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A CHARACTERIZATION OF SINGULAR MEASURES

Abstract

Denote by μ a probability Borel measure on the real line and by τ_c the translation by c. We show that μ is singular with respect to Lebesgue measure if and only if the set of those c for which μ and $\tau_c\mu$ are mutually singular is dense (Theorem 1). Another characterization of singularity (Theorem 10.) is the existence of a set of full μ measure that has continuum many disjoint translates. This result is also linked to some known results about σ-porous sets.

Kakutani [2] has an old and famous theorem concerning the singularity or mutual absolute continuity of infinite product measures. We borrow the main tool of his investigation, namely the "inner product" ρ of probability measures defined on the same measurable space,

$$\rho(\mu,\nu) = \int \left(\frac{d\mu}{d\pi}\right)^{1/2} \left(\frac{d\nu}{d\pi}\right)^{1/2} d\pi$$

where both μ, and ν are absolutely continuous with respect to the measure π. ρ does not depend on the choice of π, so if it makes the considerations easier we may assume that $\pi = \mu + \nu$.

The two probability measure μ, ν are the same if $\rho(\mu,\nu) = 1$ and mutually singular if $\rho(\mu,\nu) = 0$.

In what follows T denotes $\mathbb{R}/\mathbb{Z}$, the circle group and λ, Lebesgue (Haar) measure on T. τ_c denotes translation by c; i.e., $\tau_c(x) = x + c$ and $\tau_c\mu$ stands for measure defined by $\tau_c\mu(A) = \mu(A - c)$ for any c. Also $\bar{\mu}(A) = \mu(-A)$.

One of the main theorems of the paper is the following assertion.

Theorem 1. *Let μ be a Borel measure on T. Then μ is singular with respect to Haar measure λ if and only if the set*

$$M = \{c \in T \ : \ \mu \perp \tau_c\mu\}$$

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is a dense, G_δ, of full Haar measure; i.e., $\lambda(M) = 1$.

The proof of Theorem 1 is based on some lemmas the aim of which is to show that the non-negative real function $f(c) = \rho(\mu, \tau_c\mu)$ is upper semi-continuous and that the integral mean of f is zero on any interval. These two facts prove the necessity of the condition in Theorem 1 at once.

The opposite direction follows from two facts.

(i) If $\mu(A) = \int_A g d\lambda$, then $\rho(\mu, \tau_c\mu) = \int_T \sqrt{g(x)g(x-c)}\, dx$. Therefore

$$\int_T f(c)\, dc = \int_T \int_T \sqrt{g(x)g(x-c)}\, dx\, dc = \left(\int_T \sqrt{g} \right)^2 > 0.$$

(ii) If $\mu = \alpha\mu_{ac} + (1-\alpha)\mu_s$ where $\mu_{ac} \ll \lambda$ and $\mu_s \perp \lambda$, then $f(c) = \rho(\mu, \tau_c\mu) \geq \alpha\rho(\mu_{ac}, \tau_c\mu_{ac})$.

Denote by $\mathcal{M}$ the family of Borel probability measures on T, and by w the weak topology on $\mathcal{M}$; i.e., the weakest topology on $\mathcal{M}$ such that for all continuous functions $h : T \to \mathbb{R}$ the mapping $\nu \mapsto \int h\, d\nu$ is continuous. With this topology $\mathcal{M}$ is a compact metric space. In what follows any topological notion in connection with $\mathcal{M}$ refers to the w topology.

Lemma 2. *Let $\nu \in \mathcal{M}$. The mapping $c \mapsto \tau_c\nu$ is continuous.*

PROOF. It is enough to show that for any continuous function $h : T \to \mathbb{R}$ the composition

$$c \mapsto \int_T h\, d(\tau_c\nu) = \int_T (\tau_c h)\, d\nu$$

is a continuous real function which follows from the uniform continuity of h. □

The next lemma essentially appears in [2]. We present here a simpler martingale based proof for the sake of completeness.

Lemma 3. *Let $\mathcal{F}_n$ be an increasing sequence of σ-algebras on T such that the algebra $\cup_n \mathcal{F}_n$ generates the σ-algebra of Borel sets and let μ, ν two probability measure on T. Then the sequence $\rho(\mu|_{\mathcal{F}_n}, \nu|_{\mathcal{F}_n})$ is decreasing and*

$$\rho(\mu, \nu) = \lim_{n\to\infty} \rho(\mu|_{\mathcal{F}_n}, \nu|_{\mathcal{F}_n})$$

PROOF. Let $\pi = (\mu + \nu)/2$. Then (T, π) is a probability field and both

$$\xi_n = \frac{d\mu|_{\mathcal{F}_n}}{d\pi|_{\mathcal{F}_n}} \text{ and } \eta_n = \frac{d\nu|_{\mathcal{F}_n}}{d\pi|_{\mathcal{F}_n}}$$

are non-negative bounded martingales on it with respect to $\{\mathcal{F}_n : n \in \mathbb{N}\}$. Therefore these martingales converge almost everywhere and in L_p $(1 \leq p < \infty)$ also. Moreover their limits are $\frac{d\mu}{d\pi}$ and $\frac{d\nu}{d\pi}$ respectively, since $\cup_n \mathcal{F}_n$ generates the Borel σ-algebra.

By definition

$$\rho(\mu|_{\mathcal{F}_n}, \nu|_{\mathcal{F}_n}) = \int \sqrt{\xi_n \eta_n} d\pi|_{\mathcal{F}_n} = \int \sqrt{\xi_n \eta_n} d\pi.$$

This last integral tends to $\rho(\mu, \nu)$ by the dominated convergence theorem.

We also need that the sequence $\rho(\mu|_{\mathcal{F}_n}, \nu|_{\mathcal{F}_n})$ is decreasing. To see this we use the conditional version of the Cauchy-Schwarz inequality; i.e.,

$$E(\sqrt{\xi_{n+1}\eta_{n+1}}|\mathcal{F}_n) \leq \sqrt{E(\xi_{n+1}|\mathcal{F}_n)E(\eta_{n+1}|\mathcal{F}_n)} = \sqrt{\xi_n \eta_n}$$

holds almost everywhere. Taking the expectation of both sides we get the desired inequality. □

Lemma 4. *$\rho : \mathcal{M} \times \mathcal{M} \to \mathbb{R}$ is upper semi-continuous.*

Proof. We have to show that ρ is upper semi-continuous at any point of $\mathcal{M} \times \mathcal{M}$. So let μ_0 and ν_0 be two probability Borel measure on T and $\mathcal{F}_n$ be a sequence of finite σ-algebras of Borel sets of T, such that $\cup \mathcal{F}_n$ is generating. Moreover we can choose each $\mathcal{F}_n$ in such a way that for any $A \in \mathcal{F}_n$ we have $\mu_0(\partial A) = \nu_0(\partial A) = 0$. Let $\rho_n(\mu, \nu) = \rho(\mu|_{\mathcal{F}_n}, \nu|_{\mathcal{F}_n})$. Since $\mathcal{F}_n$ is finite

$$\rho_n(\mu, \nu) = \int \left(\frac{d\mu|_{\mathcal{F}_n}}{d\pi|_{\mathcal{F}_n}}\right)^{1/2} \left(\frac{d\nu|_{\mathcal{F}_n}}{d\pi|_{\mathcal{F}_n}}\right)^{1/2} d\pi|_{\mathcal{F}_n} = \sum_{A \in \mathcal{A}(\mathcal{F}_n)} \sqrt{\mu(A)\nu(A)} \qquad (1)$$

where π is as above and $\mathcal{A}(\mathcal{F}_n)$ denotes the collection of atoms of $\mathcal{F}_n$. Since $\mathcal{F}_n$ was chosen in such a way that for each for $A \in \mathcal{F}_n$ the evaluation mapping $\mu \mapsto \mu(A)$ is continuous at μ_0 and at ν_0, equation (1) shows that ρ_n is continuous at (μ_0, ν_0) for each n. ρ is a pointwise decreasing limit of the sequence ρ_n so ρ is upper semi-continuous at (μ_0, ν_0). □

From Lemmas 2 and 4 we have that $f(c) = \rho(\mu, \tau_c \mu)$ is upper semi continuous for any $\mu \in \mathcal{M}$. The next lemma deals with the integral mean of f.

Lemma 5. *Let I be an subinterval of T. Then $\int_I f \, d\lambda = 0$.*

PROOF. For the proof we can assume that I is an interval of positive Haar measure. Put

$$\nu(A) = \frac{1}{\lambda(I)} \int_I \tau_c \mu(A)\, dc$$

for any Borel set A. Then ν is the convolution of two probability measures one of which is absolutely continuous with respect to the Haar measure. So ν is a probability Borel measure. If $\lambda(A) = 0$, then $\lambda(A - x) = 0$ for all $x \in T$ so the application of the Fubini theorem

$$\begin{aligned}\nu(A) &= \frac{1}{\lambda(I)} \int_I \tau_c \mu(A)\, dc = \frac{1}{\lambda(I)} \int_I \int_T \chi_A(c + x)\, d\mu(x)\, dc \\ &= \frac{1}{\lambda(I)} \int_T \int_I \chi_A(c + x)\, dc\, d\mu(x) = \frac{1}{\lambda(I)} \int_T \lambda((A - x) \cap I)\, d\mu(x) = 0\end{aligned}$$

so ν is absolutely continuous with respect to the Haar measure. This means that

$$0 = \rho(\mu, \nu) = \lim_n \sum_{A \in \mathcal{A}(\mathcal{F}_n)} \sqrt{\mu(A)\nu(A)}$$

where $\mathcal{F}_n$ is a sequence of finite σ-algebras such that $\cup_n \mathcal{F}_n$ generates the Borel σ-algebra. By the concavity of square root

$$\sqrt{\nu(A)} = \sqrt{\frac{1}{\lambda(I)} \int_I \tau_c \mu(A)\, dc} \geq \frac{1}{\lambda(I)} \int_I \sqrt{\tau_c \mu(A)}\, dc,$$

so

$$\begin{aligned}\frac{1}{\lambda(I)} \int_I f(c)\, dc &= \frac{1}{\lambda(I)} \int_I \rho(\mu, \tau_c \mu)\, dc \\ &= \frac{1}{\lambda(I)} \int_I \lim_n \sum_{A \in \mathcal{A}(\mathcal{F}_n)} \sqrt{\mu(A)\tau_c \mu(A)}\, dc \\ &\leq \lim_n \sum_{A \in \mathcal{A}(\mathcal{F}_n)} \frac{\sqrt{\mu(A)}}{\lambda(I)} \int_I \sqrt{\tau_c \mu(A)}\, dc \\ &\leq \lim_n \sum_{A \in \mathcal{A}(\mathcal{F}_n)} \sqrt{\mu(A)\nu(A)} = \rho(\mu, \nu) = 0. \qquad \square\end{aligned}$$

Lemma 6. *Let μ be a singular probability measure on T, $I \subset T$ a nonempty open interval and $\varepsilon > 0$. There exists a closed set $F = -F$ such that $\mu(T \setminus F) < \varepsilon$, and $I \setminus (F + F) = I \setminus (F - F)$ is non empty.*

Proof. For an arbitrary Borel measure ν on T let $\bar{\nu}(A) = \nu(-A)$. Since $\lambda = \bar{\lambda}$, we have that $\bar{\mu}$ and $\nu = \frac{1}{2}(\mu + \bar{\mu})$ is also singular with respect to Haar measure λ. By Theorem 1 there is a $c \in I$ such that ν and $\tau_c\nu$ are mutually singular. So there is Borel set $E \subset T$ such that $\nu(E) = 0$ and $\tau_c\nu(T \setminus E) = 0$. By regularity there are open sets A, B such that

$$E \subset A \text{ and } \quad \nu(A) < \varepsilon/4,$$
$$T \setminus E \subset B \text{ and } \tau_c\nu(B) < \varepsilon/4.$$

Put $F = T \setminus (A \cup (-A) \cup (B - c) \cup (c - B))$. $F = -F$ is closed and

$$\begin{aligned}\mu(T \setminus F) &= \mu(A \cup (-A) \cup (B - c) \cup (c - B)) \\ &\leq \mu(A) + \mu(-A) + \mu(B - c) + \mu(c - B) = 2\nu(A) + 2\tau_c\nu(B) < \varepsilon.\end{aligned}$$

To prove that $F + F$ does not fill up I it is enough to show that $c \notin F + F$; that is, $F \cap (c - F) = \emptyset$, or $(T \setminus F) \cup (T \setminus (c - F)) = T$. This last equality follows from the choice of A and B, i.e. $A \cup B \supset E \cup T \setminus E = T$, and the fact that $B \subset T \setminus (c - F)$, $A \subset T \setminus F$. □

Theorem 1 and Lemma 6 have the following interesting corollary.

Theorem 7. *Let μ be a singular Borel probability measure on $\mathbb{R}$. There is a closed set $E = -E$ of positive μ measure such that $E + E = E - E$ is a nowhere dense closed set.*

Proof. Let η denote the canonical mapping from $\mathbb{R}$ to $T = \mathbb{R}/\mathbb{Z}$, ($\eta$ is the fractional part function) and $\mu_0 = \mu \circ \eta^{-1}$. Let $\{I_n : n \in \mathbb{N}\}$ be the enumeration of nonempty open subintervals of T with rational endpoints and $\varepsilon_n > 0$ such that $\sum \varepsilon_n < 1$. By Lemma 6 there are closed sets $F_n = -F_n$ such that $\mu_0(T \setminus F_n) < \varepsilon_n$, and $I_n \setminus (F_n + F_n) \neq \emptyset$. Put $\tilde{F} = \cap_n F_n$. It is clear that $\tilde{F} = -\tilde{F}$ is closed, $\mu_0(\tilde{F}) > 0$, $\tilde{F} + \tilde{F}$ is closed and $I_n \setminus (\tilde{F} + \tilde{F}) \neq \emptyset$. To complete the proof let $F = \eta^{-1}\tilde{F}$. □

As a corollary we give a new proof of the fact (proved originally by J. Tkadlec [4]) that there is a closed non-σ-porous set E such that $E + E$ is nowhere dense, so contains no interval. Later we will also prove the existence of a closed non-σ-porous set having continuumly many disjoint translates. Usually both are considered as the property of the σ-ideal of σ-porous sets. We show that these are rather the properties of the σ-ideal of the null sets of singular probability Borel measures. They hold for σ-porous sets as there are singular probability Borel measures taking value zero on porous sets (see [5] and [3]).

Corollary 8 (J. Tkadlec [4]). *There is closed non-σ-porous subset E of the real line, such that $E+E$ is of first category.*

PROOF. It is well known that there is singular probability Borel measure μ on the real line, such that each σ-porous set is of μ-measure 0 (see e.g. [5]). So Theorem 7 shows the existence of such an E. $\square$

At the end of this note we also prove another characterization of singular measures. We will use the following quite widely known lemma. For the sake of completeness we give a proof rather than a reference.

Lemma 9. *Let H be a set of first category on the real line. Then there is a perfect set C such that $(C-C)\cap H \subset \{0\}$.*

PROOF. Clearly it's enough to prove the lemma for $H = \cup_n F_n$ where F_n is closed and nowhere dense; moreover $F_n \subset F_{n+1}$. The set C will be given with the help of a so called perfect scheme; i.e., a sequence of closed set $C_n \supset C_{n+1}$ such that C_n is a disjoint union of closed intervals $I_{n,k}$ and $|\{l : I_{n+1,l} \subset I_{n,k}\}| \geq 2$. Then $C = \cap_n C_n$ is perfect. If C_n is defined in such a way that $(C_n - C_n)\cap F_n \subset [-1/n, 1/n]$, then $(C-C)\cap H \subset \{0\}$ also follows.

Let $C_1 = [0,1] = I_{1,1}$. It is clear that $C_1 - C_1 \subset [-1,1]$. Assume that $C_n = \cup_k I_{n,k}$ is defined where the union is finite and disjoint. Using the fact that F_{n+1} is nowhere dense and closed one can easily see that if I, J are two disjoint intervals, then there are intervals $\tilde{I} \subset I$, $\tilde{J} \subset J$ such that $(\tilde{I} - \tilde{J}) \cap F_{n+1} = \emptyset$.

Using this we can decrease in finitely many steps the intervals $I_{n,k}$ to $\tilde{I}_{n,k}$ such that $(\tilde{I}_{n,k} - \tilde{I}_{n,l}) \cap F_{n+1} = \emptyset$ provided that $l \neq k$, and $\left|\tilde{I}_{n,k}\right| < \frac{1}{2(n+1)}$. This latest property implies that $\tilde{I}_{n,k} - \tilde{I}_{n,k} \subset [-1/(n+1), 1/(n+1)]$. Let $I_{n+1,2k-1}$ and $I_{n+1,2k}$ be the two intervals remaining after the middle third of $\tilde{I}_{n,k}$ is deleted. It is clear that $C_{n+1} = \cup_k I_{n+1,k} \subset C_n$ and $(C_{n+1} - C_{n+1}) \cap F_{n+1} \subset [-1/(n+1), 1/(n+1)]$. So the sequence (C_n) can be defined with the desired properties and $C = \cap_n C_n$ is a perfect set such that $(C-C)\cap H \subset \{0\}$. $\square$

Theorem 10. *Let μ be a probability Borel measure on the real line. Then the following statements are equivalent:*

(i) μ is singular with respect to the Lebesgue measure.

(ii) For $\varepsilon > 0$ there is a closed set E with $\mu(E) > 1-\varepsilon$ and a perfect set C such that $\{c + E : c \in C\}$ is a family of disjoint sets.

(iii) There is an F_σ set E of μ measure one, and a perfect set C such that $\{c + E \,:\, c \in C\}$ is a family of disjoint sets.

Proof. $(ii) \implies (i)$ Let $\mu = \mu_{ac} + \mu_s$ be the Hahn decomposition of μ into singular and absolutely continuous parts with respect to the Lebesgue measure. If $\varepsilon < \mu_{ac}(\mathbb{R})$, then for any closed F, $\mu(F) > 1 - \varepsilon$ implies that $\lambda(F) > 0$. But this contradicts (ii), since we can assume that $F \subset [-K, K]$ for some $K > 0$ and L is so large that $C \cap [-L, L]$ is infinite. In that case

$$\bigcup_{c \in C \cap [-L,L]} c + F \subset [-K - L, K + L]$$

Since the sets $c + F$, $c \in C \cap [-L, L]$ are disjoint, we have that $\lambda(F) = 0$. This argument proves $(ii) \implies (i)$.

$(i) \implies (iii)$ Theorem 7 says that for any $\varepsilon > 0$ one can find a closed set F_ε such that $\mu(F_\varepsilon) > 1 - \varepsilon$ and $F_\varepsilon - F_\varepsilon$ is nowhere dense. Put $F'_n = \cap_{k>n} F_{2^{-k}}$ and $E = \cup_n F'_n$. It is clear that E is an F_σ set, and $F'_n \subset F'_{n+1}$. Therefore $E - E = \cup(F'_n - F'_n)$ is of first category. From Lemma 9 there is a perfect set C such that $C - C \cap E - E = \{\,0\,\}$, so $\{E + c \,:\, c \in C\}$ is a family of disjoint sets.

Finally $(iii) \implies (ii)$ is clear. □

Theorem 10 has the following corollary, which says that some non σ-porous sets are so small that on the real line there is enough space for continuumly many disjoint translates of them. The fact that there is a family of disjoint non σ-porous sets of continuum cardinality is known. What is probably new in this statement is that it can be chosen to be the family of translates of a given set.

Corollary 11. *There is a closed non σ-porous set $E \subset \mathbb{R}$ and a perfect set C such that $\{E + c \,:\, c \in C\}$ is a family of disjoint sets.*

Proof. Theorem 10 and the existence of singular measure taking zero on porous sets proves this statement. □

In [1] the authors investigate the so called thin subset of the real line. A compact set $C \subset \mathbb{R}$ is called *thin* if it is true in ZFC that $\mathbb{R}$ is not the union of less than continuumly many translates of C. Another view of Theorem 10 is that it provides a great many examples of thin sets. Indeed for any singular measure μ there is a set E with property (ii). We may also assume that E is compact. E is thin since less then continuum translates of it can not cover even C, as $|x + E \cap C| \leq 1$ for all $x \in \mathbb{R}$. Indeed $c_i \in C \cap (x + E)$ $(i = 1, 2)$ means that there are $e_i \in E$ with $c_i = x + e_i$, so $c_i - e_i = x$, i.e. $c_1 - c_2 = e_1 - e_2 = 0$, this means $c_1 = c_2$.

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