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ON A PROPERTY OF FUNCTIONS

Abstract

In this article, I propose a new property (a) of functions $f : X \to Y$, where X and Y are metric spaces. A function $f : X \to Y$ has the property (a) if for each real $\eta > 0$, the union $\bigcup_{x \in X}(K(x,\eta) \times K(f(x),\eta))$ contains the graph of a continuous function $g : X \to Y$ and $K(x,r)$ denotes the open ball $\{t \in X : \rho_X(t,x) < r\}$ with center x and radius $r > 0$. The class of functions with the property (a) contains all functions almost continuous in the sense of Stallings and all functions graph continuous. Moreover, I examine the sums, the products, and the uniform and discrete limits of sequences of functions from this class.

Let (X, ρ_X) and (Y, ρ_Y) be metric spaces. The symbol $K(x,r)$ denotes the open ball $\{t \in X : \rho_X(t,x) < r\}$ with center x and radius $r > 0$. For a function $f : X \to Y$ and a positive real η, let

$$A_\eta(f) = \bigcup_{x \in X}(K(x,\eta) \times K(f(x),\eta)).$$

We say that a function $f : X \to Y$ has the property (a) if for each positive real η, there is a continuous function $g : X \to Y$ such that the graph $Gr(g)$ of g is contained in $A_\eta(f)$.

In [7], Stallings introduces the notion of almost continuous functions. Recall that a function $f : X \to Y$ is almost continuous (in the sense of Stallings) if for each open set $U \subset X \times Y$ containing $Gr(f)$, there is a continuous function $g : X \to Y$ with $Gr(g) \subset U$.

Since each set $A_\eta(f)$ is open in $X \times Y$ and contains $Gr(f)$, we obtain that each almost continuous function $f : X \to Y$ has the property (a).

In [2], the notion of an A-continuous function is introduced. Later in [5, 6], K. Sakálová calls A-continuous functions graph continuous. Recall that

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a function $f : X \to Y$ is said to be graph continuous if the closure $cl(Gr(f))$ of the graph of f contains the graph $Gr(g)$ of a continuous function $g : X \to Y$.

Theorem 1. *Each graph continuous function $f : X \to Y$ has the property (a).*

Proof. Observe that for each function $f : X \to Y$, the equality

$$cl(Gr(f)) = \bigcap_{\eta>0} A_\eta(f) = \bigcap_{n\geq 1} A_{\frac{1}{n}}(f)$$

holds. Of course, if $(x, y) \in cl(Gr(f))$ and $\eta > 0$ is a real, then there is a point $(u, f(u)) \in Gr(f)$ such that $u \in K(x, \eta)$ and $f(u) \in K(y, \eta)$. Consequently, $(x, y) \in K(u, \eta) \times K(f(u), \eta)$. So, for each $\eta > 0$, we have $cl(Gr(f)) \subset A_\eta(f)$, and consequently,

$$cl(Gr(f)) \subset \bigcap_{\eta>0} A_\eta(f).$$

Now we prove the inclusion $\bigcap_{\eta>0} A_\eta(f) \subset cl(Gr(f))$. For this, fix a point $(x, y) \in \bigcap_{\eta>0} A_\eta(f)$ and a positive real ε. Since $(x, y) \in A_\varepsilon(f)$, there is a point $u \in X$ such that $x \in K(u, \varepsilon)$ and $y \in K(f(u), \varepsilon)$. But ε may be an arbitrary positive real, so $(x, y) \in cl(Gr(f))$, and consequently, $\bigcap_{\eta>0} A_\eta(f) \subset cl(Gr(f))$.

Since for $\eta_1 > \eta_2 > 0$ the inclusion $A_{\eta_1}(f) \supset A_{\eta_2}(f)$ is true, the equality

$$\bigcap_{\eta>0} A_\eta(f) = \bigcap_{n\geq 1} A_{\frac{1}{n}}(f)$$

is evident.

If $f : X \to Y$ is a graph continuous function, then there is a continuous function $g : X \to Y$ with

$$Gr(g) \subset cl(Gr(f)) \subset A_\eta(f) \text{ for each } \eta > 0,$$

so f has the property (a). This completes the proof. $\square$

Remark 1. *Let $f : X \to Y$ be a function. If there is an element $y \in Y$ such that the level set $f^{-1}(y)$ is dense in X, then f is graph continuous, and consequently has the property (a).*

Remark 2. *Let $f : X \to Y$ be a function. If there is a continuous function $g : X \to Y$ such that the set $\{x \in X : f(x) = g(x)\}$ is dense in X, then f is graph continuous, and consequently has the property (a).*

Remark 3. *Let $\mathbb{R}$ be the set of all reals. There are functions $f : [-1, 1] \to \mathbb{R}$ with the property (a) and the closed graph $Gr(f)$ which are neither almost continuous nor graph continuous.*

PROOF. Let

$$f(0) = 0 \text{ and } f(x) = \frac{1}{|x|} \text{ for } x \in [-1, 0) \cup (0, 1].$$

Fix a real $\eta > 0$ and observe that the interval

$$\left[-\frac{\eta}{3}, \frac{\eta}{3}\right] \times \left\{\frac{3}{\eta}\right\} \subset A_\eta(f).$$

Let

$$g(x) = \frac{3}{\eta} \text{ for } x \in \left[-\frac{\eta}{3}, \frac{\eta}{3}\right]$$

and

$$g(x) = f(x) \text{ otherwise on } [-1, 1].$$

Then the function g is continuous and $Gr(g) \subset A_\eta(f)$. So, f has the property (a). Moreover, $Gr(f)$ is a closed subset of $[-1, 1] \times \mathbb{R}$, but there is not a continuous function $h : [-1, 1] \to \mathbb{R}$ with $Gr(h) \subset Gr(f) = cl(Gr(f))$. So f is not graph continuous.

Since f does not have the Darboux property and since each almost continuous function $\phi : [-1, 1] \to \mathbb{R}$ has the Darboux property ([7, 4]), we obtain that f is not almost continuous and the proof is completed. □

Remark 4. *There is a function $f : \mathbb{R} \to \mathbb{R}$ with closed graph which does not have the property (a).*

PROOF. For example, such is the function

$$f(0) = 0 \text{ and } f(x) = \frac{1}{x} \text{ for } x \neq 0.$$

□

Remark 5. *There are monotone bounded and simultaneously on the right continuous functions $f : [0,1] \to [0,1]$ which do not have the property (a).*

PROOF. Let (w_n) be an enumeration of all rationals from $[0,1]$ such that $w_n \neq w_m$ for $n \neq m$ and $w_1 = \frac{1}{2}$, and let

$$f(x) = \sum_{w_n \leq x} \frac{1}{2^n} \text{ for } x \in [0,1].$$

Then f is increasing and continuous on the right hand at each point $x \in [0,1)$, but it does not have the property (a), because for $\eta \in (0, \frac{1}{10})$ and for each continuous function $h : [0,1] \to \mathbb{R}$, the difference $Gr(h) \setminus A_\eta(f)$ is nonempty. $\square$

Theorem 2. *Let (X, ρ_X) and (Y, ρ_Y) be metric spaces. If functions $f_n : X \to Y$ have the property (a) for $n \geq 1$ and if the sequence (f_n) uniformly converges to a function $f : X \to Y$, then f has also the property (a).*

PROOF. Fix a real $\eta > 0$. There is an index k with

$$\rho_Y(f_k(x), f(x)) < \frac{\eta}{3} \text{ for all } x \in X.$$

Since f_k has the property (a), there is a continuous function $h : X \to Y$ such that $Gr(h) \subset A_{\frac{\eta}{3}}(f_k)$. Observe that

$$(*) \qquad A_{\frac{\eta}{3}}(f_k) \subset A_\eta(f).$$

Of course, if $(x,y) \in A_{\frac{\eta}{3}}(f_k)$, then there is a point $u \in X$ with

$$\rho_X(u,x) < \frac{\eta}{3} \text{ and } \rho_Y(f_k(u), y) < \frac{\eta}{3}.$$

Since $\rho_Y(f_k, f) < \frac{\eta}{3}$, we obtain

$$\rho_Y(f(u), y) \leq \rho_Y(f(u), f_k(u)) + \rho_Y(f_k(u), y) < \frac{\eta}{3} + \frac{\eta}{3} < \eta.$$

So $(x,y) \in A_\eta(f)$ and the inclusion $(*)$ holds. Consequently, $Gr(h) \subset A_{\frac{\eta}{3}}(f_k) \subset A_\eta(f)$, and the proof is completed. $\square$

It is well known ([4]) that each function $f : \mathbb{R} \to \mathbb{R}$ is the pointwise limit of a sequence of almost continuous functions and the sum of two almost continuous functions. In this article, I prove the following theorems.

Theorem 3. *Let (X, ρ_X) be a metric space dense in itself, and let $(Y, \rho_Y, +)$ be a metric group. Then for each function $f : X \to Y$, there are two graph continuous functions $f_1, f_2 : X \to Y$ (so having the property (a)) such that $f = f_1 + f_2$.*

PROOF. There are two disjoint sets $A, B \subset X$ dense in X. Let

$$f_1(x) = 0 \text{ and } f_2(x) = f(x) \text{ for } x \in A,$$

$$f_2(x) = 0 \text{ and } f_1(x) = f(x) \text{ for } x \in B,$$

and

$$f_1(x) = f(x) \text{ and } f_2(x) = 0 \text{ for } x \in X \setminus (A \cup B).$$

Since the level sets $(f_1)^{-1}(0) \supset A$ and $(f_2)^{-1}(0) \supset B$ are dense in X, functions f_1 and f_2 are graph continuous. Evidently, $f = f_1 + f_2$, and the proof is completed. □

Theorem 4. *Let (X, ρ_X) be a metric space dense in itself, and let Y_1, Y_2, Z be normed spaces. Moreover, let $\Phi : (Y_1 \times Y_2) \to Z$ be a bilinear continuous function for which there are elements $a \in Y_1$ and $b \in Y_2$ such that for all elements $z \in Z$, there are elements $a'(z) \in Y_2$ with $\Phi(a, a'(z)) = z$ and $b'(z) \in Y_1$ with $\Phi(b'(z), b) = z$. Then for each function $f : X \to Z$, there are two graph continuous functions $f_1 : X \to Y_1$ and $f_2 : X \to Y_2$ such that $f = \Phi(f_1, f_2)$.*

PROOF. There are two disjoint sets $A, B \subset X$ dense in X. Let

$$f_1(x) = a \text{ and } f_2(x) = a'(f(x)) \text{ for } x \in X \setminus B$$

and

$$f_2(x) = b \text{ and } f_1(x) = b'(f(x)) \text{ for } x \in B.$$

Since the level sets $(f_1)^{-1}(a) \supset A$ and $(f_2)^{-1}(b) \supset B$ are dense in X, functions f_1 and f_2 are graph continuous. Evidently, $f = \Phi(f_1, f_2)$, and the proof is completed. □

Theorem 5. *Let (X, ρ) be a metric space, and let Y be a normed space. If $f : X \to Y$ is a continuous function and if $g : X \to Y$ has the property (a), then $f + g$ has the property (a).*

PROOF. For a function $\phi : X \to Y$ and a set $K \subset (X \times Y)$, we denote by $\phi * K$ the set $\{(x, y + \phi(x)) : (x, y) \in K\}$. Fix a real $\eta > 0$. Observe that

$$A_\eta(f+g) = \bigcup_{x \in X} (K(x,\eta) \times K(f(x)+g(x),\eta)) =$$

$$= f * \bigcup_{x \in X} (K(x,\eta) \times K(g(x),\eta)) = f * A_\eta(g).$$

Of course, if a point $(u,v) \in A_\eta(f+g)$, then there is a point $x \in X$ with $(u,v) \in K(x,\eta) \times K(f(x)+g(x),\eta)$. So, $u \in K(x,\eta)$ and $v \in K(f(x)+g(x),\eta)$, and $||v-(f(x)+g(x))|| = ||(v-f(x))-g(x)|| < \eta$. Thus, $v - f(x) \in K(g(x),\eta)$ and $v \in f(x) + K(g(x),\eta) \subset f * A_\eta(g)$. Similarly, we can prove the inverse inclusion $f * A_\eta(g) \subset A_\eta(f+g)$.

Since g has the property (a), there is a continuous function $h : X \to Y$ such that $Gr(h) \subset A_\eta(g)$. The function $f + h$ is also continuous and $Gr(f+h) \subset f * A_\eta(g) = A_\eta(f+g)$. So, $f + g$ has the property (a), and the proof is completed. □

Theorem 6. *Let (X,ρ) be a metric space, and let Y be a normed space. Assume that for a function $f_1 : X \to Y$, there is a continuous function $f : X \to Y$ such that the set $\{x \in X : f(x) = f_1(x)\}$ is dense in X. If a function $g : X \to Y$ has the property (a), then $f_1 + g$ also has the property (a).*

PROOF. Fix a real $\eta > 0$. By the proof of last theorem, we have the equality

$$A_\eta(f_1+g) = \bigcup_{x \in X} (K(x,\eta) \times K(f_1(x)+g(x),\eta)) =$$

$$= f_1 * \bigcup_{x \in X} (K(x,\eta) \times K(g(x),\eta)) = f_1 * A_\eta(g).$$

Since g has the property (a), there is a continuous function $h : X \to Y$ such that $Gr(h) \subset A(g,\eta)$. The function $f + h$ is also continuous and $Gr(f+h) \subset f_1 * A_\eta(g) = A_\eta(f_1+g)$. So, $f_1 + g$ has the property (a), and the proof is completed. □

However, the product of a continuous function and a function having the property (a) need not have the property (a).

Example. Let $X = [-1,1]$, $Y = \mathbb{R}$, and let $f(x) = x$ and

$$g(x) = \frac{1}{|x|} \text{ for } x \in X \setminus \{0\} \text{ and } g(0) = 0.$$

Then f is continuous, g has the property (a), and the product

$$f(x)g(x)=1 \text{ for } x\in(0,1],\ f(0)g(0)=0, \text{ and } f(x)g(x)=-1 \text{ for } x\in[-1,0)$$

does not have the property (a).

Remark 6. *Let (X,ρ) be a metric space, and let Y_1, Y_2, Z be normed spaces. If $\Phi:(Y_1\times Y_2)\to Z$ is a bilinear continuous function, and if a function $f_1:X\to Y_1$ is such that the set $(f_1)^{-1}(0)$ is dense in X, then for each function $f_2:X\to Y_2$, the superposition $g(x)=\Phi(f_1(x),f_2(x))$ for $x\in X$ has the property (a).*

In the article [1], the authors introduced the notion of the discrete convergence of sequences of functions and investigated the discrete limits in different families, for example, in the family $\mathcal{C}$ of all continuous functions.

Let (X,ρ_X) and (Y,ρ_Y) be metric spaces. We say that a sequence of functions $f_n:X\to Y$, $n=1,2,\ldots$, discretely converges to the limit f ($f=\mathrm{d}-\lim\limits_{n\to\infty} f_n$) if

$$\mathop{\forall}_{x}\ \mathop{\exists}_{n(x)}\ \mathop{\forall}_{n>n(x)} f_n(x)=f(x).$$

Theorem 7. *Let (X,ρ_X) be a separable metric space dense in itself, and let (Y,ρ_Y) be a metric space. Then for each function $f:X\to Y$, there is a sequence of functions $f_n:X\to Y$ having the property (a) which discretely converges to f.*

Proof. Since (X,ρ_X) is dense in itself and separable, there is an infinite countable set $A=\{a_i : i\geq 1\}$ dense in X. Fix an element $b\in Y$. For $n=1,2,\ldots,$ put

$$f_n(a_i)=b \text{ for } i\geq n \text{ and } f_n(x)=f(x) \text{ otherwise on } X.$$

Evidently, the sequence (f_n) discretely converges to f. Since the level sets $(f_n)^{-1}(b)\supset\{a_i : i\geq n\}$ are dense in X for $n\geq 1$, the functions f_n have the property (a), and the proof is completed. □

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