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ARE CONE-MONOTONE FUNCTIONS GENERICALLY INTERMEDIATELY DIFFERENTIABLE?

Abstract

On a separable Banach space, we show that a cone-monotone function is generically intermediate differentiable provided its Dini-derivatives are finite along every direction and the cone has nonempty interior.

1 Introduction

Let X be a Banach space with dual space X^*, let $A \subset X$ be a non-empty open set, and let $K \subset X$ be a closed convex cone with $\operatorname{int}(K) \neq \emptyset$. The open ball with center x and radius r is denoted by $B_r(x)$. We say that $f : A \to \mathbb{R} \cup \{+\infty\}$ is *K-increasing* on a set A if $f(x+k) \geq f(x)$ whenever $x \in A, x+k \in A$ for $k \in K$. The *upper Dini derivative* of f at $x \in A$ in the direction v is defined by

$$f^+(x; v) := \limsup_{t \downarrow 0} \frac{f(x+tv) - f(x)}{t},$$

and the *lower Dini derivative* of f at $x \in A$ in the direction v by

$$f_+(x; v) := \liminf_{t \downarrow 0} \frac{f(x+tv) - f(x)}{t}.$$

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We observe that both $f_+(x;\cdot)$ and $f^+(x;\cdot)$ are K-monotone whenever f is. Following [4] we say that f is *intermediately differentiable* at x if there exists a continuous linear functional x^* on X such that

$$f_+(x;v) \leq \langle x^*, v\rangle \leq f^+(x;v) \text{ for every } v \in X.$$

This is the same as, there exists $x^* \in X^*$ such that for every $v \in X$ there exists $t_n \downarrow 0$ with

$$\lim_{n\to\infty} \frac{f(x+t_n v) - f(x)}{t_n} = \langle x^*, v\rangle.$$

Fabian and Preiss [4] showed that for a large class of Banach spaces which includes the Asplund spaces, a locally Lipschitz function on an open subset of such a space is intermediately differentiable on a residual subset of its domain. It is our goal in this note to show that under mild assumptions, when X is separable, this also holds for cone-monotone functions.

2 Main Results

We begin with an observation on upper and lower Dini derivatives.

Lemma 1. *Let $f : X \to \mathbb{R}$ be K-increasing. Fix $x \in X$ and $e \in \operatorname{int}(K)$.*

(i) If $f^+(x;e) < +\infty$, then $f^+(x;v) < +\infty$ for every $v \in X$. Therefore, if $f^+(x;v) = +\infty$ for some $v \in X$, then $f^+(x;k) = +\infty$ for every $k \in \operatorname{int}(K)$.

(ii) If $f_+(x;-e) > -\infty$, then $f_+(x;v) > -\infty$ for every $v \in X$. Therefore, if $f_+(x;v) = -\infty$ for some $v \in X$, then $f_+(x;-k) = -\infty$ for every $k \in \operatorname{int}(K)$.

PROOF. (i) Assume $f^+(x;v) = +\infty$ for some v. Because $e \in \operatorname{int}(K)$, there exists $\epsilon > 0$ such that $e + B_\epsilon(0) \subset K$. We have

$$f^+(x;\epsilon\frac{v}{\|v\|}) \leq f^+(x;e);$$

so $f^+(x;e) = +\infty$. This contradicts the assumption.

(ii) Assume $f_+(x;v) = -\infty$ for some v. Since $e \in \operatorname{int}(K)$, there exists $\epsilon > 0$ such that

$$f_+(x;-\epsilon\frac{v}{\|v\|}) \geq f_+(x;-e),$$

so $f_+(x;-e) = -\infty$. This contradicts the assumption. □

Now we can formulate our result.

Theorem 1. *Let X be a separable Banach space and $K \subset X$ be a closed convex cone with non-empty interior. Suppose that $f : X \to \mathbb{R}$ is continuous and K-increasing. If there exists $e \in \text{int}(K)$ such that $f^+(x; e) < \infty$ and $f_+(x; -e) > -\infty$ for every $x \in X$, then f is generically intermediately differentiable on X.*

Proof. Choose a countable dense set $\{k_i\}_{i=1}^{\infty}$ from $\text{int}(K)$. For latter convenience, we let $k_1 = e$. Write

$$Y_p = \text{span}\{k_1, \ldots, k_p\}, \text{ and } B_{Y_p} := \Big\{ \sum_{i=1}^{p} l_i k_i : |l_i| \le 2 \text{ for } 1 \le i \le p \Big\}.$$

(a): **Finding intermediate derivatives on a finite dimensional space.** Define $O_n :=$

$$\Big\{ x \in X | \sup_{v \in B_{Y_p}} \left| \frac{f(x + t_x v) - f(x)}{t_x} - \langle x^*, v \rangle \right| < \frac{1}{n} \text{ for some } t_x > 0 \text{ and } x^* \in X^* \Big\}.$$

Because f is continuous and B_{Y_p} is compact, O_n is open. Indeed, let $x \in O_n$. There exists $\epsilon > 0$ such that $B_\epsilon(x) \subset O_n$. Suppose not. Then there exists $x_m \to x$ such that for every m there exists $v_m \in B_{Y_p}$ such that

$$\left| \frac{f(x_m + t_x v_m) - f(x_m)}{t_x} - \langle x^*, v_m \rangle \right| \ge \frac{1}{n}.$$

Because B_{Y_p} is compact, there exists a subsequence of $(v_m)_{m \in \mathbb{N}}$, without relabeling, say $v_m \to v \in B_{Y_p}$. Taking the limit, we have

$$\left| \frac{f(x + t_x v) - f(x)}{t_x} - \langle x^*, v \rangle \right| \ge \frac{1}{n}.$$

This contradicts the choice of x.

Borwein, Burke, and Lewis [2] show that when f is K-monotone, f is Gâteaux differentiable almost everywhere on X. This shows that O_n is dense in X. It follows that $G_p := \bigcap \{O_n | n \in \mathbb{N}\}$, is a dense G_δ in X. Let $x \in G_p$. We will show that f is intermediately differentiable at x. As $x \in G_p$, for every n, there exists $t_n > 0$ such that

$$\left| \frac{f(x + t_n v) - f(x)}{t_n} - \langle x_n^*, v \rangle \right| < \frac{1}{n} \text{ whenever } v \in B_{Y_p}. \tag{1}$$

For fixed v, we have

$$-\frac{1}{n} + \frac{f(x + t_n v) - f(x)}{t_n} \le \langle x_n^*, v \rangle \le \frac{1}{n} + \frac{f(x + t_n v) - f(x)}{t_n}.$$

So by Lemma 1

$$-\infty < f_+(x;v) \le \liminf_{n\to\infty}\langle x_n^*, v\rangle \le \limsup_{n\to\infty}\langle x_n^*, v\rangle \le f^+(x;v) < \infty. \quad (2)$$

Let $\mathbb{Q}$ denote rational numbers. Let

$$D_p := \Big\{\sum_{i=1}^{p} r_i k_i |\ r_i \in \mathbb{Q}, |r_i| \le 1\Big\}.$$

Since D_p is countable, we write $D_p := \{d_1, d_2, \ldots\}$. For d_1, by (2) we may take a subsequence of $(\langle x_n^*, d_1\rangle)_{n\in\mathbb{N}}$ such that $\langle x_{n1}^*, d_1\rangle$ converges as $n1 \to \infty$; For d_2, by (2) we may take a subsequence of $(\langle x_{n1}^*, d_2\rangle)_{n\in\mathbb{N}}$ such that $\langle x_{n2}^*, d_2\rangle$ converges as $n2 \to \infty$. Continuing in this way, we obtain $(x_{nn}^*)_{n\in\mathbb{N}}$ such that for every d_k we have

$$\langle x_{nn}^*, d_k\rangle \text{ converges as } nn \to \infty. \quad (3)$$

Associated with $(x_{nn}^*)_{n\in\mathbb{N}}$ are $t_{nn} \downarrow 0$ which verifies

$$\left|\frac{f(x+t_{nn}v)-f(x)}{t_{nn}} - \langle x_{nn}^*, v\rangle\right| < \frac{1}{nn} \text{ for all } v \in B_{Y_p}.$$

For every $v \in X$ we let

$$g(v) := \limsup_{nn\to\infty} \frac{f(x+t_{nn}v)-f(x)}{t_{nn}}.$$

Clearly, $f_+(x;v) \le g(v) \le f^+(x;v)$ for all $v \in X$. We proceed to show that g is linear on Y_p.

Now for every $d_k \in D_p$, by (3)

$$g(d_k) = \limsup_{nn\to\infty} \frac{f(x+t_{nn}d_k)-f(x)}{t_{nn}} = \lim_{nn\to\infty}\langle x_{nn}^*, d_k\rangle.$$

From (1), when $r_i \in \mathbb{Q}$ and $|r_i| \le 1$ we have

$$\left|\frac{f(x+t_{nn}\sum_{i=1}^{p} r_i k_i)-f(x)}{t_{nn}} - \langle x_{nn}^*, \sum_{i=1}^{p} r_i k_i\rangle\right| < \frac{1}{nn},$$

$$\left|\frac{f(x+t_{nn}(-k_i))-f(x)}{t_{nn}} - \langle x_{nn}^*, (-k_i)\rangle\right| < \frac{1}{nn},$$

and

$$\left|\frac{f(x+t_{nn}(-\sum_{i=1}^{p} r_i k_i)) - f(x)}{t_{nn}} - \langle x^*_{nn}, -\sum_{i=1}^{p} r_i k_i\rangle\right| < \frac{1}{nn}.$$

As $nn \to \infty$, we obtain

$$g\Big(\sum_{i=1}^{p} r_i k_i\Big) = \sum_{i=1}^{p} r_i g(k_i), \tag{4}$$

whenever $r_i \in \mathbb{Q}$ and $|r_i| \le 1$. Because g is K-increasing and K is a convex cone, for each $l_1, l_2, \ldots, l_p$ we can find rationals $\hat{l}_1 \ge l_1, \ldots, \hat{l}_p \ge l_p$ such that

$$g\Big(\sum_{i=1}^{p} l_i k_i\Big) \le g\Big(\sum_{i=1}^{p} \hat{l}_i k_i\Big) = \sum_{i=1}^{p} \hat{l}_i g(k_i),$$

where the equality follows from (4). Letting $\hat{l}_1 \to l_1, \ldots, \hat{l}_p \to l_p$, we obtain

$$g\Big(\sum_{i=1}^{p} l_i k_i\Big) \le \sum_{i=1}^{p} l_i g(k_i).$$

Similarly, we have $g\Big(\sum_{i=1}^{p} l_i k_i\Big) \ge \sum_{i=1}^{p} l_i g(k_i)$. Hence

$$g\Big(\sum_{i=1}^{p} l_i k_i\Big) = \sum_{i=1}^{p} l_i g(k_i),$$

when $|l_i| \le 1$ for $1 \le i \le p$. Because g is positive homogeneous, g is linear on Y_p.

(b): **From finite dimensional spaces to a dense linear span.**

Write $Y = \bigcup_{p=1}^{\infty} Y_p$. Because $\{k_i\}_{i=1}^{\infty}$ is dense in K, and $X = K - K$, Y is dense in X. For each Y_p, by (a) there exists G_p, a dense G_δ subset of X, such that for every $x \in G_p$ there exists $g : X \to \mathbb{R}$ satisfying

(i) g is linear on Y_p;

(ii) g is K-increasing on X and $g(v) \le f^+(x; e)$ for $v \le_K e$ with $v \in X$;

(iii) $f_+(x; v) \le g(v) \le f^+(x; v) \quad$ for $v \in X$.

Let $G := \bigcap_{p=1}^{\infty} G_p$ and $x \in G$. By (ii), there exists $g : X \to \mathbb{R}$ satisfying (i), (ii), and (iii) such that $\langle g, y\rangle \le \langle g, e\rangle \le f^+(x; e)$, when $y \le_K e$ and $y \in Y_p$. (Note here that we use $\langle g, y\rangle$ because g is linear on Y_p.) Because $e \in \operatorname{int}(K)$, there exists a $\alpha > 0$ such that $B_\alpha(0) \subset \{y \in X : y \le_K e\}$. Therefore,

$$\langle g, y\rangle \le \frac{f^+(x; e)}{\alpha}\|y\| \text{ for } y \in Y_p.$$

By the Hahn-Banach theorem, there exists $x^* \in X^*$ such that $x^*|_{Y_p} = g|_{Y_p}$ and $\langle x^*, y\rangle \le \frac{f^+(x;e)}{\alpha}\|y\|$, for $y \in X$. Set

$$C_p := \big\{x^* \in X^* |\ f_+(x; v) \le \langle x^*, v\rangle \le f^+(x; v) \text{ for } v \in Y_p, \|x^*\| \le \frac{f^+(x; e)}{\alpha}\big\}.$$

Then C_p is weak* closed and bounded, so weak* compact. By (a) we have $\{C_p : p \in \mathbb{N}\}$ has finite intersection property. Indeed, for any finite number of finite dimensional subspaces $Y_{p_1}, \ldots, Y_{p_k}$, there exists p large such that

$$Y_{p_1} \cup Y_{p_2} \cup \ldots \cup Y_{p_k} \subset Y_p.$$

Since $x \in G_p$, we know $C_p \subset \bigcap_{i=1}^{k} C_{p_i}$. It follows that $C := \bigcap_{p=1}^{\infty} C_p \ne \emptyset$. For $x^* \in C$, we have

$$f_+(x; y) \le \langle x^*, y\rangle \le f^+(x; y) \text{ for every } y \in Y.$$

(c): **From dense linear space to the separable space.**
From (b), for $x \in G$, there exists $x^* \in X^*$ such that

$$f_+(x; y) \le \langle x^*, y\rangle \le f^+(x; y) \text{ for every } y \in Y, \tag{5}$$

where Y is dense in X. For every $v \in X$, $v + \operatorname{int}(K)$ and $v - \operatorname{int}(K)$ are open. Because Y is dense in X, there exist $y_n, z_n \in Y$ arbitrary nearby v such that $y_n \in v - \operatorname{int}(K)$ and $z_n \in v + \operatorname{int}(K)$. That is, we can find $y_n, z_n \in Y$ such that $y_n \le_K v \le_K z_n$, while $y_n \to v$ and $z_n \to v$ in norm. Now by (5),

$$\begin{aligned}\langle x^*, y_n\rangle &\le f^+(x; y_n) \le f^+(x; v), \text{ and}\\ \langle x^*, z_n\rangle &\ge f_+(x; z_n)\ \ge f_+(x; v).\end{aligned}$$

Letting $n \to \infty$, we obtain $f_+(x; v) \le \langle x^*, v\rangle \le f^+(x; v)$. Therefore, x^* is an intermediate derivative of f at $x \in G$. □

Recall that a function $f : X \to \mathbb{R}$ is *quasiconvex* if the lower level sets $S_\lambda(f) = \{x \in A |\ f(x) \le \lambda\}$ is convex for every $\lambda \in \mathbb{R}$. We need the following fact from [1].

Lemma 2. *Assume f is quasiconvex and lower semicontinuous (l.s.c.) on a Banach space X. Suppose that S_λ has non-empty interior. Then for every a with $f(a) > \lambda$, there exist an open neighborhood V of a and a convex cone K with non-empty interior, such that f is K-monotone on V.*

Corollary 1. *Let X be a separable Banach space. Suppose that $f : X \to \mathbb{R}$ is continuous, quasiconvex, and $f_+(x; v) > -\infty$, $f^+(x; v) < +\infty$ for all $x, v \in X$. Then f is intermediately differentiable generically on X.*

PROOF. Consider $\overline{\lambda}$ such that whenever $\mu < \overline{\lambda} < \lambda$, the set $S_\mu(f)$ has no interior and $S_\lambda(f)$ has interior. Define

$$A := \{x \in X|\ f(x) < \overline{\lambda}\}, \quad B := \{x \in X|\ f(x) = \overline{\lambda}\},$$

$$C := \{x \in X|\ f(x) \le \overline{\lambda}\}.$$

The set $A = \bigcup_{n=1}^{\infty} A_n$ with $A_n := \{x \in X|\ f(x) \le \overline{\lambda} - 1/n\}$. Since A_n has no interior and closed, A is of first category. bdry (B) is also nowhere dense. For each $x \in (X \setminus C)$, by Lemma 2, there exists a neighborhood U_x of x such that f is K-monotone on U_x for some closed convex cone K with $\mathrm{int}(K) \neq \emptyset$. By Theorem 1, f is intermediate differentiable generically on U_x. Since X is separable, f is generically intermediate differentiable on $X \setminus C$. $\square$

A function $f : X \to \mathbb{R} \cup \{+\infty\}$ is called *directionally Lipschitz* at x in the direction $u \in X$ if there exists $\epsilon > 0$ such that when $\|z - x\| < \epsilon$, $\|h - u\| < \epsilon$, $0 < t < \epsilon$, one has

$$\frac{f(z + th) - f(z)}{t} < M.$$

In particular, $f^+(z; h) < M$ when $\|z - x\| < \epsilon, \|h - u\| < \epsilon$. Borwein, Burke, Lewis [2] show that if f is directionally Lipschitz at x, then there exists a neighborhood U_x of x, a continuous linear functional $\phi \in X^*$, and a closed convex cone K with $\mathrm{int}(K) \neq \emptyset$ such that $f + \phi$ is K-monotone on U_x. Therefore, we can apply Theorem 1 to $f + \phi$ on U_x provided that $f_+(z, v) > -\infty$ and $f^+(z; v) < +\infty$ for $z \in U_x$ and $v \in X$. With this in mind, we have the following consequence.

Corollary 2. *Let X be a separable Banach space, $A \subset X$ be nonempty open. If f is continuous, directionally Lipschitz at every point of A, and $f_+(x; v) > -\infty$, $f^+(x; v) < \infty$ for $x \in A$ and $v \in X$, then f is generically intermediate differentiable on A.*

We remark that Theorem 1 concerns finite intermediate derivatives. If we remove the finiteness of Dini derivates, the result may fail. This is illustrated by the following modified example from [3, page 288].

Example 1. Let E be a dense G_δ subset in $[0,1]$ with Lebesgue measure 0. There exists a continuous, strictly increasing function $f : [0,1] \to \mathbb{R}$ such that $f'(x) = +\infty$ for every $x \in E$. The points at which f has finite intermediate derivative must lie in $[0,1] \setminus E$, which is of first category.

3 Appendix

We say that $f : X \to \mathbb{R}$ is Lipschitz at x if

$$L(x) := \limsup_{y \to x} \frac{|f(y) - f(x)|}{\|y - x\|},$$

is finite. Prof. D. Preiss informed me of the following.

Lemma 3. *Let X be an arbitrary Banach space. Assume that $f : X \to \mathbb{R}$ is pointwise Lipschitz on X; that is, $L(x) < +\infty$ for every $x \in X$. Then there exists a dense open set O of X such that f is locally Lipschitz on O.*

PROOF. Define

$$g_n(x) := \sup_{0 < \|y - x\| < 1/n} \frac{|f(y) - f(x)|}{\|y - x\|}.$$

Then $L(x) = \inf_{n \geq 1} g_n(x)$ for every $x \in X$. Since g_n is lower semicontinuous on X, there exists a dense G_δ set D_n of X such that g_n is continuous at every point of $x \in D_n$. Define $D = \bigcap_{n=1}^{\infty} D_n$. Then D is dense G_δ in X. At every $x \in D$, L is upper semicontinuous. To see this, for $\epsilon > 0$, there exists g_N such that $g_N(x) < L(x) + \epsilon$. Since g_N is continuous at x, there exists an open neighborhood U_x of x such that $g_N(y) < L(x) + \epsilon$. Since $L \leq g_N$, we have $L(y) < L(x) + \epsilon$ for $y \in U_x$. One can take U_x to be convex. For every $y_1, y_2 \in U_x$, $[y_1, y_2] \subset U_x$. By compactness, we have

$$|f(y_2) - f(y_1)| \leq (L(x) + \epsilon)\|y_2 - y_1\|.$$

Hence f is Lipschitz on U_x. It follows that the set

$$O := \{x \in X |\ \exists \text{ an open set } U_x \text{ containing } x \text{ such that } f \text{ is Lipschitz on } U_x\}$$

is open and $D \subset O$. Thus, O is the required dense and open subset. $\square$

Lemma 4. *Let X be a finite dimensional Banach space and suppose that $f : X \to \mathbb{R}$ is K-increasing with $\operatorname{int}(K) \neq \emptyset$. Then the following are equivalent:*

(a) At $x \in X$, $f_+(x; v) > -\infty$ and $f^+(x; v) < +\infty$ for every $v \in X$.

(b) f is Lipschitz at point x.

Proof. It suffices to show (a)⇒(b). Suppose (b) does not hold. That is, there exists $y_n \to x$ such that

$$\limsup_{y_n\to x} \frac{|f(y_n)-f(x)|}{\|y_n-x\|} = \infty.$$

Without relabeling, let us assume

$$\lim_{y_n\to x} \frac{f(y_n)-f(x)}{\|y_n-x\|} = +\infty.$$

The other case is similar. Write $y_n = x + t_n v_n$ with $t_n = \|y_n - x\|$ and $v_n = (y_n - x)/t_n$. As X is finite dimensional, there exists a subsequence of $(v_n)_{n\in\mathbb{N}}$ converging. Without relabeling we assume $v_n \to v$. We have

$$\limsup_{t_n\downarrow 0, v_n\to v} \frac{f(x+t_n v_n)-f(x)}{t_n} = +\infty.$$

Take $e \in \operatorname{int}(K)$. For n sufficiently large, $(v - v_n) + e \in \operatorname{int}(K)$. Since f is K-increasing, we have

$$\frac{f(x+t_n v_n)-f(x)}{t_n} \le \frac{f(x+t_n(v+e))-f(x)}{t_n}.$$

Taking limsup gives $f^+(x; v+e) = +\infty$. This contradicts (a). □

These two lemmas show that Theorem 1 can be deduced from the results for Lipschitz functions [4, 5] when X is finite dimensional. Nevertheless, when X is infinite dimensional, it is not clear whether Lemma 4 holds.

Following [5] we say that a function $f : X \to \mathbb{R}$ is said to be *uniformly intermediately differentiable* at x if there exists a continuous linear functional x^* on X and a sequence $t_n \downarrow 0$ such that

$$\lim_{n\to\infty} \frac{f(x+t_n v)-f(x)}{t_n} = \langle x^*, v\rangle, \text{ for all } v \in X,\ \|v\| = 1.$$

Here 'uniformly' means that the same sequence is used for all $v \in X$, $\|v\| = 1$. Using Lemma 3 and Preiss' Differentiability Theorem, we can follow Giles and Sciffer's arguments in the proof of Theorem 1.4 to obtain the final result.

Theorem 2. *A pointwise Lipschitz function f on an open subset A of an Asplund space X is uniformly intermediately differentiable on a dense G_δ subset of A.*

This refines Theorem 1.4 of Giles and Sciffer [5].

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