Real Analysis Exchange
Vol. 29(2), 2003/2004, pp. 867–880

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AN EQUIVALENCE THEOREM FOR INTEGRAL CONDITIONS RELATED TO HARDY'S INEQUALITY

Abstract

Let $1 < p \leq q < \infty$. Inspired by some recent results concerning Hardy type inequalities we state and prove directly the equivalence of four scales of integral conditions. By applying our result to the original Hardy type inequality situation we obtain a new proof of a number of characterizations of the Hardy inequality and obtain also some new weight characterizations. As another application we prove some new weight characterizations for embeddings between some Lorentz spaces.

1 Introduction

We consider the general one-dimensional Hardy inequality

$$\left(\int_0^b \left(\int_0^x f(t)\,dt\right)^q u(x)\,dx\right)^{1/q} \leq C\left(\int_0^b f^p(x)v(x)\,dx\right)^{1/p} \tag{1.1}$$

with a fixed b, $0 < b \leq \infty$, for measurable functions $f \geq 0$, non-negative weights u and v and for the parameters p, q satisfying $1 < p \leq q < \infty$. The inequality (1.1) is usually characterized by the (Muckenhoupt) condition

$$A_1 := \sup_{0<x<b} A_M(x) < \infty, \tag{1.2}$$

Key Words: Inequalities, Hardy's inequality, Weights, Scales of weight characterizations, Hardy operator, Continuity, equivalent integral conditions, comparisons
Mathematical Reviews subject classification: Primary 26D10, 26D15; Secondary 47B07, 47B38
Received by the editors December 19, 2003
Communicated by: Peter Bullen

where

$$A_M(x) := \left(\int_x^b u(t)\,dt\right)^{1/q} \left(\int_0^x v^{1-p'}(t)\,dt\right)^{1/p'}.$$

Here and in the sequel $p' = p/(p-1)$. Further, let

$$U(x) := \int_x^b u(t)\,dt, \text{ and } V(x) := \int_0^x v^{1-p'}(t)\,dt,$$

and assume that $U(x) < \infty$, $V(x) < \infty$ for every $x \in (0,b)$. As was shown in [3], the validity of Hardy's inequality (1.1) for all functions $f \geq 0$ in fact can be characterized e.g. by prescribing that any of the following expressions is finite:

$$\begin{aligned}
A_M &:= \sup_{0<x<b} U^{1/q}(x) V^{1/p'}(x),\\
A_{PS} &:= \sup_{0<x<b} \left(\int_0^x u(t) V^q(t)\,dt\right)^{1/q} V^{-1/p}(x),\\
A_W(r) &:= \sup_{0<x<b} \left(\int_x^b u(t) V^{q(p-r)/p}(t)\,dt\right)^{1/q} V^{(r-1)/p}(x),\\
&\qquad \text{for any } 1 < r < p \qquad\qquad (1.3)\\
A_{PS}^* &:= \sup_{0<x<b} \left(\int_x^b v^{1-p'}(t) U^{p'}(t)\,dt\right)^{1/p'} U^{-1/q'}(x);\\
A_W^*(r) &:= \sup_{0<x<b} \left(\int_0^x v^{1-p'}(t) U^{p'(q'-r)/q'}(t)\,dt\right)^{1/p'} U^{(r-1)/q'}(x)\\
&\qquad \text{for any } 1 < r < q'.
\end{aligned}$$

This paper is organized as follows. In Section 2 we prove an equivalence theorem of independent interest (see Theorem 1). In Section 3 we use this equivalence theorem to prove some scales of weight characterizations of the Hardy inequality, which includes all results mentioned in (1.3) but also some new weight characterizations (see e.g. Corollary 1). In Section 4 we use the equivalence theorem to prove some weight characterizations for embeddings between weighted Lorentz spaces, thus extending previous results of E. Sawyer [7] and V. D. Stepanov [8] (see Theorems 4.1 and 4.2).

2 The Equivalence Theorem

Our main result in this Section is the following equivalence theorem.

Theorem 1. *For $-\infty \leq a < b \leq \infty$, α, β and s positive numbers and f, g measurable functions positive a.e. in (a, b), let*

$$F(x) := \int_x^b f(t)\,dt, \quad G(x) := \int_a^x g(t)\,dt \tag{2.1}$$

and

$$\begin{aligned}
B_1(x;\alpha,\beta) &:= F^{\alpha}(x)G^{\beta}(x),\\
B_2(x;\alpha,\beta,s) &:= \left(\int_x^b f(t)G^{\frac{\beta-s}{\alpha}}(t)\,dt\right)^{\alpha} G^{s}(x),\\
B_3(x;\alpha,\beta,s) &:= \left(\int_a^x g(t)F^{\frac{\alpha-s}{\beta}}(t)\,dt\right)^{\beta} F^{s}(x),\\
B_4(x;\alpha,\beta,s) &:= \left(\int_a^x f(t)G^{\frac{\beta+s}{\alpha}}(t)\,dt\right)^{\alpha} G^{-s}(x),\\
B_5(x;\alpha,\beta,s) &:= \left(\int_x^b g(t)F^{\frac{\alpha+s}{\beta}}(t)\,dt\right)^{\beta} F^{-s}(x).
\end{aligned}$$

The numbers $B_1 := \sup_{a<x<b} B_1(x;\alpha,\beta)$ and $B_i = \sup_{a<x<b} B_i(x;\alpha,\beta,s)$ $(i = 2,3,4,5)$ are mutually equivalent. The constants in the equivalence relations can depend on α, β and s.

Remark 1. The proof of Theorem 1 is carried out by deriving positive constants c_i and d_i so that

$$c_i \sup_{a<x<b} B_i(x;\alpha,\beta,s) \leq \sup_{a<x<b} B_1(x;\alpha,\beta) \leq d_i \sup_{a<x<b} B_i(x;\alpha,\beta,s), i = 2,3,4,5,$$

see (2.5), (2.6)-(2.12). This information is useful e.g. for obtaining good estimates of the best constant in (1.1).

PROOF.

$$\sup_{a<x<b} B_1(x;\alpha,\beta) \approx \sup_{a<x<b} B_2(x;\alpha,\beta,s) \tag{I}$$

(i) Let $s \leq \beta$. Then $\frac{\beta-s}{\alpha} \geq 0$, and since $G(x)$ is increasing, we have that for $t \geq x$

$$G^{\frac{\beta-s}{\alpha}}(t) \geq G^{\frac{\beta-s}{\alpha}}(x). \tag{2.2}$$

Consequently,

$$\begin{aligned} B_2(x;\alpha,\beta,s) &= \left(\int_x^b f(t)G^{\frac{\beta-s}{\alpha}}(t)\,dt\right)^{\alpha} G^s(x) \\ &\geq \left(\int_x^b f(t)\,dt\right)^{\alpha} \left(G^{\frac{\beta-s}{\alpha}}(x)\right)^{\alpha} G^s(x) = F^{\alpha}(x)G^{\beta}(x). \end{aligned} \tag{2.3}$$

(ii) Let $s > \beta$ and set $W(x) := \int_x^b f(t)G^{\frac{\beta-s}{\alpha}}(t)\,dt$; that is, $-dW(x) = f(x)G^{\frac{\beta-s}{\alpha}}(x)\,dx$. Then

$$\begin{aligned} F^{\alpha}(x)G^{\beta}(x) &= G^{\beta}(x)\left(\int_x^b f(t)G^{\frac{\beta-s}{\alpha}}(t)G^{\frac{s-\beta}{\alpha}}(t)W^{\frac{s-\beta}{s}}(t)W^{\frac{\beta-s}{s}}(t)\,dt\right)^{\alpha} \\ &\leq \left(\sup_{x<t<b} G^{s-\beta}(t)W^{\frac{(s-\beta)\alpha}{s}}(t)\right)G^{\beta}(x)\left(-\int_x^b W^{\frac{\beta-s}{s}}(t)dW(t)\right)^{\alpha} \\ &= \left(\sup_{x<t<b} G^s(t)W^{\alpha}(t)\right)^{\frac{s-\beta}{s}}\left(\frac{s}{\beta}\right)^{\alpha} G^{\beta}(x)W^{\frac{\beta}{s}\alpha}(x) \\ &\leq \left(\frac{s}{\beta}\right)^{\alpha}\left(\sup_{x<t<b} G^s(t)W^{\alpha}(t)\right)^{1-\frac{\beta}{s}}\left(\sup_{x<t<b} G^s(x)W^{\alpha}(t)\right)^{\frac{\beta}{s}} \\ &= \left(\frac{s}{\beta}\right)^{\alpha}\sup_{x<t<b} B_2(x;\alpha,\beta,s). \end{aligned} \tag{2.4}$$

Consequently, for every $s > 0$ it follows from (2.3) and (2.4) that

$$\sup_{a<x<b} B_1(x;\alpha,\beta) \leq \left(\max(1,\frac{s}{\beta})\right)^{\alpha}\sup_{a<x<b} B_2(x;\alpha,\beta,s). \tag{2.5}$$

Also for the proof of the opposite estimate we need to consider two cases.
(iii) Now, let $s \geq \beta$. Then we have an inequality opposite to (2.2) and hence

$$\begin{aligned} B_2(x;\alpha,\beta,s) &= G^s(x)\left(\int_x^b f(t)G^{\frac{\beta-s}{\alpha}}(t)\,dt\right)^{\alpha} \\ &\leq G^s(x)\left(\int_x^b f(t)\,dt\right)^{\alpha} G^{\beta-s}(x) = F^{\alpha}(x)G^{\beta}(x). \end{aligned}$$

(iv) For $s < \beta$ we have

$$\begin{aligned}
G^s(x)W^\alpha(x) &= G^s(x)\left(\int_x^b f(t)G^{\frac{\beta-s}{\alpha}}(t)F^{\frac{\beta-s}{\beta}}(t)F^{\frac{s-\beta}{\beta}}(t)\,dt\right)^\alpha \\
&\le \left(\sup_{x<t<b} G^{\frac{\beta-s}{\alpha}}(t)F^{\frac{\beta-s}{\beta}}(t)\right)^\alpha G^s(x)\left(\int_x^b F^{\frac{s}{\beta}-1}(t)\,(-dF(t))\right)^\alpha \\
&= \left(\sup_{x<t<b} G^\beta(t)F^\alpha(t)\right)^{\frac{\beta-s}{\beta}}\left(\frac{\beta}{s}\right)^\alpha G^s(x)F^{\frac{\alpha s}{\beta}}(x) \\
&\le \left(\sup_{x<t<b} G^\beta(t)F^\alpha(t)\right)^{\frac{\beta-s}{\beta}}\left(\frac{\beta}{s}\right)^\alpha \left(\sup_{a<x<b} G^\beta(x)F^\alpha(x)\right)^{\frac{s}{\beta}} \\
&\le \left(\frac{\beta}{s}\right)^\alpha \sup_{a<x<b} B_1(x;\alpha,\beta).
\end{aligned}$$

Consequently, for every $s > 0$ it follows that

$$\sup_{a<x<b} B_2(x;\alpha,\beta,s) \le \left(\max(1,\frac{\beta}{s})\right)^\alpha \sup_{a<x<b} B_1(x;\alpha,\beta). \tag{2.6}$$

and (I) follows from (2.5) and (2.6).

$$\sup_{a<x<b} B_1(x;\alpha,\beta) \approx \sup_{a<x<b} B_3(x;\alpha,\beta,s) \tag{II}$$

The proof of (II) follows the same idea as the proof of (I); we have only to reverse the roles of F and G. We get

$$\sup_{a<x<b} B_1(x;\alpha,\beta) \le \left(\max\left(1,\frac{s}{\alpha}\right)\right)^\beta \sup_{a<x<b} B_3(x;\alpha,\beta,s), \tag{2.7}$$

and

$$\sup_{a<x<b} B_3(x;\alpha,\beta,s) \le \left(\max\left(1,\frac{\alpha}{s}\right)\right)^\beta \sup_{a<x<b} B_1(x;\alpha,\beta). \tag{2.8}$$

$$\sup_{a<x<b} B_1(x;\alpha,\beta) \approx \sup_{a<x<b} B_4(x;\alpha,\beta,s) \tag{III}$$

If we set $\widetilde{W}(x) = \int_a^x f(t)G^{\frac{\beta+s}{\alpha}}(t)\,dt$ so that $B_4(x;\alpha,\beta,s) = G^{-s}(x)\widetilde{W}^\alpha(x)$, and use the fact that $g(t)\,dt = dG(t)$ and integration by parts, we obtain

$$\begin{aligned}
B_1(x;\alpha,\beta) &= G^{\beta}(x)\left(\int_x^b f(t)G^{\frac{\beta+s}{\alpha}}(t)G^{-\frac{\beta+s}{\alpha}}(t)\,dt\right)^{\alpha} \\
&= G^{\beta}(x)\left(\int_x^b G^{-\frac{\beta+s}{\alpha}}(t)d\widetilde{W}(t)\right)^{\alpha} \\
&\le G^{\beta}(x)\left(G^{-\frac{\beta+s}{\alpha}}(b)\widetilde{W}(b)+\frac{\beta+s}{\alpha}\int_x^b g(t)G^{-\frac{\beta+s}{\alpha}-1}(t)\widetilde{W}(t)\,dt\right)^{\alpha} \\
&\le G^{\beta}(x)\sup_{x<t<b}G^{-s}(t)\widetilde{W}^{\alpha}(t)\left(G^{-\frac{\beta}{\alpha}}(b)+\frac{\beta+s}{\alpha}\int_x^b G^{-\frac{\beta}{\alpha}-1}(t)dG(t)\right)^{\alpha} \\
&\le G^{\beta}(x)\sup_{a<t<b}B_4(t,\alpha,\beta,s)\left(G^{-\frac{\beta}{\alpha}}(b)+\frac{\beta+s}{\beta}\left(G^{-\frac{\beta}{\alpha}}(x)-G^{-\frac{\beta}{\alpha}}(b)\right)\right)^{\alpha} \\
&= \sup_{a<t<b}B_4(t,\alpha,\beta,s)\left[\frac{\beta+s}{\beta}+\left(1-\frac{\beta+s}{\beta}\right)\left(\frac{G(x)}{G(b)}\right)^{\frac{\beta}{\alpha}}\right]^{\alpha} \\
&\le \left(1+\frac{s}{\beta}\right)^{\alpha}\sup_{a<t<b}B_4(t,\alpha,\beta,s).
\end{aligned}$$

Thus,

$$\sup_{a<x<b}B_1(x;\alpha,\beta)\le\left(1+\frac{s}{\beta}\right)^{\alpha}\sup_{a<x<b}B_4(x,\alpha,\beta,s). \tag{2.9}$$

To prove the opposite inequality, we assume that $\sup\limits_{a<x<b}B_1(x;\alpha,\beta)<\infty$. Then, by using the fact that $f(t)\,dt=-dF(t)$ and integration by parts,we obtain

$$\begin{aligned}
B_4(x,\alpha,\beta,s) &= G^{-s}(x)\left(\int_a^x G^{\frac{\beta+s}{\alpha}}(t)d\left(-F(t)\right)\right)^{\alpha} \\
&= G^{-s}(x)\left(G^{\frac{\beta+s}{\alpha}}(t)F(t)\,|_x^a+\frac{\beta+s}{\alpha}\int_a^x F(t)G^{\frac{\beta+s}{\alpha}-1}g(t)\,dt\right)^{\alpha} \\
&\le G^{-s}(x)\left(\sup_{a<t<x}G^{\beta}(t)F^{\alpha}(t)\right)\left(\frac{\beta+s}{\alpha}\int_a^x G^{\frac{s}{\alpha}-1}dG(t)\right)^{\alpha} \\
&\le \left(\frac{\beta+s}{\alpha}\right)^{\alpha}\sup_{a<t<b}G^{\beta}(t)F^{\alpha}(t)G^{-s}(x)\left(\frac{\alpha}{s}G^{\frac{s}{\alpha}}(x)\right)^{\alpha} \\
&= \left(\frac{\beta+s}{s}\right)^{\alpha}\sup_{a<x<b}B_1(x;\alpha,\beta).
\end{aligned}$$

Hence we have

$$\sup_{a<x<b} B_4(x;\alpha,\beta,s) \le \left(1+\frac{\beta}{s}\right)^{\alpha} \sup_{a<x<b} B_1(x,\alpha,\beta). \tag{2.10}$$

Now (III) follows by combining (2.9) and (2.10).

$$\sup_{a<x<b} B_1(x;\alpha,\beta) \approx \sup_{a<x<b} B_5(x;\alpha,\beta,s) \tag{IV}$$

The proof of (IV) follows the same ideas as the proof of (III); we have only to reverse the roles of F and G. We have

$$\sup_{a<x<b} B_1(x;\alpha,\beta) \le \left(1+\frac{s}{\alpha}\right)^{\beta} \sup_{a<x<b} B_5(x;\alpha,\beta,s), \tag{2.11}$$

and

$$\sup_{a<x<b} B_5(x;\alpha,\beta,s) \le \left(1+\frac{\alpha}{s}\right)^{\beta} \sup_{a<x<b} B_1(x;\alpha,\beta). \tag{2.12}$$

□

3 Scales of Weight Characterizations of Hardy's Inequality

The main result in this Section is the following four scales weight characterization of Hardy's inequality:

Theorem 2. *Let $1<p\le q<\infty$, $0<s<\infty$, and define*

$$\begin{aligned}
A_1(s) &:= \sup_{0<x<b} \left(\int_x^b u(t)V^{q(\frac{1}{p'}-s)}(t)\,dt\right)^{1/q} V^s(x),\\
A_2(s) &:= \sup_{0<x<b} \left(\int_0^x u(t)V^{q(\frac{1}{p'}+s)}(t)\,dt\right)^{1/q} V^{-s}(x),\\
A_3(s) &:= \sup_{0<x<b} \left(\int_0^x v^{1-p'}(t)U^{p'(\frac{1}{q}-s)}(t)\,dt\right)^{1/p'} U^s(x),\\
A_4(s) &:= \sup_{0<x<b} \left(\int_x^b v^{1-p'}(t)U^{p'(\frac{1}{q}+s)}(t)\,dt\right)^{1/p'} U^{-s}(x).
\end{aligned} \tag{3.1}$$

Then the Hardy inequality (1.1) holds for all measurable functions $f\ge 0$ if and only if any of the quantities $A_i(s)$ is finite. Moreover, for the best constant C in (1.1) we have $C\approx A_i(s)$, $i=1,2,3,4$.

Remark 2. The conditions in (1.3) can be described in the following way.

$$\begin{aligned}
A_M &= A_1(\frac{1}{p'}),\\
A_{PS} &= A_2(\frac{1}{p}),\\
A_W(r) &= A_1(\frac{r-1}{p}) \text{ with } 1 < r < p,\\
A^*_{PS} &= A_4(\frac{1}{q'}),\\
A^*_W(r) &= A_3(\frac{r-1}{q'}) \text{ with } 1 < r < q'.
\end{aligned}$$

Hence, Theorem 2 generalizes the corresponding result in [3] and also all previous results of this type.

PROOF. In (2.1) we put $a = 0, f(x) = u(x), g(x) = v^{1-p'}(x)$, so that $F(x) = U(x)$, $G(x) = V(x)$, and choose $\alpha = \frac{1}{q}, \beta = \frac{1}{p'}$. Then the assertion follows from the fact that

$$\begin{aligned}
A_1(s) &= \sup_{a<x<b} B_2(x; \frac{1}{q}, \frac{1}{p'}, s),\\
A_2(s) &= \sup_{a<x<b} B_4(x; \frac{1}{q}, \frac{1}{p'}, s),\\
A_3(s) &= \sup_{a<x<b} B_3(x; \frac{1}{q}, \frac{1}{p'}, s),\\
A_4(s) &= \sup_{a<x<b} B_5(x; \frac{1}{q}, \frac{1}{p'}, s),
\end{aligned}$$

are all equivalent to A_1 from (1.2) according to Theorem 1 and the finiteness of A_1 is necessary and sufficient for the inequality (1.1) to hold. Moreover, since for the least constant C in (1.1) we have $C \approx A_1$ it is clear that $C \approx A_i(s)$ and the proof is complete. □

The proof of Theorem 1 (c.f. Remark 1) gives us also the possibility to estimate e.g. the quantities A_1, $A_W(r)$, $A^*_W(r)$, A_{PS} and A^*_{PS}, in terms of each other. For example by applying (2.9) and (2.10) we have

$$A_1 \left(\frac{1}{p'}\right)^{1/q} \leq A_{PS} \leq p^{1/q} A_1. \tag{3.2}$$

If we take power weights; that is, if $u(x) = x^\alpha, v(x) = x^\beta$, with $\beta < p-1$ and $\alpha > \beta\left(\frac{q}{p} - \frac{q}{p'} - 1\right)$ we simply get $A_{PS} = (p-1)^{1/q} A_1$. The equivalence constants $\left(\frac{1}{p'}\right)^{1/q}$ and $p^{1/q}$ in (3.2) can be compared with the equivalence constants $\frac{1}{p'}$ and $q^{1/q}$ obtained by G. Bennett [1] for the corresponding estimate (3.2) in the discrete case. Thus the continuous estimate is sharper.

Remark 3. For the condition $A_1(r)$ we have by (2.5) and (2.6) that if $1 < r < p$, then

$$A_1 \leq A_1(r) \leq \left(\frac{p-1}{r-1}\right)^{1/q} A_1.$$

This is the same estimate as J. Malý and L. Pick recently communicated to us. A direct calculation in the power weight case gives equality in the upper estimate. On the other hand, if $r > p$, then we have

$$A_1 \left(\frac{p-1}{r-1}\right)^{1/q} \leq A_1(r) \leq A_1.$$

By using the arguments above we can obtain new proofs and extensions of some Hardy type inequalities in the literature. As one example we state the following extension of a result of L. E. Persson and V. D. Stepanov [6, Theorem 1].

Corollary 1. *Let $1 < p \leq q < \infty$ and $s \in (0, 1/p]$. Then the inequality (1.1) holds for all measurable $f \geq 0$ if and only if $A_2(s) < \infty$, where $A_2(s)$ is defined in (3.1). Moreover, if C is the best constant in (1.1), then*

$$A_2(s)(ps)^{1/q} \leq C \leq p' A_2(s).$$

Remark 4. We note that $A_2(1/p) = A_{PS}$ (cf. Remark 2) and we conclude that Corollary 1 is a genuine generalization of [6, Theorem 1] (see also [3, p. 14]).

4 Some Embedding Results for Lorentz Spaces

Let $f^*(t)$ denote the decreasing rearrangement of f and $f^{**}(x) = \frac{1}{x}\int_0^x f^*(y)\,dy$. For $0 < p < \infty$ and v a weight function we consider the classical Lorentz spaces

$$\Lambda^p(v) := \left\{ f \in \mathbb{R} : \left(\int_0^\infty \left(f^*(x)\right)^p v(x)\,dx \right)^{1/p} < \infty \right\},$$

and

$$\Gamma^p(v) := \left\{ f \in \mathbb{R} : \left(\int_0^\infty \left(f^{**}(x) \right)^p v(x)\, dx \right)^{1/p} < \infty \right\}.$$

i) The case $\Lambda^p(v) \hookrightarrow \Lambda^q(w)$.

Let $0 < p \le q < \infty$. Then it is well-known that the inequality

$$\left(\int_0^\infty (f^*(x))^q\, w(x)\, dx \right)^{1/q} \le C \left(\int_0^\infty (f^*(x))^p\, v(x)\, dx \right)^{1/p} \tag{4.1}$$

holds if and only if

$$\mathcal{A} := \sup_{x>0} V(x)^{-\frac{1}{p}} \left(\int_0^x w(t)\, dt \right)^{\frac{1}{q}}, \tag{4.2}$$

where

$$V(t) = \int_0^t v(x)\, dx. \tag{4.3}$$

For a proof see e.g. [7] or [8]. We have the following more general result.

Theorem 3. *Let $0 < p \le q < \infty$ and assume that (4.3) holds. Then the inequality (4.1) holds for all measurable $f \ge 0$ if and only if one of the following quantities is finite with $0 < s < \infty$:*

$$\begin{aligned}
\mathcal{A}_1(s) &= \sup_{x>0} \left(\int_x^\infty V^{-p'}(t) v(t) \left(\int_0^t w(y) dy \right)^{p'(\frac{1}{q}-s)} dt \right)^{1/p'} \left(\int_0^x w(t)\, dt \right)^s, \\
\mathcal{A}_2(s) &= \sup_{x>0} \left(\int_0^x \left(\int_0^t w(y) dy \right)^{p'(\frac{1}{q}+s)} V^{-p'}(t) v(t)\, dt \right)^{1/p'} \left(\int_0^x w(y)\, dy \right)^{-s}, \\
\mathcal{A}_3(s) &= \sup_{x>0} \left(\int_0^x w(t) V^{\frac{q(sp'-1)}{p}}(t)\, dt \right)^{1/q} V^{s\left(1-p'\right)}(x), \\
\mathcal{A}_4(s) &= \sup_{x>0} \left(\int_x^\infty w(t) V^{-\frac{q(p's+1)}{p}}(t)\, dt \right)^{1/q} V^{s(p'-1)}(x).
\end{aligned} \tag{4.4}$$

Moreover, for the best constant C in (4.1) we have $C \approx \mathcal{A}_i(s)$, $i = 1, 2, 3, 4$.

PROOF. If we apply Theorem 1 with $a = 0$, $b = \infty$, $f(x) = V^{-p'}(x)v(x)$, $g(x) = w(x)$ and choose $\alpha = \frac{1}{p'}, \beta = \frac{1}{q}$, then we obtain that

$$\begin{aligned}
\mathcal{A}_1(s) &= \sup_{x>0} B_2(x; \frac{1}{p'}, \frac{1}{q}, s),\\
\mathcal{A}_2(s) &= \sup_{x>0} B_4(x; \frac{1}{p'}, \frac{1}{q}, s),\\
\mathcal{A}_3(s) &= \sup_{x>0} B_3(x; \frac{1}{p'}, \frac{1}{q}, s),\\
\mathcal{A}_4(s) &= \sup_{x>0} B_5(x; \frac{1}{p'}, \frac{1}{q}, s).
\end{aligned}$$

The first assertion follows from the fact that, according to Theorem 1, the finiteness of each quantity $\mathcal{A}_i(s)$ is equivalent to $\mathcal{A} = \sup_{x>0} B_1(x; \frac{1}{p'}, \frac{1}{q}) < \infty$ (see (4.2)) and, this condition in its turn is equivalent to (4.1). Moreover, the final equivalence statement follows from the well-known fact that $C \approx \mathcal{A}$ and Remark 1. $\square$

Remark 5. Note that the condition (4.2) is just a special case of the condition $\mathcal{A}_3(s) < \infty$ since $\mathcal{A}_3(p) = \mathcal{A}$.

ii) The case $\Lambda^p(v) \hookrightarrow \Gamma^q(w)$.

In [7] it is proved that for the case $1 < p \leq q < \infty$

$$\left(\int_0^\infty (f^{**}(x))^q \, w(x) \, dx \right)^{1/q} \leq C \left(\int_0^\infty (f^*(x))^p \, v(x) \, dx \right)^{1/p} \tag{4.5}$$

holds if and only if (4.2) holds and

$$\mathcal{B} := \sup_{x>0} \left(\int_x^\infty w(t) t^{-q} \, dx \right)^{\frac{1}{q}} \left(\int_0^x t^{p'} V^{-p'}(t) v(t) \, dt \right)^{\frac{1}{p'}} < \infty \tag{4.6}$$

We have the following more general statement.

Theorem 4. *Let $1 < p \le q < \infty$, and for $0 < s < \infty$ set,*

$$\begin{aligned}
\mathcal{B}_1(s) &:= \sup_{x>0} \left(\int_x^\infty w(t)t^{-q} \left(\int_0^t y^{p'} V^{-p'}(y)v(y)\,dy \right)^{q(1-sp')/p'} dt \right)^{1/q} \times \\
&\qquad \left(\int_0^x t^{p'} V^{-p'}(t)v(t)\,dt \right)^s, \\
\mathcal{B}_2(s) &:= \sup_{x>0} \left(\int_0^x w(t)t^{-q} \left(\int_0^t y^{p'} V^{-p'}(y)v(y)\,dy \right)^{\frac{q(1+sp')}{p'}} dt \right)^{\frac{1}{q}} \times \\
&\qquad \left(\int_0^x t^{p'} V^{-p'}(t)v(t)\,dt \right)^{-s}, \\
\mathcal{B}_3(s) &:= \sup_{x>0} \left(\int_0^x t^{p'} V^{-p'}(t)v(t)) \left(\int_t^\infty w(y)y^{-q}\,dy \right)^{p'\frac{(1-sq)}{q}} dx \right)^{1/p'} \times \\
&\qquad \left(\int_x^\infty w(t)t^{-q}\,dt \right)^s, \\
\mathcal{B}_4(s) &:= \sup_{x>0} \left(\int_x^\infty t^{p'} V^{-p'}(t)v(t) \left(\int_t^\infty w(y)y^{-q}\,dy \right)^{p'\left(\frac{1}{q}+s\right)} dt \right)^{1/p'} \times \\
&\qquad \left(\int_x^\infty w(t)t^{-q}\,dt \right)^{-s}.
\end{aligned} \tag{4.7}$$

Then the inequality (4.5) holds for all measurable $f \ge 0$ if and only if one of the points $\{\mathcal{A}_i(s), \mathcal{B}_j(s)\}$, $i, j = 1, 2, 3, 4$ with $\mathcal{A}_i(s)$ from (4.4) is finite. Moreover, for the best possible constant C in (4.5) we have

$$C \approx \max(\min \mathcal{A}_i(s), \mathcal{B}_j(s)),\ i,\ j = 1, 2, 3, 4.$$

PROOF. We have already in Theorem 3 proved that (4.2) is equivalent to that one of the quantities in (4.4) is finite. Now, in (2.1) let $a = 0$, $b = \infty$,

$f(x) = w(x)x^{-q}$, $g(x) = x^{p'}V^{-p'}(x)v(x)$ and choose $\alpha = \frac{1}{q}, \beta = \frac{1}{p'}$. We have

$$\mathcal{B}_1(s) = \sup_{x>0} B_2(x; \frac{1}{q}, \frac{1}{p'}, s),$$
$$\mathcal{B}_2(s) = \sup_{x>0} B_4(x; \frac{1}{q}, \frac{1}{p'}, s),$$
$$\mathcal{B}_3(s) = \sup_{x>0} B_3(x; \frac{1}{q}, \frac{1}{p'}, s),$$
$$\mathcal{B}_4(s) = \sup_{x>0} B_5(x; \frac{1}{q}, \frac{1}{p'}, s).$$

The first assertion follows from [7] or [8] and Theorem 3, and the fact that the finiteness of any of the quantities in (4.7) are equivalent to (4.6) by Theorem 1 since $\mathcal{A}_1 = \sup_{x>0} B_1(x, \frac{1}{q}, \frac{1}{p'})$. The final statement follows analogously as before.

Acknowledgment. The research of the first author was partially supported by the Leverhulme Grant no F/00407/E and by the grant no. 201/01/0333 of Grant Agency of the Czech Republic (GA CR).The research of the second author was partially supported by the grants A1019305 of the Academy of Science of the Czech Republic and 201/03/0671 of the Grant Agency of the Czech Republic. □

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