$ at any stage except the first one. We have $\bar{B}\bigl((\tilde{L}_n), \tilde{b}, \tilde{s}\bigr) \in \mathcal{B}$ and $\bar{B}_{\nearrow}\bigl((L_n), b, s\bigr) \in \mathcal{B}_{\nearrow}$ at the previous stage. Since $\bar{B}\bigl((K_n), a, r\bigr) \subset \bar{B}\bigl((\tilde{L}_n), \tilde{b}, \tilde{s}\bigr)$, for each $x \in \bigcup_{n=1}^{\tilde{b}} K_n$ there exists a unique $y \in \bigcup_{n=1}^{\tilde{b}} \tilde{L}_n = L_b$ satisfying $\rho(x, y) \leqq \tilde{s}$, where uniqueness follows from the assumption that any two distinct points in $\bigcup_{n=1}^{\tilde{b}} \tilde{L}_n$ have distance at least $3\tilde{s}$. This y is called the *parent* of x. Observe that if $x \in K_n$ then $y \in \tilde{L}_n$. We give a similar definition also when we look at $\bar{B}_{\nearrow}\bigl((L_n), b, s\bigr) \in \mathcal{B}_{\nearrow}$ and $\bar{B}\bigl((\tilde{L}_n), \tilde{b}, \tilde{s}\bigr) \in \mathcal{B}$: the *parent* of $x \in L_{\tilde{a}}$ is the unique $y \in \bigcup_{n=1}^{a} K_n = \tilde{K}_{\tilde{a}}$ satisfying $\rho(x, y) \leqq \tilde{r}$.

3.2.2 Transfers from the $\mathcal{K}^{\mathbb{N}}$-BM game to the $\mathcal{K}^{\mathbb{N}}_{\nearrow}$-BM game

Given a move $\bar{B}\bigl((K_n), a, r\bigr) \in \mathcal{B}$, we shall construct its transfer $\bar{B}_{\nearrow}\bigl((\tilde{K}_n), \tilde{a}, \tilde{r}\bigr) \in \mathcal{B}_{\nearrow}$. If it is the first move of Player I, then we put $\tilde{a} = a$, $\tilde{r} = r/2$, and $\tilde{K}_n = \bigcup_{j=1}^{n} K_j$ for every $n \in \mathbb{N}$, and we can easily see that the conditions $(*)$ are fulfilled. So suppose otherwise. Then we already know $\bar{B}_{\nearrow}\bigl((L_n), b, s\bigr) \in \mathcal{B}_{\nearrow}$ and its transfer $\bar{B}\bigl((\tilde{L}_n), \tilde{b}, \tilde{s}\bigr) \in \mathcal{B}$, and we have $\bar{B}\bigl((K_n), a, r\bigr) \subset \bar{B}\bigl((\tilde{L}_n), \tilde{b}, \tilde{s}\bigr)$.

Put $\tilde{a} = a$ and $\tilde{r} = \min\{s - \tilde{s}, r/2\}$, and define $\tilde{K}_n = \bigcup_{j=1}^{n} K_j$ for $n > \tilde{b}$. We define $\tilde{K}_n$ for $n \leqq \tilde{b}$ by declaring that the second affiliation of each $x \in \bigcup_{n=1}^{\tilde{b}} K_n$ is the same as that of the parent of x.

Claim 1. *We have $d(\tilde{K}_n, L_n) \leqq \tilde{s}$ for $n = 1, \dots, b$.*

Proof. Fix such an integer n.

Let $x \in \tilde{K}_n$ and denote its affiliation by (n_1, n_2). Then the parent y of x has affiliation (n_1, n_2) and so belongs to L_{n_2}. It follows from $y \in L_{n_2} \subset L_n$ and $\rho(x, y) \leqq \tilde{s}$ that $x \in L_n[\tilde{s}]$.

Conversely let $y \in L_n$ and denote its affiliation by (n_1, n_2). Then there exists a point $x \in K_{n_1}$ with $\rho(x, y) \leqq \tilde{s}$ because $d(K_{n_1}, \tilde{L}_{n_1}) \leqq \tilde{s}$. Since y is the parent of x, the affiliation of x is (n_1, n_2). Therefore $x \in \tilde{K}_{n_2} \subset \tilde{K}_n$ and so $y \in \tilde{K}_n[\tilde{s}]$. $\square$

We may deduce from this claim that $\bar{B}_{\nearrow}\big((\tilde{K}_n), \tilde{a}, \tilde{r}\big) \subset \bar{B}_{\nearrow}\big((L_n), b, s\big)$ using the triangle inequality and $\tilde{r} + \tilde{s} \leqq s$. Therefore $\bar{B}_{\nearrow}\big((\tilde{K}_n), \tilde{a}, \tilde{r}\big)$ is a valid reply in the $\mathcal{K}^{\mathbb{N}}_{\nearrow}$-BM game. It is easy to see that the conditions $(*)$ are fulfilled.

3.2.3 Transfers from the $\mathcal{K}^{\mathbb{N}}_{\nearrow}$-BM game to the $\mathcal{K}^{\mathbb{N}}$-BM game

Given a move $\bar{B}_{\nearrow}\big((L_n), b, s\big) \in \mathcal{B}_{\nearrow}$, we shall construct its transfer $\bar{B}\big((\tilde{L}_n), \tilde{b}, \tilde{s}\big) \in \mathcal{B}$. If it is the first move of Player I, then we put $\tilde{b} = b$, $\tilde{s} = s/2$, $\tilde{L}_1 = L_1$, and $\tilde{L}_n = L_n \setminus L_{n-1}$ for every $n \geqq 2$. We can easily see that the conditions $(*)$ are fulfilled in this case. So suppose otherwise. Then we already know $\bar{B}\big((K_n), a, r\big) \in \mathcal{B}$ and its transfer $\bar{B}_{\nearrow}\big((\tilde{K}_n), \tilde{a}, \tilde{r}\big) \in \mathcal{B}_{\nearrow}$, and we have $\bar{B}_{\nearrow}\big((L_n), b, s\big) \subset \bar{B}_{\nearrow}\big((\tilde{K}_n), \tilde{a}, \tilde{r}\big)$.

Put $\tilde{b} = b + 1$ and $\tilde{s} = \min\{r - \tilde{r}, s/2\}$, and define $\tilde{L}_n = L_{n-1}$ for $n > \tilde{b}$. We define $\tilde{L}_n$ for $n \leqq \tilde{b}$ by determining the first affiliation of each point in L_b as follows. Let $x \in L_b$ and denote its second affiliation by n_2. If $n_2 > \tilde{a}$, then the first affiliation of x is n_2. Suppose $n_2 \leqq \tilde{a}$, and let $y \in \tilde{K}_{n_2}$ denote the parent of x. If the second affiliation of y is n_2, then the first affiliation of x is the same as that of y; otherwise the first affiliation of x is $\tilde{b}$.

Claim 2. *We have $d(\tilde{L}_n, K_n) \leqq \tilde{r}$ for $n = 1, \dots, a$.*

Proof. Fix such an integer n.

Let $x \in \tilde{L}_n$ and denote its parent by y. Then it follows that x and y have the same affiliation, and so $y \in K_n$. Hence we may infer from $\rho(x, y) \leqq \tilde{r}$ that $x \in K_n[\tilde{r}]$.

Conversely let $y \in K_n$ and denote its second affiliation by n_2. Then there exists a point $x \in L_{n_2}$ with $\rho(x, y) \leqq \tilde{r}$ because $d(\tilde{K}_{n_2}, L_{n_2}) \leqq \tilde{r}$. Since y is the parent of x and has the same second affiliation as x, the first affiliation of x is n. Therefore $y \in \tilde{L}_n[\tilde{r}]$. $\square$

We may deduce from the claim that $\bar{B}\big((\tilde{L}_n),\tilde{b},\tilde{s}\big) \subset \bar{B}\big((K_n),a,r\big)$ using the triangle inequality and $\tilde{r}+\tilde{s} \leqq r$. Therefore $\bar{B}\big((\tilde{L}_n),\tilde{b},\tilde{s}\big)$ is a valid reply in the $\mathcal{K}^{\mathbb{N}}$-BM game. It is easy to see that the conditions $(*)$ are fulfilled.

3.3 Proof of $\bigcup_{n=1}^{\infty} P_n = \bigcup_{n=1}^{\infty} Q_n$

We shall prove that $\bigcup_{n=1}^{\infty} P_n = \bigcup_{n=1}^{\infty} Q_n$, which will complete the proof of Theorem 2.8 and hence of our main theorem. Recall that $(K_n^{(m)})$ and $(\tilde{K}_n^{(m)})$ converge to (P_n) and (Q_n) respectively as m tends to infinity. In other words we have $\lim_{m\to\infty} K_n^{(m)} = P_n$ and $\lim_{m\to\infty} \tilde{K}_n^{(m)} = Q_n$ for every $n \in \mathbb{N}$.

In order to prove $\bigcup_{n=1}^{\infty} P_n \subset \bigcup_{n=1}^{\infty} Q_n$, it is enough to show that $\bigcup_{j=1}^{n} P_j \subset Q_n$ for every $n \in \mathbb{N}$. The set $\{\,(A,B) \in \mathcal{K}^2 \mid A \subset B\,\}$ is closed in $\mathcal{K}^2$ and contains $(\bigcup_{j=1}^{n} K_j^{(m)}, \tilde{K}_n^{(m)})$ for all $m \in \mathbb{N}$. Since $(\bigcup_{j=1}^{n} K_j^{(m)}, \tilde{K}_n^{(m)})$ converges to $(\bigcup_{j=1}^{n} P_j, Q_n)$ as m tends to infinity, which follows from the continuity of the map $(A_1,\dots,A_n) \longmapsto \bigcup_{j=1}^{n} A_j$ from $\mathcal{K}^n$ to $\mathcal{K}$, we obtain $\bigcup_{j=1}^{n} P_j \subset Q_n$.

Now we shall prove $\bigcup_{n=1}^{\infty} Q_n \subset \bigcup_{n=1}^{\infty} P_n$. Let $x \in \bigcup_{n=1}^{\infty} Q_n$, and denote by n the least positive integer with $x \in Q_n$. Since it is easy to observe that $K_1^{(m)} = \tilde{K}_1^{(m)}$ for every $m \in \mathbb{N}$, which implies $P_1 = Q_1$, we may assume that $n \geqq 2$. Because Q_{n-1} is closed and $x \notin Q_{n-1}$, there exists a positive real number r less than 1 satisfying $\bar{B}(x,4r) \cap Q_{n-1} = \emptyset$, that is, $x \notin Q_{n-1}[4r]$. Fix a positive integer m_0 such that $\tilde{a}^{(m)} \geqq n$, $\tilde{r}^{(m)} \leqq r$, and $d(\tilde{K}_{n-1}^{(m)}, Q_{n-1}) \leqq r$ for every $m \geqq m_0$. Observe that $x \notin \tilde{K}_{n-1}^{(m)}[3r]$ for every $m \geqq m_0$.

Set $k_0 = \lceil 1/r \rceil$. For each $k \geqq k_0$, choose $m_k \geqq m_0$ satisfying $d(\tilde{K}_n^{(m)}, Q_n) \leqq 1/k$ for every $m \geqq m_k$, and for each $m \geqq m_k$ take $y_{km} \in \tilde{K}_n^{(m)}$ with $\rho(x, y_{km}) \leqq 1/k$ and let $z_{km} \in \tilde{K}_n^{(m_0)}$ denote the unique point satisfying $\rho(y_{km}, z_{km}) \leqq \tilde{r}^{(m_0)}$.

Claim 3. *The two points y_{km} and z_{km} have the same affiliation.*

PROOF. By an *ancestor* of y_{km} we mean a point that can be written as 'the parent of … the parent of y_{km}.' Observe that z_{km} is an ancestor of y_{km}. Indeed if we denote by z'_{km} the ancestor of y_{km} in $\tilde{K}_n^{(m_0)}$, then

$$\rho(y_{km}, z'_{km}) < \tilde{r}^{(m_0)} + \frac{\tilde{r}^{(m_0)}}{2} + \frac{\tilde{r}^{(m_0)}}{2^2} + \cdots = 2\tilde{r}^{(m_0)}$$

and so $\rho(z_{km}, z'_{km}) < 3\tilde{r}^{(m_0)}$, which implies $z_{km} = z'_{km}$.

In order to prove our claim, it suffices to prove that the second affiliation of the ancestor $w \in \tilde{K}_n^{(m')}$ of y_{km} is n for any $m' \in \{m_0,\dots,m\}$. We can see

$\rho(w, y_{km}) \leqq 2\tilde{r}^{(m')} \leqq 2r$ by the same reasoning as above. Therefore we have

$$\rho(w, x) \leqq \rho(w, y_{km}) + \rho(y_{km}, x) \leqq 2r + \frac{1}{k} \leqq 3r.$$

Thus the second affiliation of w cannot be less than n because $x \notin \tilde{K}_{n-1}^{(m')}[3r]$. $\square$

Note that all z_{km} belong to the single finite set $\tilde{K}_n^{(m_0)}$. We can choose $z_k \in K_n^{(m_0)}$ for $k \geqq k_0$ inductively so that the set

$$\{\, m \geqq m_k \mid z_{k_0 m} = z_{k_0}, \ldots, z_{km} = z_k \,\}$$

is infinite for any $k \geqq k_0$. Then we take $z \in K_n^{(m_0)}$ for which $\{\, k \geqq k_0 \mid z_k = z \,\}$ is infinite, and put $\{\, k \geqq k_0 \mid z_k = z \,\} = \{k_1, k_2, \ldots\}$, where $k_1 < k_2 < \cdots$. Since the set

$$\{\, m \geqq m_{k_j} \mid z_{k_1 m} = \cdots = z_{k_j m} = z \,\}$$

is infinite for every $j \in \mathbb{N}$, we may construct a strictly increasing sequence m'_1, $m'_2, \ldots$ of positive integers satisfying $z_{k_1 m'_j} = \cdots = z_{k_j m'_j} = z$.

Let l denote the first affiliation of z. Then the foregoing claim shows that whenever $i \leqq j$, the first affiliation of $y_{k_i m'_j}$ is l, which implies that $x \in K_l^{(m'_j)}[1/k_i]$. For any $i \in \mathbb{N}$, since $d(K_l^{(m'_j)}, P_l) \leqq 1/k_i$ for sufficiently large j, we have $x \in P_l[2/k_i]$. Hence $x \in \bigcap_{i=1}^\infty P_l[2/k_i] = P_l$. This completes the proof.

References

[1] John C. Oxtoby, *The Banach-Mazur Game and Banach Category Theorem*, Contributions to the Theory of Games, vol. 3, Annals of Mathematics Studies **39** (1957), 159–163, Princeton University Press.

[2] Robert R. Phelps, *Convex Functions, Monotone Operators and Differentiability*, second edition, Lecture Notes in Mathematics **1364** (1993), Springer-Verlag.

[3] Tudor Zamfirescu, *How Many Sets Are Porous*?, Proceedings of the American Mathematical Society **100** (1987), vol. 2, 383–387.