Real Analysis Exchange
Vol. 29(2), 2003/2004, pp. 921–930

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REGULAR DEPENDENCE OF TOTAL VARIATION ON PARAMETERS

Abstract

Let X be an interval, Y a metric space, T a set of parameters, and $f : T \times X \to Y$ a function. For given $t \in T$ denote by $v(t)$ the total variation of $f(t, \cdot)$ on X. We look for sufficient conditions for regular (measurable, continuous, etc.) dependence of v on t.

1 Preliminaries

Throughout this paper X is an interval (open, closed, half-closed, bounded or not) on the real line, and (Y, d) is a metric space. Given a mapping $g : X \to Y$ we define the *total variation of g on X* as

$$V(g, X) = \sup_{\Pi} \sum_{i=1}^{n} d(g(x_i), g(x_{i-1})),$$

where the supremum is taken over all partitions $\Pi = \{x_0, x_1, \ldots, x_n\}$ of X (i.e., $n \in \mathbb{N}$, $x_0 < x_1 < \cdots < x_n$ and $x_i \in X$, $i = 0, 1, \ldots, n$). We say that g is of *bounded variation* if $V(g, X) < \infty$.

Let T be a nonempty set of parameters and $f : T \times X \to Y$ a mapping. Let $v : T \to [0, \infty]$ be given by $v(t) = V(f(t, \cdot), X)$, i.e., $v(t)$ is the total variation of $f(t, \cdot)$ on X. We shall look for sufficient conditions for regular (measurable, continuous, etc.) dependence of v on t. This problem appeared in the

Key Words: function of bounded variation, measurability, continuity.
Mathematical Reviews subject classification: Primary 26A45; Secondary 26B99.
Received by the editors July 29, 2003
Communicated by: B. S. Thomson

study of regular selectors for multifunctions of bounded variation depending on parameters ([2]).

Let Z be a topological space. We say that Z is *Polish* if it is separable and can be metrizable by a complete metric. By $\mathcal{B}(Z)$ we denote the Borel σ-field on Z. Suppose T is endowed with a σ-field $\mathcal{T}$. Then $\mathcal{T} \otimes \mathcal{B}(Z)$ denotes the product σ-field on $T \times Z$.

We shall need some further terminology. Denote by $\mathcal{N}$ and $\mathcal{N}^*$, respectively, the sets of infinite and finite sequences of positive integers. Let $\mathcal{F}$ be a family of sets. We say that A is *obtained from $\mathcal{F}$ by the Suslin operation* if

$$A = \bigcup_{\sigma \in \mathcal{N}} \bigcap_{n=1}^{\infty} F(\sigma_1, \ldots, \sigma_n)$$

for some function $F : \mathcal{N}^* \to \mathcal{F}$. Denote by $S(\mathcal{F})$ the family of all sets obtained from $\mathcal{F}$ by the Suslin operation. The family $\mathcal{F}$ is *closed with respect to the Suslin operation* if $S(\mathcal{F}) = \mathcal{F}$ (cf. [4],[6])

Recall that a σ-field $\mathcal{T}$ is closed with respect to the Suslin operation provided one of the following conditions is satisfied: (i) $\mathcal{T}$ is complete with respect to a σ-finite measure; (ii) T is a topological space and $\mathcal{T}$ is the Baire σ-field, i.e., $\mathcal{T}$ is the family of all subsets of T having the Baire property; (iii) T is a locally compact space and $\mathcal{T}$ is the family of all subsets of T measurable with respect to a Radon measure (see e.g. [4],[6]).

Let A be a subset of $T \times Z$. Given $t \in T$, we denote by A_t the t-section of A; i.e., $A_t = \{z : (t, z) \in A\}$. By $\mathrm{Proj}_T A$ we mean the projection of A on T; i.e., $\mathrm{Proj}_T A = \{t \in T : (t, z) \in A \text{ for some } z \in Z\}$.

Let $\mathcal{T}$ be a σ-field on T and $A \in \mathcal{T} \otimes \mathcal{B}(Z)$. In general, $\mathrm{Proj}_T A$ does not belong to $\mathcal{T}$. We shall use the following well known results:

Projection Theorem. *(see e.g. [4, Theorem 1.3]) If $(T, \mathcal{T})$ is a measurable space, Z is a Polish space and $A \in \mathcal{T} \otimes \mathcal{B}(Z)$, then $\mathrm{Proj}_T A \in S(\mathcal{T})$.*

Arsenin-Kunugui-Novikov Theorem. *(e.g. [6, Theorem 18.18]) Suppose T and Z are Polish spaces, and $A \in \mathcal{B}(T \times Z)$ has σ-compact t-sections. Then $\mathrm{Proj}_T A \in \mathcal{B}(T)$.*

2 Measurability

Throughout this section $(T, \mathcal{T})$ is a measurable space, (Y, d) is a separable metric space, $f : T \times X \to Y$ is a function, and v is the total variation of $f(t, \cdot)$ on X. We are interested in the measurability properties of the function v. Let us start with some examples, where $T = X = Y = [0, 1]$.

Example 2.1. Lebesgue measurable function $f : [0,1]^2 \to [0,1]$ such that v is not Lebesgue measurable.

Let A be a nonmeasurable subset of $(0,1)$, $B = \{(t,x) : t = x \in A\}$, and $f = \chi_B$, the characteristic function of B. Clearly, f is Lebesgue measurable, and $v(t) = 2$ for $t \in A$ and $v(t) = 0$ for $t \notin A$.

Example 2.2. Function f measurable with respect to the Baire σ-field on $[0,1]^2$ such that v is not Baire measurable.

This is an obvious modification of the previous example with $A \subset (0,1)$ without the Baire property.

Example 2.3. Borel measurable function $f : [0,1]^2 \to [0,1]$ such that v is not Borel measurable.

Let P be the set of all irrationals from [0,1], and $A \subset [0,1]$ an analytic non-Borel set. There is a continuous and onto function $h : P \to A$. Let $D = \{(h(x), x) : x \in P\}$; i.e., D is the graph of h. Since h is continuous, D is a closed subset of $[0,1] \times P$ and, consequently, a Borel subset of $[0,1]^2$. Let $f = \chi_D$. Then $v(t) \geq 2$ for $t \in A$ and $v(t) = 0$ for $t \notin A$. Hence, v is not Borel measurable.

Since $\mathcal{B}([0,1]^2) = \mathcal{B}([0,1]) \otimes \mathcal{B}([0,1])$, the last example shows that if f is $\mathcal{T} \otimes \mathcal{B}(X)$-measurable, then v need not be $\mathcal{T}$-measurable.

We start with a theorem which gives sufficient conditions for the measurability of the total variation for an arbitrary measurable space $(T, \mathcal{T})$.

Theorem 2.1. *Suppose there exists a countable dense subset E of X such that the following conditions are satisfied:*

(i) For each $(t,x) \in T \times X$ the value $f(t,x)$ is the limit of $(f(t,e_k))$ for some sequence (e_k) of points of E convergent to x.

(ii) For each $e \in E$, $f(\cdot, e)$ is measurable.

Then v is measurable.

Proof. Fix $t \in T$. For each partition $\Pi = \{x_0, \ldots, x_n\}$ of X and each $\varepsilon > 0$ there exists a partition $\Pi' = \{x'_0, \ldots, x'_n\}$ consisting of points of E, such that

$$\sum_{i=1}^{n} d(f(t,x_i), f(t,x_{i-1})) \leq \varepsilon + \sum_{i=1}^{n} d(f(t,x'_i), f(t,x'_{i-1})).$$

Hence,

$$v(t) = \sup_{\Pi'} \sum_{i=1}^{n} d(f(t,x'_i), f(t,x'_{i-1})),\ t \in T,$$

where supremum is taken over all partitions of X consisting of elements of E. Under our assumptions, for each such a partition Π' the function

$$t \mapsto \sum_{i=1}^{n} d(f(t,x'_i), f(t,x'_{i-1})),\ t \in T$$

is measurable. Thus v is measurable, as the pointwise supremum of a countable family of measurable functions. □

Remark 2.1. We list some cases, when the condition (i) of the last theorem is satisfied for each dense subset $E \subset X$:

1. f is continuous in x.
2. f is one-sided continuous in x (possibly, left-continuous at some points and right-continuous at others).
3. f is quasi-continuous in x (see e.g. [9]).

We shall use the following technical lemmas.

Lemma 2.1. *For any $r \geq 0$ we have $\{t \in T : v(t) > r\} = \bigcup\{\mathrm{Proj}_T A_n : n \in \mathbb{N}\}$, where*

$$A_n = \{(t, x_0, \ldots, x_n) \in T \times X^{n+1} : x_0 < x_1 < \cdots < x_n \quad \sum_{i=1}^{n} d(f(t,x_i), f(t,x_{i-1})) > r\}.$$

PROOF. Indeed, $v(t) > r$ iff there is a partition $\{x_0, x_1, \ldots, x_n\}$ of X such that $\sum_{i=1}^{n} d(f(t,x_i), f(t,x_{i-1})) > r$. □

Lemma 2.2. *If f is $\mathcal{T} \otimes \mathcal{B}(X)$-measurable, then $A_n \in \mathcal{T} \otimes \mathcal{B}(X^{n+1})$.*

PROOF. Let an auxiliary function $g : T \times X^{n+1} \to \mathbb{R}$ be defined by

$$g(t, x_0, \ldots, x_n) = \sum_{i=1}^{n} d(f(t,x_i), f(t,x_{i-1})).$$

Since Y is separable, g is $\mathcal{T} \otimes \mathcal{B}(X^{n+1})$-measurable. Now the measurability of A_n follows from $A_n = T \times U_n \cap g^{-1}((r,\infty))$, where $U_n = \{x \in X^{n+1} : x_0 < \cdots < x_n\}$ is open in X^{n+1}. □

Theorem 2.2. *If f is $\mathcal{T} \otimes \mathcal{B}(X)$-measurable then v is $\sigma(S(\mathcal{T}))$-measurable.*

Proof. Fix $r \geq 0$. By Lemma 2.2, $A_n \in \mathcal{T} \otimes \mathcal{B}(X^{n+1})$, $n \in \mathbb{N}$. Since X^{n+1} is a Polish space, we can use the Projection Theorem. Thus $\mathrm{Proj}_T A_n \in S(\mathcal{T})$ and, consequently, v is measurable with respect to the σ-field generated by $S(\mathcal{T})$. □

Corollary 2.1. *If the σ-field $\mathcal{T}$ is closed under the Suslin operation and f is $\mathcal{T} \otimes \mathcal{B}(X)$-measurable, then v is $\mathcal{T}$-measurable.*

Corollary 2.2. *Suppose T is a Lebesgue measurable subset of $\mathbb{R}^m$ and $\mathcal{T} = \mathcal{L}_m(T)$. If f is $\mathcal{L}_m(T) \otimes \mathcal{B}(X)$-measurable, then v is Lebesgue measurable.*

Corollary 2.3. *If T is a Polish space and f is Borel measurable, then v is measurable with respect to the σ-field generated by analytic subsets of T.*

Remind that a function $h : Z \to Y$, where Z is metrizable, is of the *1st class of Baire* if $h^{-1}(U) \in F_\sigma(Z)$ for each open $U \subset Y$. If h is the pointwise limit of a sequence of continuous functions, then it is of the 1st class of Baire. If Y is an interval, then these two conditions are equivalent. The function h is of the *2nd class of Baire* if for each open $U \subset Y$, $h^{-1}(U) \in G_{\delta\sigma}(Z)$.

Theorem 2.3. *Suppose T is a Polish space, $f : T \times X \to Y$ is Borel measurable, and for each $t \in T$, $f(t, \cdot)$ is of the 1st class of Baire. Then v is Borel measurable.*

Proof. Since Y is separable, for fixed t the function

$$(x_0, \ldots, x_n) \mapsto \sum_{i=1}^{n} d(f(t, x_i), f(t, x_{i-1}))$$

is of the 1st class of Baire on X^{n+1}. Let A_n be the set defined in Lemma 2.1 for fixed $r \geq 0$. By Lemma 2.2, $A_n \in \mathcal{T} \otimes \mathcal{B}(X^{n+1})$. Moreover, for each $t \in T$ we have

$$(A_n)_t = \{(x_0, \ldots, x_n) \in X^{n+1} : x_0 < \cdots < x_n,\ \sum_{i=1}^{n} d(f(t, x_i), f(t, x_{i-1})) > r\},$$

and this set belongs to $F_\sigma(X^{n+1})$. Since X^{n+1} is σ-compact, $(A_n)_t$ is also σ-compact. By the Arsenin-Kunugui-Novikov Theorem, $\mathrm{Proj}_T A_n \in \mathcal{B}(T)$. Thus v is Borel measurable. □

Corollary 2.4. *If T is a Polish space, f is Borel measurable, and for each $t \in T$ $f(t, \cdot)$ is of bounded variation, then v is Borel.*

Proof. Being of bounded variation, $f(t,\cdot)$ has at most countable set of discontinuity points (see e.g., [1],[5]). Consequently, $f(t,\cdot)$ is of the 1st class of Baire (cf. [7, 34.VII]). Now we can apply Theorem 2.3. $\square$

Example 2.3 shows that the assumption of the bounded variation of $f(t,\cdot)$ is essential for the Borel measurability of v.

If we assume that f is of the 1st class of Baire as a function of two variables, then we can strengthen the thesis of Theorem 2.3.

Theorem 2.4. *If T is metrizable and σ-compact, and f is of the 1st class of Baire, then v is of the 2nd class.*

Proof. Fix $r \geq 0$, and let A_n be such as in Lemma 2.1. Under our assumptions, the function g defined in the proof of Lemma 2.2 is of the 1st class of Baire on $T \times X^{n+1}$. Consequently, $A_n \in F_\sigma(T \times X^{n+1})$. Since $T \times X^{n+1}$ is σ-compact, A_n is σ-compact, and $\mathrm{Proj}_T A_n$ is also σ-compact. Hence, $\{t \in T : v(t) > r\} \in F_\sigma(T)$. Moreover, $\{t \in T : v(t) < r\} = T \setminus \{t \in T : v(t) \geq r\} = T \setminus \bigcap_{k\in\mathbb{N}} \{t \in T : v(t) > r + \frac{1}{k}\} \in G_{\delta\sigma}(T)$. It completes the proof. $\square$

Remark 2.2. Example 2.3 shows that there is no analogous result for functions of the next classes of Baire. The function $f = \chi_D$ from this example is of the 2nd class of Baire, and v is not Borel measurable. It follows from the fact that D is a G_δ-set in $[0,1]^2$, as a closed subset of $[0,1] \times P$.

3 Continuity

In this section we collect some simple observations concerning the continuity properties of the function v. We shall assume that T is a topological space. First consider the following example:

Example 3.1. Continuous function f such that v is discontinuous.
Let $T = [0,1]$, $X = (0,1]$, $Y = \mathbb{R}$ and $f(t,x) = x^t$. Then $v(0) = 0$ and $v(t) = 1$ for $t > 0$.

Note that the function v in this example is lower semicontinuous.

Theorem 3.1. *If f is continuous in t, then v is lower semicontinuous.*

Proof. For each partition $\Pi = \{x_0, \ldots, x_n\}$ of X the function

$$t \mapsto \sum_{i=1}^{n} d(f(t,x_i), f(t,x_{i-1}))$$

is continuous. Hence, v is lower semicontinuous, as the pointwise supremum of continuous functions. $\square$

As a consequence, we obtain the following well known property of the total variation:

Corollary 3.1. *If $g_k, g : X \to Y$ and the sequence (g_k) pointwise converges to g, then*

$$V(g, X) \leq \liminf_{k\to\infty} V(g_k, X).$$

PROOF. In order to apply Theorem 3.1, we put $T = \{\frac{1}{k} : k \in \mathbb{N}\} \cup \{0\}$, $f(\frac{1}{k}, x) = g_k(x)$ for $k \in \mathbb{N}$, and $f(0, x) = g(x)$. Then $v(\frac{1}{k}) = V(g_k, X)$ and $v(0) = V(g, X)$. By the lower semicontinuity of v at 0, $\liminf_{k\to\infty} v(\frac{1}{k}) \geq v(0)$. $\square$

Now we give a condition equivalent to the continuity of the function v.

Theorem 3.2. *Let T be a compact metric space, (Y, d) a metric space, and $f : T \times [a, b] \to Y$. Assume that $f(\cdot, x)$ is continuous for each $x \in [a, b]$, and $f(t, \cdot)$ is continuous of bounded variation for each $t \in T$. The function v is continuous on T if and only if the sequence (φ_n) of functions $\varphi_n : T \to \mathbb{R}$ given by*

$$\varphi_n(t) = \sum_{k=1}^{2^n} d\left(f\left(t, a + (k-1)\frac{b-a}{2^n}\right), f\left(t, a + k\frac{b-a}{2^n}\right)\right), \ t \in T,$$

is uniformly convergent on T.

PROOF. Since for a fixed $t \in T$ the function $f(t, \cdot)$ is continuous on $[a, b]$, $v(t) = \lim_{n\to\infty} \varphi_n(t)$ (see [8]; the proof given there for the real-valued case also holds for functions with values in a metric space). ¿From the continuity of $f(\cdot, x)$ (for every x) it follows that the functions φ_n are continuous on T.

By the above remarks, the proof of sufficiency is obvious. Let us show necessity. Observe that for $k \in \{1, \ldots, 2^n\}$ we have

$$\begin{aligned} & d\left(f\left(t, a + (k-1)\frac{b-a}{2^n}\right), f\left(t, a + k\frac{b-a}{2^n}\right)\right) \\ \leq\ & d\left(f\left(t, a + (2k-2)\frac{b-a}{2^{n+1}}\right), f\left(t, a + (2k-1)\frac{b-a}{2^{n+1}}\right)\right) \\ & + d\left(f\left(t, a + (2k-1)\frac{b-a}{2^{n+1}}\right), f\left(t, a + 2k\frac{b-a}{2^{n+1}}\right)\right). \end{aligned}$$

Consequently, $0 \leq \varphi_n \nearrow v$, and φ_n, v are continuous on T. Hence by the Dini theorem, (φ_n) converges to v uniformly on T. $\square$

Let (Y, d) be a metric space. Remind that a function $g : [a, b] \to Y$ is *absolutely continuous* if for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any finite number of points $a \le a_1 < b_1 \le a_2 < b_2 \le \cdots \le a_n < b_n \le b$ the condition $\sum_{i=1}^{n}(b_i - a_i) < \delta$ implies $\sum_{i=1}^{n} d(g(b_i), g(a_i)) < \varepsilon$. It appears that such functions have analogous properties as in the real-valued case. Recently V. V. Chistyakov and R. E. Svetic [3] obtained the following result.

Theorem 3.3. *If $g : [a, b] \to Y$ is absolutely continuous, then for almost all $x \in (a, b)$ there exists the limit (called the metric derivative of g at x)*

$$|g'|(x) = \lim_{h \to 0} \frac{d(g(x + h), g(x))}{h},$$

and the following integral formula holds

$$V(g, [a, b]) = \int_a^b |g'|(x)\, dx.$$

This integral formula is an extension of the corresponding result for continuously differentiable functions with values in a normed space (cf. [1]).

For $f : T \times [a, b] \to Y$ absolutely continuous in the second variable we denote by $|f'_x|(t, \cdot)$ the metric derivative of $f(t, \cdot)$. Hence,

$$v(t) = \int_a^b |f'_x|(t, x)\, dx.$$

In order to apply this formula in the study of the continuity of v, we have to assume that the metric derivative $|f'_x|$ depends continuously on t. But for fixed $x \in (a, b)$, the function $|f'_x|(\cdot, x)$ may be defined not for all $t \in T$ (or even undefined). Therefore we shall assume that $|f'_x|$ can be extended to a function g defined on $T \times [a, b]$ which satisfies some regularity conditions. For example, for $f(t, x) = |t - x|$ the function $g \equiv 1$ is such an extension. More precisely, we impose the following assumption:

(A) The function $f(t, \cdot)$ is absolutely continuous for every $t \in T$, and there exists a function $g : T \times [a, b] \to \mathbb{R}$ continuous in t and such that for each $t \in T$, $g(t, x) = |f'_x|(t, x)$ for almost all $x \in [a, b]$.

In particular, the assumption (A) is satisfied if Y is a normed space, f is differentiable in the second variable and this derivative is continuous on $T \times [a, b]$.

Theorem 3.4. *Let T and Y be metric spaces and $f : T \times [a, b] \to Y$. Assume (A). Moreover, assume that there exists an integrable function $h : [a, b] \to \mathbb{R}$ such that $g(t, x) \le h(x)$ for all $(t, x) \in T \times [a, b]$. Then v is continuous.*

Proof. Let $t_0 \in T$ and a sequence $(t_n) \subset T$ converging to t_0 be arbitrary but fixed. By the Lebesgue Dominated Convergence Theorem we obtain

$$\begin{aligned}\lim_{k\to\infty} v(t_k) &= \lim_{k\to\infty} \int_a^b |f'_x|(t_k,x)|\,dx = \lim_{k\to\infty} \int_a^b g(t_k,x)\,dx \\ &= \int_a^b g(t_0,x)\,dx = v(t_0).\end{aligned}$$

It shows the continuity of v. $\square$

In the next theorem we prove the Lipschitz continuity of the function v.

Theorem 3.5. *Let (T,ϱ) and (Y,d) be metric spaces, and $f : T\times[a,b]\to Y$. Assume (A). Moreover, suppose that $g(\cdot,x)$ satisfies the Lipschitz condition; i.e., $|g(s,x)-g(t,x)| \le h(x)\varrho(s,t)$ for all $s,t\in T$, $x\in[a,b]$, where $h : [a,b]\to \mathbb{R}$ is an integrable function. Then the function v is Lipschitzian.*

Proof. Fix $s,t\in T$. Then

$$\begin{aligned}|v(s)-v(t)| &= \left|\int_a^b |f'_x|(s,x)\,dx - \int_a^b |f'_x|(t,x)\,dx\right| \\ &= \left|\int_a^b g(s,x)\,dx - \int_a^b g(t,x)\,dx\right| \le \int_a^b |g(s,x)-g(t,x)|\,dx \\ &\le \left(\int_a^b h(x)\,dx\right)\varrho(s,t).\end{aligned}$$ $\square$

Remark 3.1. Since we use the Lebesgue integral, it suffices if the estimations in the assumptions of Theorems 3.4 and 3.5 hold for almost all $x\in[a,b]$.

Acknowledgement. We are grateful to V. V. Chistyakov (Nizhny Novgorod, Russia) for letting us know Theorem 3.3 from his joint paper with R. E. Svetic [3] and for helpful comments. The research of the second and third author was supported by the Department of Mathematics of the Silesian University in Katowice.

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