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Piotr Szuca, Department of Mathematics, Gdańsk University, ul. Wita Stwosza 57, 80-952 Gdańsk, Poland. email: pszuca@radix.com.pl

LOOPS OF INTERVALS AND DARBOUX BAIRE 1 FIXED POINT PROBLEM

Abstract

We show that the existence of a loop of covering intervals for connected G_δ real functions implies the existence of a periodic point with its period equal to the length of the cycle. As a corollary we get that the composition of N connected G_δ real functions from a compact interval into itself has a fixed point for every natural number N. In particular, the composition of finitely many derivatives from $[0,1]$ to $[0,1]$ has a fixed point. This solves a problem from [3].

1 Introduction

We will say that for a given function $f\colon \mathbb{R} \to \mathbb{R}$ a compact interval I_1 f-covers a compact interval I_2 if $f(I_1) \supset I_2$. We will write $I_1 \to_f I_2$ to denote such covering.

It is easy to observe that if f is continuous and $I \to_f I$, then f has a fixed point in I. Generally, if all functions f_i are continuous and $I_1 \to_{f_1} I_2 \to_{f_2} \cdots \to_{f_{N-1}} I_N \to_{f_N} I_1$, then there exists an $x \in I_1$ such that $f_1(x) \in I_2, (f_2 \circ f_1)(x) \in I_3, \ldots, (f_{N-1} \circ f_{N-2} \circ \cdots \circ f_1)(x) \in I_N$ and $(f_N \circ f_{N-1} \circ \cdots \circ f_1)(x) = x$. The technique of loops of covering intervals is widely used in combinatorial dynamics (see e.g. [1]); so generalizing this can give an easy way to rewrite such theorems for wider classes of functions.

The natural way to generalize continuity is to consider functions with the Darboux property, but that class is too big—there are known examples of Darboux functions that map an interval onto itself without any fixed points (see e.g. [2]). A smaller class of functions is the class of Darboux Baire 1 functions. In this paper we will show that the technique of loops of covering

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intervals works for the class of all real functions having a connected G_δ graph in $\mathbb{R}^2$. This class is a proper superclass of DB_1.

One of the reasons for studying the properties of such functions comes from the fact that there is no known characterization of derivatives. Certainly derivatives are in DB_1, but not every DB_1 function is a derivative. It is easy to see that the composition of two derivatives has the Darboux property, but it need not to be a derivative. Since functions with connected graphs have the Darboux property, it seems interesting to find out if the composition of two derivatives has a connected graph. That the composition of two derivatives has a fixed point was proved by Csörnyei, O'Neil and Preiss [3], and independently, by Elekes, Keleti and Prokaj [4]. A consequence of one of the results of this paper is that the composition of more than two derivatives has a fixed point, which solves Problem 2 from [3].

2 Preliminaries

By $\mathbb{I}$ we will denote the interval $[0,1]$. For every function f we will identify f with its graph. We will write f^N to denote $f \circ f \circ \cdots \circ f$ (N times). We will consider the following classes of functions from $\mathbb{R}$ into $\mathbb{R}$.

- D We will say that f is Darboux ($f \in \mathrm{D}$) if f has the intermediate value property; i.e. $[a,b] \to_f [f(a), f(b)]$ for every $a, b \in \mathbb{R}$; equivalently $f(I)$ is an interval for every interval $I \subset \mathbb{R}$.

- Conn We say that f is connected ($f \in \mathrm{Conn}$) if f is a connected subset of $\mathbb{R}^2$. We will also call such a function connectivity.

- B_1 The function f is Baire class 1 if f is a pointwise limit of a sequence of continuous functions. This is equivalent to the condition that $f^{-1}(G)$ is an F_σ subset of $\mathbb{R}$ for every open set $G \subset \mathbb{R}$.

- DB_1 We say that f is Darboux Baire 1 ($f \in \mathrm{DB}_1$) if f is Darboux and Baire 1.

- G_δ The function f is in the class G_δ if f is a G_δ subset of $\mathbb{R}^2$; i.e. $f = \bigcap_{n \in \mathbb{N}} G_n$, where all $G_n \subset \mathbb{R}^2$ are open.

For properties of these and other Darboux-like classes of functions see e.g. the survey [5]. In particular, it is known that $\mathrm{Conn} \subset \mathrm{D}$ and $\mathrm{DB}_1 \subset \mathrm{Conn} \cap G_\delta$ and that none of these inclusions is reversible. It follows that $\mathrm{Conn} = \mathrm{D}$ if we consider only Baire 1 functions. In the sequel we will also use the fact that every Darboux real function is bilaterally dense in itself and that the class D is closed under the operation of composition.

We will say that a class $\mathcal{F} \subset \mathrm{D}$ has the property $\mathcal{I}$ if for every finite sequence of functions $\{f_k\}_{k \le N} \subset \mathcal{F}$ and compact intervals $\{I_k\}_{k \le N}$ with $I_1 \to_{f_1} I_2 \to_{f_2} \cdots \to_{f_{N-1}} I_N \to_{f_N} I_1$ there exists an $x \in I_1$ such that $(f_N \circ f_{N-1} \circ \cdots \circ f_1)(x) = x$ and $(f_i \circ f_{i-1} \circ \cdots \circ f_1)(x) \in I_{i+1}$ for every $i = 1, \ldots, N-1$.

Remark 2.1. The class $\mathcal{C}$ of all continuous functions has the property $\mathcal{I}$.

The main aim of this paper is to strengthen Remark 2.1 and show that the class of connected G_δ real functions has the property $\mathcal{I}$ (see Theorem 3.2). The proof of Sharkovskiĭ's theorem presented in [6] shows that some facts known from the combinatorial dynamics of continuous functions are also valid for every class of functions with property $\mathcal{I}$.

Remark 2.2. Suppose that $M \in \mathbb{N}$ and $f = g_M \circ g_{M-1} \circ \cdots \circ g_1$ is the composition of Darboux functions. If $I_1, I_2 \subset \mathbb{R}$ are compact intervals and $I_1 \to_f I_2$, then there exist compact intervals $J_1, J_2, \ldots, J_{M+1}$ such that $I_1 = J_1 \to_{g_1} J_2 \to_{g_2} \cdots \to_{g_M} J_{M+1} = I_2$.

Using this remark we can easily get the following.

Remark 2.3. Suppose that $\mathcal{G} \subset \mathrm{D}$ has property $\mathcal{I}$. Then the class $\mathcal{F} = \{g_M \circ g_{M-1} \circ \cdots \circ g_1 \mid M \in \mathbb{N} \text{ and } g_i \in \mathcal{G} \text{ for all } i \le M\}$ also has property $\mathcal{I}$.

3 Property $\mathcal{I}$ for the Class of all Connected G_δ Functions

The proof of Theorem 3.2 is based on the following lemma. (Note that Theorem 3.2 is a generalization of Lemma 3.1.)

Lemma 3.1 ([6], Co. 1)**.** *If $f : \mathbb{R} \to \mathbb{R}$ is a connected G_δ function, $N \in \mathbb{N}$, $I_1, \ldots, I_N$ are compact intervals with $I_1 \to_f I_2 \to_f \cdots \to_f I_N \to_f I_1$, then there exists an $x \in I_1$ such that $f^N(x) = x$ and $f^i(x) \in I_{i+1}$ for every $i < N$.*

Theorem 3.2. *The class of connected G_δ real functions has property $\mathcal{I}$; i.e. for every loop of compact intervals $I_1 \to_{g_1} I_2 \to_{g_2} \to \cdots \to_{g_{N-1}} I_N \to_{g_N} I_1$ there exists an $x \in I_1$ such that $(g_N \circ g_{N-1} \circ \cdots \circ g_1)(x) = x$ and $(g_i \circ g_{i-1} \circ \cdots \circ g_1)(x) \in I_{i+1}$ for every $i < N$.*

PROOF. Suppose that $I_1 \to_{g_1} I_2 \to_{g_2} \cdots \to_{g_{N-1}} I_N \to_{g_N} I_1$. During the remainder of the proof for any $\tau \in \mathbb{R}$ we will use the notation $I_i + \tau$ for the interval $\{x + \tau \mid x \in I_i\}$.

Let $\delta \in \mathbb{R}$ be such that $\bigcup_{i \le N} I_i \subset (-\frac{\delta}{2}, \frac{\delta}{2})$. Obviously, intervals from the family $\mathcal{J} = \{I_i + i\delta\}_{i \le N}$ are pairwise disjoint. Define $h : \bigcup \mathcal{J} \to \mathbb{R}$ by

$$h(x) = \begin{cases} g_i(x - i\delta) + (i+1)\delta & \text{if } x \in I_i + i\delta \text{ for some } i < N; \\ g_N(x - N\delta) + \delta & \text{if } x \in I_N + N\delta. \end{cases}$$

Let $f\colon \mathbb{R} \to \mathbb{R}$ be the extension of h such that f is linear on the closure of any component of the set $\mathbb{R} \setminus \bigcup \mathcal{J}$. Clearly, f is connected G_δ and

$$(I_1 + \delta) \to_f (I_2 + 2\delta) \to_f \cdots \to_f (I_N + N\delta) \to_f (I_1 + \delta);$$

so by Lemma 3.1 there exists an $y \in I_1 + \delta$ such that $f^N(y) = y$ and $f^i(y) \in I_{i+1} + (i+1)\delta$ for each $i < N$. Then for $x = y - \delta$ it is easy to observe that $(g_N \circ g_{N-1} \circ \cdots \circ g_1)(x) = x$ and $(g_i \circ g_{i-1} \circ \cdots \circ g_1)(x) \in I_{i+1}$ for every $i < N$. $\square$

From Remark 2.3 we get also the following fact, which is stronger than Theorem 3.2 because the composition of connected G_δ functions does not have to be G_δ (see an example in [6]).

Corollary 3.3. *The class of all compositions of connected G_δ functions has property $\mathcal{I}$.*

4 Fixed Point of a Composition of Connected G_δ Functions

The existence of a fixed point of a composition of finitely many connected G_δ real functions is a consequence of Corollary 3.3 and the following lemma.

Lemma 4.1. *Suppose that $f\colon \mathbb{I} \to \mathbb{I}$ is a Darboux function. Then there exists a compact interval $I \subset \mathbb{I}$ such that $I \to_f I$.*

Proof. Suppose that $f\colon \mathbb{I} \to \mathbb{I}$ is a Darboux function. There are two possibilities:

1. The function f has a fixed point x_0.
2. For every $x \in \mathbb{I}$ either $f(x) < x$ or $f(x) > x$.

In the first case the degenerate interval $\{x_0\}$ covers itself: so for the rest of the proof we will assume that f has no fixed point.

Let

- $A = \{x \in \mathbb{I} \mid x < f(x)\}$;
- $B = \{x \in \mathbb{I} \mid x > f(x)\}$.

Clearly, $A \cup B = \mathbb{I}$, $A \cap B = \emptyset$, $0 \in A$ and $1 \in B$. Let $s = \sup A$. Since f is bilaterally dense in itself, $s \in B$ and $s > 0$. Again, since f is bilaterally dense in itself and $s \in B$, there exists an $p < s$ such that $p > f(p)$. Take $q \in (p, s) \cap A$. Thus, $f(p) < p < q < f(q)$ and by the Darboux property of f, $[p, q] \to_f [p, q]$. $\square$

Theorem 4.2. *Suppose that $N \in \mathbb{N}$ and $g_i \colon \mathbb{I} \to \mathbb{I}$ is a connected G_δ function for each $i \leq N$. Then the composition $g_1 \circ g_2 \circ \cdots \circ g_N$ possesses a fixed point.*

PROOF. Let $g = g_1 \circ g_2 \circ \cdots \circ g_N$. Since g is Darboux, by Lemma 4.1 there exists a $I \subset \mathbb{I}$ such that $I \to_g I$. So, by Corollary 3.3 there exists an $x \in I$ such that $g(x) = x$. $\square$

The next corollary answers the question from [3].

Corollary 4.3. *Suppose that $N \in \mathbb{N}$ and $g_i \colon \mathbb{I} \to \mathbb{I}$ is* DB_1 *(in particular, derivative) for every $i \leq N$. Then $g_1 \circ g_2 \circ \cdots \circ g_N$ has a fixed point.*

5 Problems

It is known that an assertion analogous to Theorem 3.2 (or 4.2) does not hold for Darboux Borel measurable functions (see examples in [6]). The main reason for this fact is that the proof of Lemma 3.1 strongly uses the connectivity of f. However, it is natural to ask whether the assumption that f is G_δ in Theorems 3.2 and 4.2 can be weakened to Borel measurability of f.

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