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Zbigniew Grande, Institute of Mathematics, Bydgoszcz Academy, Plac Weyssenhoffa 11, 85-072 Bydgoszcz, Poland.
email: `grande@wsp.bydgoszcz.pl`

BORSÍK'S BILATERAL QUASICONTINUITY OF FUNCTIONS OF TWO VARIABLES

Abstract

In his lecture [1] Borsík introduces some notions of the bilateral quasicontinuity of applications of topological spaces. In this paper I define analogously the bilateral semi-quasicontinuities and investigate some functions of two variables whose sections are bilaterally quasicontinuous or bilaterally semiquasicontinuous.

Let $(\mathbb{R}, T_e)$ be the set of all reals with the Euclidean topology T_e and let (X, T_X), (Y, T_Y) be topological spaces.

A function $h : X \mapsto \mathbb{R}$ is quasicontinuous [res. upper semiquasicontinuous] {lower semiquasicontinuous} at a point $x \in X$ if for every positive real η and for every open set I containing x there is a nonempty open set $J \subset I$ such that $h(J) \subset (h(x)-\eta, h(x)+\eta)$ [resp. $h(J) \subset (-\infty, h(x)+\eta)$ $\{h(J) \subset (h(x)-\eta, \infty)\}$ (see [5, 6]).

A function $f : \mathbb{R} \to \mathbb{R}$ is said to be left-hand (right-hand) sided quasicontinuous at a point x if for each real $r > 0$ and for each neighborhood $V \in T_e$ of $f(x)$ there exists a nonempty set $G \subset (x - r, x) \cap f^{-1}(V)$ ($G \subset (x, x+r) \cap f^{-1}(V)$) belonging to T_e. A function $f : \mathbb{R} \to \mathbb{R}$ is bilaterally quasicontinuous at x if it is both left-hand and right-hand sided quasicontinuous at this point.

Analogously, we can define the bilateral upper (or lower) semiquasicontinuity of functions of a real variable.

In his talk [1] J. Borsík shows some possibilities to define a bilateral quasicontinuity for functions defined on metric or topological spaces. His definitions are the following:

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A function $f : X \to Y$ is B-quasicontinuous at a point $x \in X$ if for every neighborhood V of $f(x)$ and for every open connected set $A \in T_X$ such that $x \in \mathrm{cl}(A)$ (here $\mathrm{cl}(A)$ denotes the closure of the set A) there exists an open nonempty set $G \subset A \cap f^{-1}(V)$.

In [2] Borsik observes that if (X, T_X) is a locally connected space, then B-quasicontinuity implies quasicontinuity.

In the case where X is a metric space, Borsík proposes also other generalizations of the bilateral quasicontinuity of functions of real variable. One of these is the following.

A function $f : X \to Y$ is S-quasicontinuous at $x \in X$ if for every neighborhood $V \in T_Y$ of $f(x)$ and for every $x \neq y \in X$ there is an open nonempty set $G \subset S(y, d(x, y)) \cap f^{-1}(V)$, where d denotes the metric in X and $S(y, d(x, y)) = \{t \in X; d(t, y) < d(x, y)\}$.

One of the more important theorems concerning quasicontinuity is the following theorem of Kempisty ([5]).

Theorem 1. *If the sections $f_x(t) = f(x, t)$ and $f^y(t) = f(t, y)$, $x, y, t \in \mathbb{R}$, of a function $f : \mathbb{R}^2 \to \mathbb{R}$ are quasicontinuous, then f is also quasicontinuous on $(\mathbb{R}^2, T_e \times T_e)$.*

There are, however, functions $f : \mathbb{R}^2 \to \mathbb{R}$ having continuous sections f_x and f^y, $x, y \in \mathbb{R}$, which are not B-quasicontinuous ([2], Th. 7). Moreover the following theorem is proved in [2].

Theorem 2. *Let (X, T_X) and (Y, T_Y) be Baire locally separable metric spaces and let (Z, T_Z) be a regular space. If the sections f_x and f^y, $x \in X$ and $y \in Y$, of a function $f : X \times Y \to Z$ are S-quasicontinuous, then f is S-quasicontinuous.*

In regards to this theorem, we observe that there is a function $f : \mathbb{R}^2 \to \mathbb{R}$ which has continuous sections f_x and f^y, where $x, y \in \mathbb{R}$, and which is not B-quasicontinuous. The following theorem, however, is true.

Theorem 3. *Suppose that (X, T_X) is a locally connected Baire space, (Y, T_Y) is a topological space such that each point $y \in Y$ has a neighborhood with a countable basis and (Z, T_Z) is a topological regular space. Let $f : X \times Y \to Z$ be a function such that the sections $f_x(t) = f(x, t)$ and $f^y(z) = f(z, y)$, $x, z \in X$ and $t, y \in Y$, are B-quasicontinuous. Then f satisfies the following condition:*

(a) for each point $(x, y) \in X \times Y$ and each neighborhood $W \in T_Z$ of $f(x, y)$ and for all connected sets $U \in T_X$ and $V \in T_Y$ such that $x \in \mathrm{cl}(U)$ and $y \in \mathrm{cl}(V)$ there are nonempty sets $G \in T_X$ and $H \in T_Y$ such that $G \times H \subset f^{-1}(W) \cap (U \times V)$.

PROOF. Fix a point $(x,y) \in X \times Y$, a set $W \in T_Z$ containing $f(x,y)$ and connected sets $U \in T_X$ and $V \in T_Y$ such that $x \in \mathrm{cl}(U)$ and $y \in \mathrm{cl}(V)$. The topological space (Z, T_Z) is regular, so there are disjoint sets $W_1, W_2 \in T_Z$ such that $f(x,y) \in W_1 \subset W$ and $Z \setminus W \subset W_2$. Since the section f^y is B-quasicontinuous at x, there is a nonempty set $A \in T_X$ which is contained in $(f^y)^{-1}(W_1) \cap U$. From the hypothesis of our theorem it follows that there is a set $V_1 \in T_Y$ having a countable basis (B_n) of subsets of T_Y and such that $y \in V_1 \subset V$. Since the sections f_t are B-quasicontinuous at y, for each point $t \in A$ there is a set $B_{n(t)}$ such that $f_t(B_{n(t)}) \subset W_1$. But (X, T_X) is a Baire space, so the set A is of the second category. Since the basis (B_n) is countable, there is a positive integer k such that the set $E = \{t \in A; n(t) = k\}$ is of the second category. There is a nonempty set $G \in T_X$ contained in A such that the intersection $E \cap G$ is dense in G. Let $H = B_k$. We will prove that $f(G \times H) \subset W$. Of course, assume in order to obtain a contradiction, that there is a point $(x_1, y_1) \in G \times H$ such that $f(x_1, y_1) \notin W$. Then $f(x_1, y_1) \in W_2$. Since the section f^{y_1} is quasicontinuous at x_1, there is a nonempty set $U_1 \in T_X$ contained in G such that $f^{y_1}(U_1) \subset W_2$. But the set $E \cap G$ is dense in G and $U_1 \in T_X$ is nonempty and contained in G, so there is a point $x_2 \in E \cap U_1$. Then $f(x_2, y_1) \in W_1$, is a contraction with $f^{y_1}(U_1) \subset W_2$ and $W_1 \cap W_2 = \emptyset$. So, $f(G \times H) \subset W$ and the proof is completed. □

In next example we show that some hypotheses of the last theorem are important.

Example 1. *Let $X = Y = Z = \mathbb{R}$, let $T_X = T_Y$ be the density topology in $\mathbb{R}$ ([2, 7]) and let T_Z be the Euclidean topology in $\mathbb{R}$. There is ([3]) an approximately continuous function $g : \mathbb{R} \to [0,1]$ such that the set $g^{-1}(0)$ is dense and of Lebesgue measure zero and $g(\mathbb{R}) = [0,1]$. Let $f(x,y) = g(x-y)$. Then the sections f_x and f^y, $x, y \in \mathbb{R}$, are continuous (as mappings from $(\mathbb{R}, T_d)$ to $(\mathbb{R}, T_e)$). There is a point (u,v) such that $f(u,v) = 1$.*

PROOF. We will prove that f is not quasicontinuous at (u,v). For this assume, to the contrary, that f is quasicontinuous at (u,v). Since $f(u,v) = 1$, there are nonempty sets $A, B \in T_d$ such that $f(x,y) > \frac{1}{2}$ for $(x,y) \in A \times B$. But the sets $A, B \in T_d$ are nonempty, so they are of positive Lebesgue measure and consequently, by the Steinhaus theorem from [7], the set $A - B = \{x - y; x \in A,\ y \in B\}$ has nonempty Euclidean interior $\mathrm{int}(A-B)$. Since the set $g^{-1}(0)$ is dense, there is a point $z \in g^{-1}(0) \cap (A - B)$. Then $z = x - y$, where $x \in A,\ y \in B$ and $g(z) = g(x-y) = f(x,y) = 0 \leq \frac{1}{2}$. This contradicts the relation $(x,y) \in A \times B$ and this contradiction proves that f is not quasicontinuous at (u,v). □

Now we can introduce, similarly to Borsík's B-quasicontinuity, the notions of upper and lower B-semiquasicontinuities of real functions.

A function $f : X \to \mathbb{R}$ is upper (resp. lower) B-semiquasicontinuous at a point $x \in X$ if for each real $\eta > 0$ and each connected set $A \in T_X$ such that $x \in \mathrm{cl}(A)$ there is a nonempty set $G \in T_X$ contained in A for which $f(G) \subset (-\infty, f(x) + \eta)$ (resp. $f(G) \subset (f(x) - \eta, \infty)$).

As in Borsík's article [2] for the quasicontinuity, we can observe that if (X, T_X) is a locally connected space, then the upper (lower) B semiquasicontinuity of $f : X \to Y$ implies the upper (lower) semiquasicontinuity of f.

Let $\mathcal{E}(X, \mathbb{R})$ be the family of all functions $g : X \to \mathbb{R}$ which are upper and lower B-semiquasicontinuous at each point $x \in X$.

In Theorems 4 and 5 we suppose that (Y, T_Y) is a locally connected Baire topological space and (X, T_X) is a second countable topological space.

Theorem 4. *Let $f : X \times Y \to \mathbb{R}$ be a function such that the sections $f^y \in \mathcal{E}(X, \mathbb{R})$ for $y \in Y$. If the sections f_x, $x \in X$, are upper (lower) B-semiquasicontinuous, then for all points $(u, v) \in X \times Y$, for all connected sets $V \in T_X$ and all connected sets $W \in T_Y$ with $u \in \mathrm{cl}(V)$ and $y \in \mathrm{cl}(W)$, and for each real $\eta > 0$ there are nonempty sets $G \in T_X$ and $H \in T_Y$ such that $G \times H \subset V \times W$ and $f(G \times H) \subset (-\infty, f(u, v) + \eta)$ $(f(G \times H) \subset (f(u, v) - \eta, \infty))$.*

Proof. Fix a point $(x, y) \in X \times Y$, a positive real η and connected sets $V \in T_X$ and $W \in Y$ such that $x \in \mathrm{cl}(V)$ and $y \in \mathrm{cl}(W)$. Let (V_n) be a countable basis of T_X open sets in V.

Since the section f_x is upper B-semiquasicontinuous at y, there is a nonempty open set $P \subset W$ such that $f(x, u) < f(x, y) + \frac{\eta}{4}$ for $u \in P$. The sections f^u, $u \in P$, are also upper B-semiquasicontinuous at x, so for each point $u \in P$ there is a nonempty open set $V_{n(u)} \subset V$ such that $f(v, u) < f(x, u) + \frac{\eta}{4}$ for $v \in V_{n(u)}$. Since (Y, T_Y) is a Baire space, the set P is of the second category. So there is a positive integer n such that the set $P_n = \{u \in P; n(u) = n\}$ is of the second category. There is a nonempty set $Q \subset \mathrm{int}(\mathrm{cl}(P_n))$ belonging to T_X.

If $(v, u) \in V_n \times P_n$, then $f(v, u) < f(x, u) + \frac{\eta}{4} < f(x, y) + \frac{\eta}{4} + \frac{\eta}{4} = f(x, y) + \frac{\eta}{2}$. If $(v, u) \in V_n \times Q$, then $f(v, u) \leq f(x, y) + \frac{\eta}{2} < f(x, y) + \eta$. Really, if there is a point $(v_0, u_0) \in V_n \times Q$ with $f(u_0, v_0) > f(x, y) + \frac{\eta}{2}$, then from the lower semi-quasicontinuity of the section f_{v_0} at u_0 follows then there is a nonempty set $Q_1 \subset Q$ belonging to T_Y such that $f(v_0, u) > f(x, y) + \frac{\eta}{2}$ for $u \in Q_1$. Consequently, there is a point $u_1 \in P_n \cap Q_1$. Since $f(v_0, u_1) > f(x, y) + \frac{\eta}{2}$, we obtain a contradiction and the proof in the case of the upper semi-quasicontinuity is completed. The proof of the lower semi-quasicontinuity is analogous. $\square$

As an immediate corollary we obtain the following.

Theorem 5. *If the sections $f_x \in \mathcal{E}(Y,\mathbb{R})$, $x \in X$ and $f^y \in \mathcal{E}(X,\mathbb{R})$, $y \in Y$, then for each real $\eta > 0$, for each point $(u,v) \in X \times Y$ and for all connected sets $V \in T_X$ and $W \in T_Y$ with $u \in \mathrm{cl}(V)$ and $y \in \mathrm{cl}(W)$ there are nonempty sets $G_1, G_2 \in T_X$ and $H_1, H_2 \in T_Y$ such that $G_1 \cup G_2 \subset V$, $H_1 \cup H_2 \subset W$, $f(G_1 \times H_1) \subset (-\infty, f(u,v)+\eta)$ and $f(G_2 \times H_2) \subset (f(u,v)-\eta, \infty)$.*

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