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CLUSTER SETS AND APPROXIMATION PROPERTIES OF QUASI-CONTINUOUS AND CLIQUISH FUNCTIONS

Abstract

The crucial concept for studying quasi-continuous and cliquish functions on arbitrary topological spaces X is the concept of a semi-open subset of X. On the one hand, it gives rise to the cluster set $SO\text{-}C(f;x)$ of a function $f : X \to \mathbb{R}$ at a point $x \in X$, which turns out to be an appropriate tool for investigating both local and global properties of f. On the other hand, the concept of a semi-open set is used for introducing so-called semi-open partitions of X. A central result of the paper says that every quasi-continuous function can be represented as a uniform limit of step functions defined on a chain of semi-open partitions of X. Similarly, every cliquish function is proved to be the uniform limit of step functions defined on a chain of so-called almost semi-open partitions of X.

1 Survey and Historical Remarks

The notion of a *quasi-continuous function* was introduced by Kempisty in 1932 (cf. [16]). An independent definition was given by Bledsoe in 1952, who created the notion of a *neighborly function* (cf. [4]). Thielman in his 1953 paper [29] expanded the class of neighborly functions into the class of so-called

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cliquish functions on arbitrary topological spaces. Finally, Marcus in 1961 gave a summary of the various definitions and developed further properties of quasi-continuous and cliquish functions, illustrating them by functions of real variables (cf. [19]).

A real-valued function f on a topological space X is called *quasi-continuous at the point* $x_0 \in X$ if, for every $\varepsilon > 0$ and for every neighborhood $U \in \mathcal{U}(x_0)$ of x_0, there exists a non-empty open set $G \subseteq U$ such that $|f(x) - f(x_0)| < \varepsilon$ for all $x \in G$. The function f is called *cliquish at the point* $x_0 \in X$ if under the same conditions as above $|f(x) - f(x')| < \varepsilon$ for all $x, x' \in G$. The function f is called *quasi-continuous* or *cliquish on* X if it is quasi-continuous or cliquish, respectively, at each point of X.

A new view on functions quasi-continuous in the large was opened in 1963 when Levine introduced the notion of a semi-open set (cf. [17]). According to his definition a subset S of a topological space X is *semi-open* if $S \subseteq \mathrm{cl}(\mathrm{int}\, S)$; that is, if S is contained in the closure of its interior. A *semi-continuous function* f in [17] is defined to be a function such that the inverse image $f^{-1}(V)$ of every open subset $V \subseteq \mathbb{R}$ is semi-open in X. Levine's local characterization says that a function f is semi-continuous at a point $x_0 \in X$ if, for every open subset $V \subseteq \mathbb{R}$ with $f(x_0) \in V$, there exists a semi-open set $S \subseteq X$ with $x_0 \in S$ and $f(x) \in V$ for all $x \in S$. In the local situation, however, an open set $S = G$ with $x_0 \in \mathrm{cl}(G)$ would do.

Independently, Njåstad in 1965 introduced so-called "β-sets" (cf. [22]). His definition coincides with Levine's definition of semi-open sets. Moreover, Njåstad in his 1965 paper points out that quasi-continuous mappings in the sense of Kempisty can be characterized by the property that the inverse image $f^{-1}(V)$ of every open set V is a β-set; that is, a semi-open set. Without referring to semi-open sets or β-sets explicitly, Bruteanu 1970 (cf. [7]) ends up with the same characterization of quasi-continuity (cf. also [20]) .

Though Levine's notation "semi-continuous function" fits in with the notation "semi-open subset", we shall stick to the original notation of a "quasi-continuous function" due to Kempisty, that has been preserved in the 70th and 80th (cf. e.g. [7], [18], [20], [21]). Only in the 90th another notation came up, when Shi, Zheng, and Zhuang rediscovered quasi-continuous functions and semi-open sets calling them "robust functions" and "robust sets", respectively (cf. [27]). They were motivated by numerical problems arising in global optimization in the case of discontinuous functions. The title "Discontinuous robust mappings are approximatable" of their paper is to announce the following local property of quasi-continuous functions f on a Baire space X. For every point $x_0 \in X$, there exists a directed system $\{x_\alpha\} \subseteq X$ of continuity points x_α of f such that $\lim_\alpha x_\alpha = x_0$ and $\lim_\alpha f(x_\alpha) = f(x_0)$.

Let us mention that in the case of a real-valued function f of one or several real variables the idea of calculating the value of f in a discontinuity point by an appropriate sequence of continuity points has already been developed by Schoenflies in 1900 (cf. [26], pp. 125–144). In this very article Schoenflies, on the basis of results of Brodén (cf. [6]), studies the discontinuities of pointwise discontinuous functions (functions with a dense set of continuity points).

The main result of the present paper is a global one. It says that a quasi-continuous function f on an arbitrary topological space X can be represented as a uniform limit of a sequence of piecewise constant functions φ_n on a chain $K = (\mathcal{P}_n)_{n=1}^{\infty}$ of *semi-open partitions* $\mathcal{P}_n$ of X (partitions into semi-open sets). If f is locally bounded or bounded on X, the partitions $\mathcal{P}_n$ of the chain K can be chosen to be locally finite or finite, respectively (see Section 3).

According to this result quasi-continuous functions on a topological space X can be arranged with respect to their discontinuities. Indeed, the chain $K = (\mathcal{P}_n)_{n=1}^{\infty}$ of semi-open partitions $\mathcal{P}_n$ is the common characteristic for a linear space $A_K(X)$ of quasi-continuous functions which can be approximated by step functions on the partitions of K, the discontinuity points of all the functions $f \in A_K(X)$ being boundary points of partition elements of K. Since the partitions $\mathcal{P}_n$ of the chain K can be chosen to be finite if f is bounded, under these circumstances a space $A_K(X)$ consisting of bounded quasi-continuous functions is obtained which, moreover, turns out to be a Banach space with respect to the supremum norm. In the paper [27] a separate section is devoted to "Banach spaces of bounded robust mappings". The construction developed there, in contrast with ours, refers to a common set S of points of continuity for a subclass of quasi-continuous functions, which has to be dense in X. The space X itself is supposed to be a complete metric space in [27].

Finally, the results of Section 3 concerning quasi-continuous functions can be employed to derive similar results for cliquish functions. In Section 5 we show that a cliquish function on an arbitrary topological space can be represented as a uniform limit of a sequence of piecewise constant functions φ_n on a chain $K = (\mathcal{P}_n)_{n=1}^{\infty}$ of so-called *almost semi-open partitions* $\mathcal{P}_n$ of X. The crucial point of the proof is the transformation of cliquish functions into quasi-continuous functions, which can be carried out under rather general conditions by changing the values in the discontinuity points (see Section 4).

Thielman in 1953 stated that the points of discontinuity of a cliquish function on an arbitrary topological space form a set of the first category (cf. [29]). For pointwise discontinuous functions of several real variables this property had already been proved by Schoenflies in the survey article quoted above (cf. pp. 128–129). Hahn in his book from 1921 (cf. [14], p. 204) admitted pointwise discontinuous functions on arbitrary metric spaces. For cliquish functions on metric spaces a proof was given by Marcus in 1961 (cf. [19]). Levine 1963 was working in arbitrary topological spaces, however, he considered quasi-continuous functions only, called semi-continuous in his terminology (cf. [17]). A detailed proof of Thielman's general claim was worked out by Neubrunnová in 1974 (cf. [21]). We shall refer to the set of discontinuity points of a cliquish function as a set of the first category in Section 4 when studying cliquish functions on Baire spaces. According to the definition of a Baire space the complement of any subset of the first category is dense in the space. This implies that an arbitrary cliquish function on a Baire space X is pointwise discontinuous.

The global connection between cliquish functions and quasi-continuous functions represented in Section 4 rests on the local characterizations given in Section 2. The so-called *SO-cluster set* $SO\text{-}C(f;x_0)$ of a real-valued function f in a point $x_0 \in X$ introduced in Section 2 strengthens the notion of the classical cluster set $C(f;x_0)$. While the cluster set $C(f;x_0)$ can be used to study upper and lower semi-continuity of the function f in the point x_0 (cf. [1], pp. 150–151), the *SO*-cluster set $SO\text{-}C(f;x_0)$ turns out to be an appropriate tool for characterizing quasi-continuity and cliquishness of f in x_0 (see Section 2).

To end this introductory section, let us remark that the emphasis of the present paper is on the approximation of quasi-continuous and cliquish functions by semi-open and almost semi-open step functions, respectively. For this reason we have confined ourselves to quasi-continuous and cliquish mappings with values in $\mathbb{R}$.

2 The *SO*-Cluster Set

Hahn in his book from 1921 associated a so-called "cluster function" to an arbitrary real-valued function on a metric space (cf. [14], p. 185). However, only in his 1932 book he gave the first explicit definition of a cluster set (cf. [15], p. 188). Of course, cluster phenomena appearing with real functions of one and of several real variables had already been studied 40 years earlier by Bettazzi 1892 (cf. [2]) and by Brodén 1897 (cf. [6]) and were presented by Schoenflies in his survey article [26]. The general theory of cluster sets was finally continued by Aumann in 1954 who considered maps between arbitrary topological spaces (cf. [1], pp. 140–141). But Aumann's notation did not survive. Collingwood with his papers on functions meromorphic in the unit disc, which appeared in the 50s (cf. [8], [9], [10], [11], [12]), pursued the mainstream originating in ideas of Painlevé (cf. [23], p. 438) and was more successful by this. Weston 1958 (cf. [31]) preserved Collingwood's notation when transferring the notion of a cluster set to maps between arbitrary topological spaces, as Aumann had done four years earlier. The standard symbol $C(f;x_0)$ for the *cluster set of a function f at a point $x_0 \in X$* used since that time stands for

$$C(f;x_0) = \bigcap_{U\in\mathcal{B}(x_0)} \mathrm{cl}(f(U)),$$

the neighborhoods U of x_0 running through an arbitrary basis $\mathcal{B}(x_0)$ of the neighborhood system $\mathcal{U}(x_0)$ of x_0. Obviously, one always has $f(x_0) \in C(f;x_0)$. In the case of a real-valued function f on a topological space X the above definition amounts to

$$C(f;x_0) = \{\gamma \in \mathbb{R} : \text{for every } \varepsilon > 0 \text{ there exists a subset } A \subseteq X \text{ with } x_0 \in \mathrm{cl}(A) \text{ and } |f(x)-\gamma| < \varepsilon \text{ for all } x \in A\}.$$

Instead of working with arbitrary subsets A of X, we shall confine ourselves to semi-open sets $S \subseteq X$ in the following and thus end up with another type of cluster set of a function $f : X \to \mathbb{R}$.

Definition. *Given a real-valued function f on a topological space X, the set*

$$SO\text{-}C(f;x_0) = \{\gamma \in \mathbb{R} : \textit{for every } \varepsilon > 0 \textit{ there exists a semi-open subset } S \subseteq X \textit{ with } x_0 \in \mathrm{cl}(S) \textit{ and } |f(x)-\gamma| < \varepsilon \textit{ for all } x \in S\}.$$

is to be called the SO-cluster set of the function f at the point $x_0 \in X$.

Let us remark that, owing to the definition of a semi-open set S, the definition of the *SO*-cluster set $SO\text{-}C(f;x_0)$ can be formulated by using open subsets $G \subseteq X$ with $x_0 \in \mathrm{cl}(G)$ instead of semi-open sets S. The notion of a semi-open subset, however, proves to be an appropriate tool for studying quasi-continuous functions in the large as already mentioned in Section 1.

The local version of quasi-continuity can be derived from the so-called “local sieve” generated by the semi-open sets in the sense of Ţevy/Bruteanu (cf. [28], pp. 122, 130). The interest of these authors lies in “pseudotopologies” and corresponding continuity notions. Thomson in his 1985 book pursues the same aim for functions $f : \mathbb{R} \to \mathbb{R}$. He operates with “local systems” $\mathcal{S}$ on $\mathbb{R}$ and introduces the notion of “$\mathcal{S}$-continuity” as well as “$\mathcal{S}$-cluster sets” for real-valued functions (cf. [30], pp. 3, 45, 70). Among others, he also briefly refers to quasi-continuous functions in this context (cf. [30], pp. 26–27). But an explicit and detailed treatment of the *SO*-cluster set $SO\text{-}C(f;x_0)$, as we call it, is missing in the literature so far.

One recognizes that $SO\text{-}C(f;x_0) \subseteq C(f;x_0)$. Whereas $f(x_0)$ belongs to $C(f;x_0)$ in any case, the *SO*-cluster set $SO\text{-}C(f;x_0)$ may be empty. This can easily be demonstrated by the function

$$f(x) = \begin{cases} \frac{1}{x} & \text{for } x \neq 0, \\ 0 & \text{for } x = 0 \end{cases}$$

on $\mathbb{R}$. It is $C(f;0) = \{0\}$ but, on the other hand, $SO\text{-}C(f;0) = \emptyset$.

If f is locally bounded at x_0, the fact $C(f;x_0) = \{f(x_0)\}$ indicates continuity of f at x_0. In the general situation the membership $f(x_0) \in SO\text{-}C(f;x_0)$ reflects quasi-continuity of f at x_0, as can be seen immediately from the definitions. For later use we formulate a first proposition.

Proposition 1. *A real-valued function f on a topological space X is quasi-continuous at a point $x_0 \in X$ if and only if $f(x_0) \in SO\text{-}C(f;x_0)$.*

The following claim concerns an interesting global phenomenon.

Proposition 2. *If a real-valued function f on a topological space X is quasi-continuous, then $SO\text{-}C(f;x) = C(f;x)$ for all $\in X$.*

PROOF. We consider an arbitrary element $\gamma \in C(f;x)$ and fix a neighborhood $U \in \mathcal{U}(x)$ as well as $\varepsilon > 0$. We then can find a point $y \in \operatorname{int} U$ with $|f(y) - \gamma| < \frac{\varepsilon}{2}$. On the other hand, there exists a non-empty open subset $G \subseteq U$ such that $|f(z) - f(y)| < \frac{\varepsilon}{2}$ for all $z \in G$, since f is quasi-continuous at y and U is a neighborhood of y, too. The triangular inequality yields $|f(z) - \gamma| < \varepsilon$ for all $z \in G$, which proves that $\gamma \in SO\text{-}C(f;x)$ and hence $C(f;x) \subseteq SO\text{-}C(f;x)$. □

If the function f is not quasi-continuous on the whole space, the SO-cluster set $SO\text{-}C(f;x_0)$ may be a proper subset of the cluster set $C(f;x_0)$ at a point x_0 of quasi-continuity. To demonstrate this phenomenon let us consider the function

$$f(x) = \begin{cases} 1 & \text{for } x = 2^{-k}, \\ 0 & \text{otherwise} \end{cases}$$

on the interval $[0,1]$. It is cliquish at the points $x_k = 2^{-k}$, $k = 1, 2, 3, \ldots$, quasi-continuous at $x_0 = 0$, and even continuous everywhere else. Nevertheless, $SO\text{-}C(f;0) = \{0\}$ is properly contained in the cluster set $C(f;0) = \{0,1\}$.

Comparing the two examples we notice that $SO\text{-}C(f;x) \neq \emptyset$ everywhere in the case of the second one. This turns out to be a characteristic property for locally bounded functions to be cliquish.

Proposition 3. *Let f be a real-valued function on a topological space X. If $SO\text{-}C(f;x_0) \neq \emptyset$, then f is cliquish at $x_0 \in X$. Conversely, if f is cliquish and locally bounded at x_0, then $SO\text{-}C(f;x_0) \neq \emptyset$.*

Proof. Clearly, f is cliquish at x_0 if $SO\text{-}C(f;x_0) \neq \emptyset$. So let f be cliquish and locally bounded at x_0. Then there exists a base $\mathcal{B}(x_0)$ for the neighborhood system $\mathcal{U}(x_0)$ of x_0 such that the sets $f(B)$ are bounded for $B \in \mathcal{B}(x_0)$. Accordingly, the corresponding sets

$$H(f;B,\varepsilon) = \{\gamma \in \mathbb{R} : \text{there exists a non-empty open set } G \subseteq B \text{ with } |f(x)-\gamma| < \varepsilon \text{ for all } x \in G\}$$

are bounded, too. Moreover, they turn out to be non-empty, because f is cliquish at x_0. Obviously, the SO-cluster set $SO\text{-}C(f;x_0)$ admits the representation

$$SO\text{-}C(f;x_0) = \bigcap\nolimits_{B\in\mathcal{B}(x_0),\,\varepsilon>0} H(f;B,\varepsilon),$$

which can easily be changed into

$$SO\text{-}C(f;x_0) = \bigcap\nolimits_{B\in\mathcal{B}(x_0),\,\varepsilon>0} \operatorname{cl}(H(f;B,\varepsilon)),$$

since $\operatorname{cl}(H(f;B,\varepsilon)) \subseteq H(f;B,2\varepsilon)$. This way $SO\text{-}C(f;x_0)$ appears as an intersection of bounded and closed sets and hence proves to be a bounded and closed set itself; that is, a compact set of real numbers. Thus, if $SO\text{-}C(f;x_0)$ were empty, even a finite intersection $\bigcap_{i=1}^{n} \operatorname{cl}(H(f;B_i,\varepsilon_i))$ had to be empty. However, one easily checks that

$$\begin{aligned}\operatorname{cl}(H(f;B_1,\varepsilon_1)) \cap \operatorname{cl}(H(f;B_2,\varepsilon_2)) &\supseteq H(f;B_1,\varepsilon_1) \cap H(f;B_2,\varepsilon_2)\\ &\supseteq H(f;B_1\cap B_2,\min\{\varepsilon_1,\varepsilon_2\})\\ &\supseteq \operatorname{cl}\left(H\big(f;B_1\cap B_2,\tfrac{1}{2}\min\{\varepsilon_1,\varepsilon_2\}\big)\right)\\ &\neq \emptyset.\end{aligned}$$ □

As a by-product of the proof of Proposition 3 we have obtained that $SO\text{-}C(f;x_0)$ is a compact set of real numbers if f is locally bounded at x_0.

3 Quasi-Continuous Functions as Uniform Limits of Semi-Open Step Functions

Usually a step function is defined to be a function with finitely many values only. For the interests of the present paper, however, we shall give a definition which takes the topology of the underlying space X into account.

Definition. *A real-valued function φ on a topological space X is called a semi-open step function if there exists a partition $\mathcal{P} = \{P_\iota : \iota \in I\}$ of X into semi-open subsets P_ι such that φ is constant on the sets P_ι.*

In what follows we shall often speak of *semi-open partitions* of a topological space X, thereby automatically thinking of partitions of X into semi-open subsets. Given a quasi-continuous step function φ in the usual sense; that is, a step function having finitely many values only, a simple consideration shows that φ is a semi-open step function on a finite semi-open partition of X. Conversely, semi-open step functions quite generally turn out to be quasi-continuous.

Proposition 4. *Every semi-open step function is quasi-continuous.*

PROOF. Let the step function φ be defined on a semi-open partition $\mathcal{P} = \{P_\iota : \iota \in I\}$. If $V \subseteq \mathbb{R}$ is an open set, then $f^{-1}(V)$ is the union of certain semi-open sets P_ι and thus is semi-open itself. □

It is a well-known fact that the uniform limit of any sequence of quasi-continuous functions is also quasi-continuous (cf. [4], [17]). Hence, in particular, the uniform limit of any sequence of semi-open step functions is quasi-continuous. The converse claim, however, is surprising, the proof being rather involved.

Theorem 1. *Let f be a real-valued quasi-continuous function on a topological space X. Then f can be represented as the uniform limit of a sequence $(\varphi_n)_{n=1}^\infty$ of semi-open step functions which are defined on a chain $K = (\mathcal{P}_n)_{n=1}^\infty$ of semi-open partitions $\mathcal{P}_n = \{P_\iota^{(n)} : \iota \in I_n\}$. If f is locally bounded, then there exists a chain K of locally finite partitions with the above property. If f is bounded, then K can be chosen to be a chain of finite partitions.*

Preliminary Remarks. 1. A chain K of partitions of X is a sequence of partitions $\mathcal{P}_n$ such that $\mathcal{P}_{n+1}$ is a refinement of $\mathcal{P}_n$, though not necessarily a proper one. A partition $\mathcal{P}$ or, more generally, a covering $\mathcal{C} = \{C_\iota : \iota \in I\}$ of a topological space X is called *locally finite* if, for every $x \in X$, there exists a neighborhood $U \in \mathcal{U}(x)$ intersecting only finitely many covering sets C_ι. A covering, in particular a partition, is called finite if it consists of finitely many elements only.

2. The proof of Theorem 1 will be combined with the proof of Theorem 2, which makes a still more detailed claim concerning quasi-continuous functions on compact metrizable spaces. Let us remark here that, in contrast with a continuous function, a quasi-continuous function f on a compact metric space need not be bounded. This phenomenon can be demonstrated by the function

$$f(x) = \begin{cases} \frac{1}{x} \sin \frac{1}{x} & \text{for } x \neq 0, \\ 0 & \text{for } x = 0 \end{cases}$$

on the compact interval $[0, 1]$.

Theorem 2. *Let f be a real-valued quasi-continuous function on a compact metrizable space X. Then there exists a chain $K = (\mathcal{P}_n)_{n=1}^{\infty}$ of semi-open partitions $\mathcal{P}_n = \{P_\iota^{(n)} : \iota \in I_n\}$ of the space X such that f as well as any real-valued continuous function on X can be attained as the uniform limit of a sequence of semi-open step functions which are defined on K. If f is bounded, then K can be chosen to be a chain of finite partitions.*

The lemma to follow is the basis for the proof of both the Theorems 1 and 2. The sets $A_{f,\varepsilon}(x)$ appearing in this context are understood to be the semi-open subsets

$$A_{f,\varepsilon}(x) = f^{-1}\big((f(x) - \varepsilon, f(x) + \varepsilon)\big)$$

defined by a quasi-continuous function f on X for arbitrary $\varepsilon > 0$.

Lemma 1. *Let f be a real-valued quasi-continuous function on a topological space X and let $\mathcal{P}$ be a semi-open partition of X which corresponds to f in the sense that, for every $x \in X$, every neighborhood $U \in \mathcal{U}(x)$, and every $\varepsilon > 0$,*

$$\operatorname{int} P(x) \cap \operatorname{int} U \cap \operatorname{int} A_{f,\varepsilon}(x) \neq \emptyset, \tag{1}$$

where $P(x)$ is the partition element from $\mathcal{P}$ that contains x. Moreover, suppose that $\mathcal{C} = \{C_1, C_2, \ldots, C_k\}$ is an arbitrary finite open covering of X and let $\delta > 0$. Then there exists a semi-open refinement $\mathcal{Q}$ of $\mathcal{P}$ satisfying the following conditions:

(i) The partition $\mathcal{Q}$ again corresponds to f in the above sense; that is,

$$\operatorname{int} Q(x) \cap \operatorname{int} U \cap \operatorname{int} A_{f,\varepsilon}(x) \neq \emptyset$$

for every $x \in X$, $U \in \mathcal{U}(x)$, and $\varepsilon > 0$, where $Q(x)$ is the set from $\mathcal{Q}$ that contains x.

(ii) For all $Q \in \mathcal{Q}$ there exists a set $C_i \in \mathcal{C}$ such that $Q \subseteq \operatorname{cl}(C_i)$.

(iii) For all $Q \in \mathcal{Q}$ there exists an integer l such that $f(Q) \subseteq [l\delta, (l+1)\delta]$.

(iv) If f is locally bounded and $\mathcal{P}$ locally finite, then $\mathcal{Q}$ can be chosen to be locally finite as well. If f is bounded and $\mathcal{P}$ finite, then $\mathcal{Q}$ can be chosen to be finite.

PROOF. 1. Let $\mathcal{D} = \{D_1, D_2, \ldots, D_k\}$ consist of the sets $D_i = C_i \backslash \operatorname{cl}\big(\bigcup_{j=1}^{i-1} C_j\big)$. Obviously, the sets from $\mathcal{D}$ are open and pairwise disjoint their union being dense in X.

2. Similarly, the system $\mathcal{E} = \{E_l : l \in \mathbb{Z}\}$ with $E_l = \operatorname{int} f^{-1}\big([l\delta, (l+1)\delta)\big)$ consists of open and pairwise disjoint sets and the union $\bigcup_{l\in\mathbb{Z}} E_l$ is dense in X.

This second claim is obvious up to the fact that $\bigcup_{l\in\mathbb{Z}} E_l$ is dense.

We have to show that every non-empty open subset $G \subseteq X$ has a non-empty intersection with $\bigcup_{l\in\mathbb{Z}} E_l$. So let us choose $x_0 \in G$ and let $l_0 \in \mathbb{Z}$ be such that $f(x_0) \in ((l_0-1)\delta, (l_0+1)\delta)$. Then, by the quasi-continuity of f at x_0, there exists a non-empty open subset $G_1 \subseteq G$ such that $f(G_1) \subseteq ((l_0-1)\delta, (l_0+1)\delta)$. In the case $f(G_1) = \{l_0\delta\}$ we obtain $G_1 \subseteq E_{l_0}$ and $G \cap \bigcup_{l\in\mathbb{Z}} E_l \supseteq G_1 \neq \emptyset$. Otherwise we find $x_1 \in G_1$ with $f(x_1) \neq l_0\delta$, say $f(x_1) \in (l_0\delta, (l_0+1)\delta)$. Then there exists a non-empty open set $G_2 \subseteq G_1$ with $f(G_2) \subseteq (l_0\delta, (l_0+1)\delta)$ according to the quasi-continuity of f at x_1. Hence $G_2 \subseteq E_{l_0}$ and $G \cap \bigcup_{l\in\mathbb{Z}} E_l \supseteq G_2 \neq \emptyset$. Thus $\bigcup_{l\in\mathbb{Z}} E_l$ is dense in X.

3. For $x \in X$ we define a set $H(x)$ of pairs of indices by

$$H(x) = \{(i,l) \in \{1,2,\dots,k\} \times \mathbb{Z} : \text{ for all } U \in \mathcal{U}(x) \text{ and all } \varepsilon > 0, \\ \operatorname{int} P(x) \cap \operatorname{int} U \cap \operatorname{int} A_{f,\varepsilon}(x) \cap D_i \cap E_l \neq \emptyset\}.$$

Then $H(x) \neq \emptyset$.

Let us assume the contrary, i.e. $H(x) = \emptyset$. We choose $l_0 \in \mathbb{Z}$ such that $f(x) \in ((l_0-1)\delta, (l_0+1)\delta)$. Putting $\varepsilon_0 = \min\{f(x) - (l_0-1)\delta, (l_0+1)\delta - f(x)\}$ we obtain

$$\operatorname{int} A_{f,\varepsilon_0}(x) \cap E_l = \emptyset \ \text{ for } \ l \notin \{l_0-1, l_0\}.$$

For $(i,l) \in \{1,2,\dots,k\} \times \{l_0-1, l_0\}$ there exist $U(i,l) \in \mathcal{U}(x)$ and $\varepsilon(i,l) > 0$ such that

$$\operatorname{int} P(x) \cap \operatorname{int} U(i,l) \cap \operatorname{int} A_{f,\varepsilon(i,l)}(x) \cap D_i \cap E_l = \emptyset,$$

since $H(x) = \emptyset$. Now we define $U_1 = \bigcap_{1\le i\le k,\, l_0-1\le l\le l_0} U(i,l) \in \mathcal{U}(x)$ and $\varepsilon_1 = \min(\{\varepsilon_0\} \cup \{\varepsilon(i,l) : 1 \le i \le k, l_0-1 \le l \le l_0\}) > 0$. Then

$$\operatorname{int} P(x) \cap \operatorname{int} U_1 \cap \operatorname{int} A_{f,\varepsilon_1}(x) \cap D_i \cap E_l = \emptyset$$

for all pairs $(i,l) \in \{1,2,\dots,k\} \times \mathbb{Z}$. But, by the first two steps of the proof, the union $\bigcup_{1\le i\le k,\, l\in\mathbb{Z}} (D_i \cap E_l)$ is dense in X. Hence

$$\operatorname{int} P(x) \cap \operatorname{int} U_1 \cap \operatorname{int} A_{f,\varepsilon_1}(x) = \emptyset.$$

This contradicts property (1) and thus proves that $H(x) \neq \emptyset$.

4. We obtain

$$H(y) = \{(i_0, l_0)\} \ \text{ if } \ y \in D_{i_0} \cap E_{l_0}.$$

Indeed, if $(i,l) \in H(y)$ then $(D_{i_0} \cap E_{l_0}) \cap D_i \cap E_l = U_0 \cap D_i \cap E_l \neq \emptyset$, since $U_0 = D_{i_0} \cap E_{l_0} \in \mathcal{U}(y)$. The systems $\mathcal{D}$ and $\mathcal{E}$ both consisting of disjoint sets, we obtain $(i,l) = (i_0, l_0)$. Hence $H(y) \subseteq \{(i_0, l_0)\}$ and, by $H(y) \neq \emptyset$, $H(y) = \{(i_0, l_0)\}$.

5. For every $x \in X$ we fix $i(x)$ and $l(x)$ such that $(i(x), l(x)) \in H(x)$. Then we define a new partition

$$\mathcal{Q} = \left\{Q_{(P,i,l)} : P \in \mathcal{P},\, i \in \{1, 2, \ldots, k\},\, l \in \mathbb{Z}\right\} \setminus \{\emptyset\}$$

where

$$Q_{(P,i,l)} = \{x \in P : i(x) = i, l(x) = l\}.$$

Clearly, $\mathcal{Q}$ is a refinement of $\mathcal{P}$.

6. $\mathcal{Q}$ fulfils condition (i). (This implies in particular that $\mathcal{Q}$ is a semi-open partition, since $\operatorname{int} Q(x) \cap U \neq \emptyset$ for all $x \in X$ and $U \in \mathcal{U}(x)$).

Let $x \in X$, $U \in \mathcal{U}(x)$, and $\varepsilon > 0$ be fixed. Then $Q(x) = Q_{(P(x), i(x), l(x))}$. Step 4 of the proof shows that $\operatorname{int} P(x) \cap D_{i(x)} \cap E_{l(x)} \subseteq \operatorname{int} \{y \in P(x) : i(y) = i(x), l(y) = l(x)\} = \operatorname{int} Q(x)$. Accordingly,

$$\operatorname{int} Q(x) \cap \operatorname{int} U \cap \operatorname{int} A_{f,\varepsilon}(x) \supseteq \operatorname{int} P(x) \cap \operatorname{int} U \cap \operatorname{int} A_{f,\varepsilon}(x) \cap D_{i(x)} \cap E_{l(x)} \neq \emptyset$$

because of $(i(x), l(x)) \in H(x)$. This proves (i).

7. $\mathcal{Q}$ fulfils condition (ii). Namely,

$$Q \subseteq \operatorname{cl}(C_i) \ \text{ if } \ Q = Q_{(P,i,l)} \in \mathcal{Q}.$$

Let $x \in Q = Q_{(P,i,l)}$. Then $(i,l) = (i(x), l(x)) \in H(x)$. This yields in particular $U \cap D_i \neq \emptyset$ for all $U \in \mathcal{U}(x)$; that is, $x \in \operatorname{cl}(D_i)$. Hence $Q \subseteq \operatorname{cl}(D_i) \subseteq \operatorname{cl}(C_i)$ by the definition of D_i.

8. $\mathcal{Q}$ fulfils condition (iii). More precisely,

$$f(Q) \subseteq [l\delta, (l+1)\delta] \ \text{ if } \ Q = Q_{(P,i,l)} \in \mathcal{Q}.$$

Again we start with $x \in Q$ and obtain $(i,l) = (i(x), l(x)) \in H(x)$. This implies that, for all $\varepsilon > 0$, there exists at least one point $y_\varepsilon \in A_{f,\varepsilon}(x) \cap E_l$. Consequently, $|f(x) - f(y_\varepsilon)| < \varepsilon$, since $y_\varepsilon \in A_{f,\varepsilon}(x)$, and $f(y_\varepsilon) \in [l\delta, (l+1)\delta)$ because of $y_\varepsilon \in E_l$. Hence $f(x) \in (l\delta - \varepsilon, (l+1)\delta + \varepsilon)$ for all $x \in Q$ and $\varepsilon > 0$. This proves that $f(Q) \subseteq [l\delta, (l+1)\delta]$.

9. $\mathcal{Q}$ fulfils condition (iv).

Assume first that f is bounded and $\mathcal{P}$ finite. Then $\mathcal{P} = \{P_1, P_2, \ldots, P_{m_0}\}$. Moreover, only finitely many of the sets E_l are non-empty, say $E_l = \emptyset$ if $|l| > l_0$. Hence the sets $Q_{(P,i,l)}$ are empty if $|l| > l_0$, since $|l(x)| \leq l_0$ for all $x \in X$. This yields

$$\mathcal{Q} \subseteq \left\{Q_{(P_m,i,l)} : 1 \leq m \leq m_0,\, 1 \leq i \leq k,\, -l_0 \leq l \leq l_0\right\}$$

showing that $\mathcal{Q}$ is finite.

If f is locally bounded and $\mathcal{P}$ locally finite, then the above arguments apply to a suitable open neighborhood U of x for arbitrary $x \in X$. Thus $\mathcal{Q}$ is locally finite. This completes the proof of Lemma 1. $\square$

PROOF OF THEOREM 1. Given a quasi-continuous function f, the chain K can be constructed inductively by the aid of Lemma 1. We start with the trivial partition $\mathcal{P}_0 = \{X\}$ which fulfils property (1), since f is quasi-continuous. For $n \geq 1$, we obtain the partition $\mathcal{P}_n$ by applying Lemma 1 to the previous partition $\mathcal{P}_{n-1}$, the trivial covering $\mathcal{C}_n = \{X\}$, and $\delta_n = \frac{1}{n}$. Obviously, $K = (\mathcal{P}_n)_{n=1}^{\infty}$ is a chain of semi-open partitions which are locally finite or finite if f is locally bounded or bounded, respectively.

Now let $\mathcal{P}_n = \{P_\iota^{(n)} : \iota \in I_n\}$ be a fixed partition from K. By claim (iii) of the lemma, there exist reals $\lambda_\iota^{(n)}$ such that $f(P_\iota^{(n)}) \subseteq [\lambda_\iota^{(n)} - \frac{1}{2}\delta_n, \lambda_\iota^{(n)} + \frac{1}{2}\delta_n]$ for $\iota \in I_n$. Putting $\varphi_n = \sum_{\iota \in I_n} \lambda_\iota^{(n)} \mathbf{I}_{P_\iota^{(n)}}$ we obtain $\sup_{x \in X} |f(x) - \varphi_n(x)| \leq \frac{1}{2}\delta_n = \frac{1}{2n}$. Here $\mathbf{I}_{P_\iota^{(n)}}$ denotes the characteristic function of the set $P_\iota^{(n)}$.

Hence f is the uniform limit of the semi-open step functions φ_n, $n \geq 1$, which are defined on the partitions from K. $\square$

PROOF OF THEOREM 2. Let us assume that X is equipped with a metric d. Given a quasi-continuous function f on X, the construction of K differs from the procedure in the above proof in so far as we use a covering $\mathcal{C}_n = \{B(x_1^{(n)}; \frac{1}{n}), B(x_2^{(n)}; \frac{1}{n}), \ldots, B(x_{k_n}^{(n)}; \frac{1}{n})\}$ by open balls of radius $\frac{1}{n}$ instead of the trivial covering $\{X\}$ when generating the refinement $\mathcal{P}_n$ of $\mathcal{P}_{n-1}$.

In addition to the proof of Theorem 1 we have to show that any real-valued continuous function g on X is the uniform limit of a sequence $(\psi_n)_{n=1}^{\infty}$ of semi-open step functions defined on the partitions $\mathcal{P}_n = \{P_\iota^{(n)} : \iota \in I_n\}$ from K. Claim (ii) of Lemma 1 says that any set $P_\iota^{(n)}$ from $\mathcal{P}_n$ is contained in the closure of a suitable ball from $\mathcal{C}_n$, say $P_\iota^{(n)} \subseteq \operatorname{cl}(B(x_{i(\iota,n)}^{(n)}; \frac{1}{n}))$. This yields $d(x, x_{i(\iota,n)}^{(n)}) \leq \frac{1}{n}$ for all $x \in P_\iota^{(n)}$. We define semi-open step functions by $\psi_n = \sum_{\iota \in I_n} g(x_{i(\iota,n)}^{(n)}) \mathbf{I}_{P_\iota^{(n)}}$. For arbitrary $x \in X$, say $x \in P_\iota^{(n)}$, we can estimate

$$|g(x) - \psi_n(x)| = |g(x) - g(x_{i(\iota,n)}^{(n)})| \leq \sup\{|g(s) - g(t)| : d(s,t) \leq \tfrac{1}{n}\}.$$

Accordingly,

$$\sup_{x \in X} |g(x) - \psi_n(x)| \leq \sup\{|g(s) - g(t)| : d(s,t) \leq \tfrac{1}{n}\} = \omega(g; \tfrac{1}{n}),$$

where $\omega(g; \cdot)$ denotes the modulus of continuity of g. This proves that the functions ψ_n uniformly approach g. $\square$

Remarks. 1. Theorem 2, in contrast with Theorem 1, claims the existence of a chain $K = (\mathcal{P}_n)_{n=1}^{\infty}$ of semi-open partitions on X which in a sense is universal. Indeed, under the suppositions of Theorem 2, simultaneously with the given quasi-continuous function f, any continuous function on X can be uniformly approximated by step functions on K. The assumption that X be a compact metrizable space can be shown to be necessary in the following sense: If X is a completely regular space which possesses a sequence of finite partitions $\mathcal{P}_n$ such that any bounded real-valued continuous function on X appears as a uniform limit of a sequence of step functions defined on the partitions $\mathcal{P}_n$, then X is compact metrizable (cf. [25]).

2. Theorem 1 enables us to arrange quasi-continuous functions on a topological space X in such a way that they appear as elements of certain linear spaces. In fact, given a quasi-continuous function f, we can refer to a chain $K = (\mathcal{P}_n)_{n=1}^{\infty}$ of semi-open partitions $\mathcal{P}_n$ of X such that f turns out to be the uniform limit of a sequence of step functions φ_n defined on the partitions $\mathcal{P}_n$. The class $A_K(X)$ of all uniform limits arising this way then obviously is a linear space of quasi-continuous functions which contains the original function f. A common property of the functions belonging to the space $A_K(X)$ is that their discontinuities all are boundary points of partition elements of K. To demonstrate this, we consider the chain of partition elements $P_n(x_0) \in \mathcal{P}_n$ containing a point $x_0 \in X$. If x_0 is supposed to be an inner point of all the sets $P_n(x_0)$, $n = 1, 2, 3, \ldots$, an easy computation shows that every function $g \in A_K(X)$ is continuous at x_0. Consequently, the set D_g of discontinuity points of a function $g \in A_K(X)$ is a subset of the set B_K of boundary points of partition elements of K. The set B_K admits a representation $B_K = \bigcup_{n=1}^{\infty} B_n$ where B_n stands for the set of points appearing as boundary points of partition elements $P \in \mathcal{P}_n$. Since the partition elements $P \in \mathcal{P}_n$ are semi-open, the sets B_n prove to be nowhere dense. Accordingly, the set B_K is of the first category. As a by-product of Theorem 1 we thus once more obtain that the points of discontinuity of any quasi-continuous function form a set of the first category.

3. In the case of a bounded quasi-continuous function f on a topological space X the partitions $\mathcal{P}_n$ of the chain K can be chosen to be finite. Then the whole space $A_K(X)$ consists of bounded functions exclusively. Moreover, $A_K(X)$ turns out to be a Banach space with respect to the supremum norm. But on the other hand, a Banach space of bounded quasi-continuous functions on X is not necessarily contained in a space of type $A_K(X)$. We refer to quasi-continuous regulated functions f on $[-1, 1]$ which are continuous from the left (see [1], p. 236). These functions are characterized by the properties

that

$$\begin{aligned}&\lim_{t\to x+0} f(t) \ \text{ exists for } -1<x<1,\\ &\lim_{t\to -1+0} f(t)=f(-1), \ \text{ and}\\ &\lim_{t\to x-0} f(t)=f(x) \ \text{ for } -1<x\le 1.\end{aligned}$$

They obviously form a Banach space E of bounded quasi-continuous functions with respect to the supremum norm. But this Banach space cannot be involved in a space of type $A_K(X)$, for any point $x_0 \in (-1,1)$ happens to be a discontinuity of an appropriate function $f_0 \in E$ (cf. [24]).

4 Transforming Cliquish Functions into Quasi-Continuous Functions

In this section we shall study real-valued functions f on a topological space X with $SO\text{-}C(f;x) \neq \emptyset$ for all $x \in X$. According to Proposition 3 these functions are cliquish and, vice versa, locally bounded cliquish functions fulfil the condition $SO\text{-}C(f;x) \neq \emptyset$ for all $x \in X$. The mechanism for transforming functions f with $SO\text{-}C(f;x) \neq \emptyset$ into quasi-continuous functions to be presented rests upon the concept of an *admissible modification* $\tilde f$ *of a function* f. Denoting the set of continuity points of f and $\tilde f$ by C_f and $C_{\tilde f}$, respectively, one can describe this concept as follows (cf. [1], p. 162): *A function* $\tilde f$ *is said to be an admissible modification of a function* f *if*

$$\tilde f(x) = f(x) \ \textit{ for all } \ x \in C_f \ \textit{ and } C_f \subseteq C_{\tilde f}.$$

Theorem 3. *Let* f *be a real-valued function on a topological space* X *such that* $SO\text{-}C(f;x) \neq \emptyset$ *for all* $x \in X$*. Then every function* $\tilde f$ *with*

$$\tilde f(x) \in SO\text{-}C(f;x) \ \textit{ for all } \ x \in X$$

is a quasi-continuous admissible modification of f *with*

$$SO\text{-}C\big(\tilde f;x\big) = SO\text{-}C(f;x) \ \textit{ for all } \ x \in X.$$

PROOF. Let $\tilde f$ be such that $\tilde f(x) \in SO\text{-}C(f;x)$ for all $x \in X$. If f is continuous at x, then $SO\text{-}C(f;x) = \{f(x)\}$ and thus $\tilde f(x) = f(x)$. In order to show that even $\tilde f$ is continuous at x, we fix $\varepsilon > 0$ arbitrarily. Owing to the continuity of f, there exists an open neighborhood $U \in \mathcal{U}(x)$ such that $f(U) \subseteq (f(x) - \varepsilon, f(x)+\varepsilon)$. Because of $\tilde f(y) \in SO\text{-}C(f;y)$ and $U \in \mathcal{U}(y)$ for all $y \in U$, the value $\tilde f(y)$ can be approximated by values $f(z)$ with $z \in U$ for all $y \in U$. This means that $\tilde f(U) \subseteq \operatorname{cl}(f(U))$ and thus $\tilde f(U) \subseteq [f(x)-\varepsilon, f(x)+\varepsilon] =$

$\big[\tilde{f}(x)-\varepsilon,\tilde{f}(x)+\varepsilon\big]$. Accordingly, $\tilde{f}$ is continuous at x. This proves that $\tilde{f}$ is an admissible modification of f.

In order to show the coincidence of $SO\text{-}C\big(\tilde{f};x\big)$ and $SO\text{-}C(f;x)$, we first consider $\gamma\in SO\text{-}C\big(\tilde{f};x\big)$. Given $\varepsilon>0$ and $U\in\mathcal{U}(x)$, we can determine $y\in\operatorname{int}U$ such that $\big|\tilde{f}(y)-\gamma\big|<\frac{\varepsilon}{2}$. Because of $\tilde{f}(y)\in SO\text{-}C(f;y)$ and $U\in\mathcal{U}(y)$, there exists a non-empty open set $G\subseteq U$ with $\big|f(z)-\tilde{f}(y)\big|<\frac{\varepsilon}{2}$ for all $z\in G$. The triangle inequality finally gives the desired estimate $|f(z)-\gamma|<\varepsilon$ for all $z\in G$, which yields $SO\text{-}C\big(\tilde{f};x\big)\subseteq SO\text{-}C(f;x)$.

Now let $\gamma\in SO\text{-}C(f;x)$. For proving $\gamma\in SO\text{-}C\big(\tilde{f};x\big)$, we fix $\varepsilon>0$ and $U\in\mathcal{U}(x)$ and determine a non-empty open set $G\subseteq U$ such that $f(G)\subseteq(\gamma-\varepsilon,\gamma+\varepsilon)$. As we have seen above, the function $\tilde{f}$ then satisfies the inclusion $\tilde{f}(G)\subseteq\operatorname{cl}(f(G))$, which implies that $\tilde{f}(G)\subseteq[\gamma-\varepsilon,\gamma+\varepsilon]$ and thus shows that $\gamma\in SO\text{-}C\big(\tilde{f};x\big)$. This finishes the proof of $SO\text{-}C\big(\tilde{f};x\big)=SO\text{-}C(f;x)$.

Finally, the choice of $\tilde{f}$ yields $\tilde{f}(x)\in SO\text{-}C(f;x)=SO\text{-}C\big(\tilde{f};x\big)$ for all $x\in X$. Applying Proposition 1 we recognize that the function $\tilde{f}$ is quasi-continuous on X, as asserted. □

From Proposition 3 we know that a real-valued function f on a topological space X with $SO\text{-}C(f;x)\neq\emptyset$ for all $x\in X$ is cliquish on X. Hence the set D_f of discontinuity points of f is of the first category in X. Passing to an admissible modification $\tilde{f}$ of f therefore amounts to changing the values of f in a set of the first category only.

As a counterpart to Theorem 3 we prove that the validity of $SO\text{-}C\big(\tilde{f};x\big)=SO\text{-}C(f;x)$ for all $x\in X$ is a necessary condition for any admissible modification $\tilde{f}$ of f, provided that f is a cliquish function on a Baire space.

Theorem 4. *Let f be a real-valued cliquish function on a Baire space X. Then every admissible modification $\tilde{f}$ of f satisfies the condition*

$$SO\text{-}C\big(\tilde{f};x\big)=SO\text{-}C(f;x)\ \ \textit{for all}\ \ x\in X.$$

PROOF. Let $\tilde{f}$ be an admissible modification of f. Suppose $\gamma\in SO\text{-}C\big(\tilde{f};x\big)$. For any $\varepsilon>0$ and any neighborhood $U\in\mathcal{U}(x)$, we then can determine a non-empty open set $\tilde{G}\subseteq U$ such that $\big|\tilde{f}(\tilde{z})-\gamma\big|<\frac{\varepsilon}{2}$ for all $\tilde{z}\in\tilde{G}$. According to the supposition the set of continuity points of the function f is dense in X. Therefore we in particular find a continuity point $y=\tilde{z}$ of f in $\tilde{G}$. This implies $\tilde{f}(y)=f(y)$. Furthermore we may conclude that there exists a non-empty open set $G\subseteq\tilde{G}$ which contains the point y and guarantees that $|f(y)-f(z)|<\frac{\varepsilon}{2}$ for all $z\in G$. Via the triangle inequality this yields $|f(z)-\gamma|<\varepsilon$ for all $z\in G$ and thus proves that $\gamma\in SO\text{-}C(f;x)$.

Next we consider $\gamma \in SO\text{-}C(f;x)$. Given $\varepsilon > 0$ and a neighborhood $U \in \mathcal{U}(x)$, we can apply the same steps as before. By using $C_f \subseteq C_{\tilde{f}}$ and $\tilde{f}(y) = f(y)$ for $y \in C_f$ we this time end up with a non-empty open subset $\tilde{G} \subseteq U$ such that $\left|\tilde{f}(\tilde{z}) - \gamma\right| < \varepsilon$ for all $\tilde{z} \in \tilde{G}$. This estimate tells us that $\gamma \in SO\text{-}C\big(\tilde{f};x\big)$ and thus completes the proof of the assertion. $\square$

If X is not a Baire space, then there even may exist quasi-continuous functions f on X having an admissible modification $\tilde{f}$ with

$$SO\text{-}C\big(\tilde{f};x\big) \neq SO\text{-}C(f;x) \text{ for all } x \in X.$$

For example, consider the function $f(x) = \sum_{x_i \leq x} 2^{-i}$ on the rational numbers $\mathbb{Q} = \{x_1, x_2, x_3, \ldots\}$ equipped with the usual topology induced by the real line. The function f is quasi-continuous, since it is continuous from the right. But f does not have any continuity point, so that every function $\tilde{f} : \mathbb{Q} \to \mathbb{R}$ is an admissible modification of f.

5 Cliquish Functions as Uniform Limits of Almost Semi-Open Step Functions

In order to obtain a representation theorem for cliquish functions that corresponds to Theorem 1 for quasi-continuous functions, we first have to generalize the notion of a semi-open step function. This requires a type of partitions on a topological space which generalize semi-open partitions.

Definition. *A partition $\mathcal{P} = \{P_\iota : \iota \in I\}$ of a topological space X is to be called almost semi-open if the union $\bigcup_{\iota \in I} \operatorname{int} P_\iota$ is dense in X. An almost semi-open step function is understood to be a real-valued function φ on X which is piecewise constant on the partition elements of an almost semi-open partition $\mathcal{P}$ of X.*

Note that, in contrast with the notion of a semi-open partition, the concept of an almost semi-open partition $\mathcal{P} = \{P_\iota : \iota \in I\}$ does not refer to a property of the single partition sets P_ι, but to a topological property of the whole partition $\mathcal{P}$. Owing to this fact almost semi-open partitions can be handled in a simpler way than semi-open partitions as we shall see right now.

Lemma 2. *If $\mathcal{P} = \{P_\iota : \iota \in I\}$ and $\mathcal{Q} = \{Q_\kappa : \kappa \in K\}$ are two almost semi-open partitions of a topological space X, then the merged partition $\mathcal{R} = \{P_\iota \cap Q_\kappa : P_\iota \cap Q_\kappa \neq \emptyset, \iota \in I, \kappa \in K\}$ is almost semi-open as well.*

PROOF. The open sets $\bigcup_{\iota\in I} \operatorname{int} P_\iota$ and $\bigcup_{\kappa\in K} \operatorname{int} Q_\kappa$ are dense in X, since $\mathcal{P}$ and $\mathcal{Q}$ are almost semi-open. Then $\bigcup_{R\in\mathcal{R}} \operatorname{int} R = \bigcup_{\iota\in I,\, \kappa\in K} \operatorname{int}(P_\iota \cap Q_\kappa) = (\bigcup_{\iota\in I} \operatorname{int} P_\iota) \cap (\bigcup_{\kappa\in K} \operatorname{int} Q_\kappa)$ is dense as well. This proves that $\mathcal{R}$ is almost semi-open. □

Almost semi-open step functions turn out to be the simplest types of cliquish functions.

Proposition 5. *Every almost semi-open step function on a topological space X is cliquish.*

PROOF. Given an almost semi-open step function φ, there exists an almost semi-open partition $\mathcal{P} = \{P_\iota : \iota \in I\}$ such that φ is constant on the partition elements P_ι. The union $\bigcup_{\iota\in I} \operatorname{int} P_\iota$ being dense in X, for arbitrary $x \in X$ and $U \in \mathcal{U}(x)$, a partition element P_ι can be determined such that $\operatorname{int} U \cap \operatorname{int} P_\iota \neq \emptyset$. The non-empty open set $G = \operatorname{int} U \cap \operatorname{int} P_\iota \subseteq U$ now can serve to prove that φ is cliquish at x, for φ is constant on G. □

If a step function φ in the usual sense; that is, a function with finitely many values only, is known to be cliquish, then a straight-forward consideration leads to a finite almost semi-open partition $\mathcal{P}$ of X such that φ is constant on the partition elements of $\mathcal{P}$. However, it is surprising again that arbitrary cliquish functions on a topological space X can be traced back to almost semi-open step functions, namely in so far as every cliquish function on X admits a representation as a uniform limit of a sequence of almost semi-open step functions defined on a chain of almost semi-open partitions. The following lemma concerning locally bounded cliquish functions is the key to this general result.

Lemma 3. *Let f be a locally bounded real-valued cliquish function on a topological space X and let $\delta > 0$. Then there exists a locally finite almost semi-open partition $\mathcal{P}$ of X and an almost semi-open step function φ on $\mathcal{P}$ such that $\sup_{x\in X} |f(x) - \varphi(x)| \leq \delta$. If f is bounded, then $\mathcal{P}$ can be chosen to be finite.*

PROOF. 1. By Proposition 3, we have $SO\text{-}C(f;x) \neq \emptyset$ for all $x \in X$. Hence Theorem 3 applies telling us that f admits a quasi-continuous admissible modification $\tilde{f}$. Clearly, $\tilde{f}$ is locally bounded and even bounded if f is bounded.

2. Theorem 1 yields the existence of a locally finite semi-open partition $\tilde{\mathcal{P}}$ and of a step function $\tilde{\varphi}$ on $\tilde{\mathcal{P}}$ with $\sup_{x\in X} \left|\tilde{f}(x) - \tilde{\varphi}(x)\right| \leq \frac{\delta}{2}$. Moreover, we can assume that $\tilde{\mathcal{P}}$ is finite if f is bounded, since $\tilde{f}$ is bounded in this case as well.

3. Let

$$N_{f,\delta} = \big\{x \in X : \text{ for all } U \in \mathcal{U}(x) \text{ there exists a point } y \in U \text{ with } |f(x) - f(y)| > \tfrac{\delta}{2}\big\}.$$

Then $N_{f,\delta}$ is nowhere dense.

Indeed, if V is a non-empty open subset of X, then there exists a non-empty open set $G \subseteq V$ with $|f(y) - f(z)| < \frac{\delta}{2}$ for all $y, z \in G$, since f is cliquish at any point $x_0 \in V$. Hence $G \cap N_{f,\delta} = \emptyset$, which shows that $N_{f,\delta}$ is not dense in V.

4. The system $\mathcal{P}_1 = \big\{\tilde{P} \setminus N_{f,\delta} : \tilde{P} \in \tilde{\mathcal{P}}\big\}$ is a locally finite partition of $X \setminus N_{f,\delta}$. It is even finite if f is bounded, for $\tilde{\mathcal{P}}$ is finite in this case in accordance with Step 2. Moreover, the union $\bigcup_{\tilde{P}\in\tilde{\mathcal{P}}} \operatorname{int} \tilde{P}$ being dense and $N_{f,\delta}$ being nowhere dense in X, the union $\bigcup_{P^{(1)}\in\mathcal{P}_1} \operatorname{int} P^{(1)}$ is dense in X as well.

5. Let the partition $\mathcal{P}_2 = \{P_k : k \in \mathbb{Z}\} \setminus \{\emptyset\}$ of $N_{f,\delta}$ be defined by

$$P_k = \{x \in N_{f,\delta} : f(x) \in [k\delta, (k+1)\delta)\}.$$

Clearly, $\mathcal{P}_2$ is locally finite and even finite if f is bounded.

6. Now $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2$ is a locally finite partition of X which is finite if f is bounded. $\mathcal{P}$ is almost semi-open, since $\bigcup_{P\in\mathcal{P}} \operatorname{int} P = \bigcup_{P^{(1)}\in\mathcal{P}_1} \operatorname{int} P^{(1)}$ is dense in X.

7. The function

$$\varphi(x) = \begin{cases} \tilde{\varphi}(x) & \text{for } x \in X \setminus N_{f,\delta}, \\ k\delta & \text{for } x \in P_k \ (\subseteq N_{f,\delta}\,) \end{cases}$$

is an almost semi-open step function defined on $\mathcal{P}$. We shall finish the proof by showing that

$$|f(x) - \varphi(x)| \le \delta \ \text{ for all } \ x \in X.$$

If $x \in P_k$ then this estimate is obvious, since $f(x) \in [k\delta, (k+1)\delta)$ and $\varphi(x) = k\delta$.

So let $x \in X \setminus N_{f,\delta}$. The definition of $N_{f,\delta}$ implies that $SO\text{-}C(f;x) \subseteq \big[f(x) - \frac{\delta}{2}, f(x) + \frac{\delta}{2}\big]$. Hence $\big|f(x) - \tilde{f}(x)\big| \le \frac{\delta}{2}$ because of $\tilde{f}(x) \in SO\text{-}C(f;x)$. Thus we get the estimate

$$|f(x) - \varphi(x)| = |f(x) - \tilde{\varphi}(x)| \le \big|f(x) - \tilde{f}(x)\big| + \big|\tilde{f}(x) - \tilde{\varphi}(x)\big| \le \tfrac{\delta}{2} + \tfrac{\delta}{2} = \delta.$$

This completes the proof of Lemma 3. □

Though the claim of Lemma 3 is restricted to locally bounded cliquish functions, the representation theorem announced above can be proved for arbitrary cliquish functions on a topological space X.

Theorem 5. *Let f be a real-valued cliquish function on a topological space X. Then f can be represented as the uniform limit of a sequence $(\varphi_n)_{n=1}^{\infty}$ of almost semi-open step functions which are defined on a chain $K = (\mathcal{P}_n)_{n=1}^{\infty}$ of almost semi-open partitions. If f is locally bounded, then there exists a chain K of locally finite partitions with the above property. If f is bounded, then K can be chosen to be a chain of finite partitions.*

PROOF. First we assume that f is locally bounded. By Lemma 3, there exist a sequence of locally finite almost semi-open partitions $\mathcal{Q}_n$, $n \geq 1$, and step functions φ_n on the partitions $\mathcal{Q}_n$ with $\lim_{n\to\infty} \sup_{x\in X} |f(x) - \varphi_n(x)| = 0$. We define partitions $\mathcal{P}_n$, $n \geq 1$, inductively by $\mathcal{P}_1 = \mathcal{Q}_1$ and $\mathcal{P}_{n+1} = \big\{P^{(n)} \cap Q^{(n+1)} : P^{(n)} \in \mathcal{P}_n, Q^{(n+1)} \in \mathcal{Q}_{n+1}\big\} \setminus \{\emptyset\}$. Then $K = (\mathcal{P}_n)_{n=1}^{\infty}$ is a chain of almost semi-open partitions according to Lemma 2. For each $n \geq 1$, the step function φ_n on $\mathcal{Q}_n$ can be considered as a step function on $\mathcal{P}_n$ as well, since $\mathcal{P}_n$ is a refinement of $\mathcal{Q}_n$. Hence f is the uniform limit of step functions defined on K. Moreover, all partitions are locally finite, since they were formed by merging locally finite partitions. If f is bounded then, by Lemma 3, we can assume the partitions $\mathcal{Q}_n$ to be finite. In this case the merged partitions $\mathcal{P}_n$ from K are finite as well. This completes the proof of Theorem 5 if f is locally bounded.

Now we consider a cliquish function f which is not locally bounded. Let

$$N_f = \{x \in X : f(U) \text{ is unbounded for all } U \in \mathcal{U}(x)\}$$

be the set of all points $x \in X$ where f is not locally bounded. Obviously, N_f is closed. Moreover, every non-empty open set $V \subseteq X$ contains a non-empty open set $G \subseteq V$ such that f is bounded on G, say $|f(y) - f(z)| < 1$ for all $y, z \in G$, since f is cliquish at any point $x_0 \in V$ and V is a neighborhood of x_0. Hence $N_f \cap G = \emptyset$. This shows that N_f is nowhere dense.

The restriction $f_1 = f|_{X_1}$ of f to $X_1 = X \setminus N_f$ is a locally bounded cliquish function on X_1. According to the first part of the proof there exist a chain $K_1 = (\mathcal{P}_{n,1})_{n=1}^{\infty}$ of almost semi-open partitions of X_1 and step functions $\varphi_{n,1}$ on $\mathcal{P}_{n,1}$ with $\lim_{n\to\infty} \sup_{x\in X_1} |f_1(x) - \varphi_{n,1}(x)| = 0$. We define the chain $K = (\mathcal{P}_n)_{n=1}^{\infty}$ of partitions of the whole space X by $\mathcal{P}_n = \mathcal{P}_{n,1} \cup \{\{x\} : x \in N_f\}$. Then $\mathcal{P}_n$, $n \geq 1$, is almost semi-open, since $\bigcup_{P^{(n)}\in\mathcal{P}_n} \operatorname{int} P^{(n)} = \bigcup_{P^{(n,1)}\in\mathcal{P}_{n,1}} \operatorname{int} P^{(n,1)}$ is dense in X_1 and thus also in X, because N_f is nowhere dense. Finally, we define step functions φ_n on $\mathcal{P}_n$ by

$$\varphi_n(x) = \begin{cases} \varphi_{n,1}(x) & \text{for } x \in X_1 = X \setminus N_f, \\ f(x) & \text{for } x \in N_f. \end{cases}$$

So $\lim_{n\to\infty} \sup_{x\in X} |f(x) - \varphi_n(x)| = \lim_{n\to\infty} \sup_{x\in X_1} |f_1(x) - \varphi_{n,1}(x)| = 0$. This proves Theorem 5. □

For compact metrizable spaces X, in analogy with Theorem 2, the existence of "universal" chains $K = (\mathcal{P}_n)_{n=1}^{\infty}$ of almost semi-open partitions $\mathcal{P}_n$ can be proved in the sense that a given cliquish function f can be approximated by step functions on K simultaneously with all continuous functions on X.

Theorem 6. *Let f be a real-valued cliquish function on a compact metrizable space X. Then there exists a chain $K = (\mathcal{P}_n)_{n=1}^{\infty}$ of almost semi-open partitions of the space X such that f as well as any real-valued continuous function on X can be attained as the uniform limit of a sequence of almost semi-open step functions defined on K. If f is bounded, then K can be chosen to be a chain of finite partitions.*

PROOF. By Theorem 5, there exist a chain $(\mathcal{Q}_n)_{n=1}^{\infty}$ of almost semi-open partitions of X and step functions φ_n, $n \geq 1$, defined on the partitions $\mathcal{Q}_n$ which uniformly approach f. Furthermore we can assume that the partitions are finite if f is bounded.

Application of Theorem 2 to the constant function $f_0 \equiv 0$ yields a chain $(\mathcal{R}_n)_{n=1}^{\infty}$ of finite semi-open partitions of X such that every real-valued continuous function g on X can be written as the uniform limit of a sequence of step functions ψ_n, $n \geq 1$, which are defined on the partitions $\mathcal{R}_n$.

Now we can define the required chain $K = (\mathcal{P}_n)_{n=1}^{\infty}$ by putting $\mathcal{P}_n = \{Q \cap R : Q \cap R \neq \emptyset, Q \in \mathcal{Q}_n, R \in \mathcal{R}_n\}$; that is, with $\mathcal{P}_n$ formed by merging $\mathcal{Q}_n$ and $\mathcal{R}_n$. The chain K consists of almost semi-open partitions according to Lemma 2.

For fixed $n \geq 1$ the above mentioned step functions φ_n and ψ_n, which were defined on $\mathcal{Q}_n$ and $\mathcal{R}_n$, respectively, are step functions on the partition $\mathcal{P}_n$ as well, since $\mathcal{P}_n$ is a refinement of both $\mathcal{Q}_n$ and $\mathcal{R}_n$. Thus f as well as any real-valued continuous function g is the uniform limit of a sequence of step functions defined on the partitions from K.

If f is bounded, then K consists of finite partitions $\mathcal{P}_n$, since in this case the partitions $\mathcal{P}_n$ have been obtained by merging finite partitions. This completes the proof of Theorem 6. □

Remarks. 1. The uniform limit of a sequence of cliquish functions obviously is cliquish again. Combining this fact with the claims of Proposition 5 and

Theorem 5 yields a characterization of cliquish functions on arbitrary topological spaces X as uniform limits of almost semi-open step functions defined on chains of almost semi-open partitions of X.

2. According to Theorem 5, cliquish functions, similarly as quasi-continuous functions, can be arranged to form linear spaces, a chain $K = (\mathcal{P}_n)_{n=1}^{\infty}$ of almost semi-open partitions $\mathcal{P}_n$ being a common characteristic of cliquish functions belonging to the same linear space. Denoting the space of cliquish functions defined by a chain K of almost semi-open partitions $\mathcal{P}_n$ by $A_K(X)$, in correspondence with the notation of Remark 2 in Section 3, we again see that the set D_g of discontinuity points of a function $g \in A_K(X)$ is contained in the set B_K of boundary points of the partition elements of K. By the same arguments as in Section 3, we can reproduce the well-known result that the points of discontinuity of any cliquish function form a set of the first category.

3. A bounded cliquish function f on a topological space X can always be considered as an element of a space $A_K(X)$ defined by a chain K of finite almost semi-open partitions $\mathcal{P}_n$. Then all the functions $g \in A_K(X)$ are bounded, the space $A_K(X)$ itself being a Banach space with respect to the supremum norm.

4. In contrast with quasi-continuous functions, cliquish functions on a topological space X in their totality form a linear space. This well-known property appears as an immediate consequence of Theorem 5 and Lemma 2. If we restrict our attention to bounded cliquish functions, we recognize that they form a Banach space with respect to the supremum norm.

5. A central result of [5] and [13] says that every cliquish function f mapping a Baire space X into a separable metric space Y can be expressed as the uniform limit of a sequence of simply continuous functions f_n, $n \geq 1$. We recall that f_n is *simply continuous* if $f_n^{-1}(V)$ is simply open for every open set $V \subseteq Y$; that is, if $f_n^{-1}(V)$ is the union of an open set and a nowhere dense set (see [3]). Clearly, every almost semi-open step function φ is simply continuous. Thus Theorem 5 generalizes the result mentioned above for the case $Y = \mathbb{R}$ in so far as it refers to a very special kind of simply continuous functions, namely to almost semi-open step functions. Besides that, in contrast with the papers [5] and [13] which concern Baire spaces and perfect Baire spaces, respectively, no additional supposition on the underlying topological space X is required.

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References

[1] G. Aumann, *Reelle Funktionen*, Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen 68, Springer-Verlag, Berlin, Göttingen, Heidelberg, 1954.

[2] R. Bettazzi, *Sui punti di discontinuità delle funzioni di variabile reale*, Rendiconti del Circolo Matematico di Palermo, **6** (1892), 173–195.

[3] N. Biswas, *On some mappings in topological spaces*, Bull. Calcutta Math. Soc., **61** (1969), 127–135.

[4] W. W. Bledsoe, *Neighborly functions*, Proc. Amer. Math. Soc., **3** (1952), 114–115.

[5] J. Borsík, *Limits of simply continuous functions*, Real Anal. Exchange, **18** (1992/93), 270–275.

[6] T. Brodén, *Beiträge zur Theorie der stetigen Funktionen einer reellen Veränderlichen*, J. Reine Angew. Math., **118** (1897), 1–60.

[7] C. Bruteanu, *Asupra unor proprietăţi ale funcţiilor cvasicontinue*, Stud. Cerc. Mat., **22** (1970), 983–991.

[8] E. F. Collingwood, M. L. Cartwright, *Boundary theorems for a function meromorohic in the unit circle*, Acta Math., **87** (1952), 83–146.

[9] E. F. Collingwood, *On the linear and angular cluster sets of functions meromorphic in the unit circle*, Acta Math., **91** (1954), 165–185.

[10] E. F. Collingwood, *Sur le comportement à la frontière, d'une fonction méromorphe dans le cercle unité*, C. R. Acad. Sci. Paris, **240** (1955), 1502–1504.

[11] E. F. Collingwood, *On sets of maximum indetermination of analytic functions*, Math. Z., **67** (1957), 377–396.

[12] E. F. Collingwood, *Addendum: On sets of maximum indetermination of analytic functions*, Math. Z., **68** (1958), 498–499.

[13] J. Ewert, *Note on limits of simply continuous and cliquish functions*, Internat. J. Math. Math. Sci., **17** (1994), 447–450.

[14] H. Hahn, *Theorie der reellen Funktionen*, Band I, Verlag Julius Springer, Berlin, 1921.

[15] H. Hahn, *Reelle Funktionen, Erster Teil: Punktfunktionen*, Akademische Verlagsgesellschaft m.b.H., Leipzig, 1932, reprinted by Chelsea Publishing Company, New York, 1948.

[16] S. Kempisty, *Sur les fonctions quasicontinues*, Fund. Math., **19** (1932), 184–197.

[17] N. Levine, *Semi-open sets and semi-continuity in topological spaces*, Amer. Math. Monthly, **70** (1963), 36–41.

[18] J. S. Lipiński, T. Šalát, *On the points of quasicontinuity and cliquishness of functions*, Czechoslovak Math. J., **21 (96)** (1971), 484–489.

[19] S. Marcus, *Sur les fonctions quasicontinues au sens de S. Kempisty*, Colloq. Math., **8** (1961), 47–53.

[20] A. Neubrunnová, *On certain generalizations of the notion of continuity*, Mat. Časopis Sloven. Akad. Vied, **23** (1973), 374–380.

[21] A. Neubrunnová, *On quasicontinuous and cliquish functions*, Časopis Pěst. Mat., **99** (1974), 109–114.

[22] 0. Njåstad, *On some classes of nearly open sets*, Pacific J. Math., **15** (1965), 961–970.

[23] P. Painlevé, *Leçons sur la théorie analytique des équations différentielles professées à Stockholm 1895*, Herrmann, Paris, 1897.

[24] C. Richter, *n-Term approximation by controllable step functions*, Math. Nachr., **239-240** (2002), 177–197.

[25] C. Richter, I. Stephani, *Characterizing compact metrizable spaces among completely regular spaces*, Preprint, Jena, 1999.

[26] A. Schoenflies, *Die Entstehung der Lehre von den Punktmannigfaltigkeiten*, Jahresber. Deutsch. Math.-Verein, **8** (1900), Zweites Heft, 1–251.

[27] S. Shi, Q. Zheng, D. Zhuang, *Discontinuous robust mappings are approximatable*, Trans. Amer. Math. Soc., **347** (1995), 4943–4957.

[28] I. Ţevy, C. Bruteanu, *On some continuity notions*, Rev. Roumaine Math. Pures Appl., **18** (1973), 121–135.

[29] H. P. Thielman, *Types of functions*, Amer. Math. Monthly, **60** (1953), 156–161.

[30] B. S. Thomson, *Real functions*, Lecture Notes in Mathematics, 1170, Springer-Verlag, Berlin, 1985.

[31] J. D. Weston, *Some theorems on cluster sets*, J. London Math. Soc., **33** (1958), 435–441.