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ON POINTWISE HÖLDER FUNCTIONS*

Abstract

Let $f : X \to \mathbb{R}$, where X is a subset of $\mathbb{R}^k$. We observe that in order for f be almost everywhere pointwise Hölder it is enough that f satisfy the Hölder condition inside angles of a fixed width. In this analysis, density points of X play a primary role. This has some interesting consequences concerning summability of a naturally defined coefficient.

1 Introduction

The classical theorems due to Rademacher and Stepanoff clearly show the link between locally Lipschitz functions and differentiable functions. Lately, there have been numerous investigations in this direction ([1], [2], [4], [8], [13], [17], [18]). On the other hand, the important role played by directions has appeared to be fundamental, and it was already noticed in the classical papers by Blumberg and Haslam Jones ([3], [11]), and successively in the papers by de Lucia, Guariglia and Mukhopadhyay ([6], [14], [10]).

We think it useful to investigate pointwise Hölder functions, and in this context rather than working to examine a spanning set of directions, we focus our attention on intervals of $\mathbb{R}^k$ with an opportune parameter of regularity. In Section 2 we provide a characterization of almost everywhere pointwise Hölder functions. In Section 4 we prove a condition that assures the summability of the Hölder coefficient of an almost everywhere pointwise Hölder function. Finally, in Section 3, using properties analogous to the classical notions

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of absolute continuity and bounded variation, we establish further conditions that a function be almost everywhere pointwise Hölder.

In what follows λ and $\lambda^\star$ are the Lebesgue measure and the Lebesgue outer measure of $\mathbb{R}^k$ ($k \geq 2$), respectively.

If $x = (x_1, \ldots, x_k)$, $y = (y_1, \ldots, y_k)$ are two points of $\mathbb{R}^k$, by $|x - y|$ we denote the Euclidean distance between x and y.

We write $x < y$ if $x_i < y_i$, for all $i \in \{1, \ldots, k\}$.

If $x < y$, by $r([x, y])$ we mean the *parameter of regularity* of the interval $[x, y]$ defined as $r([x, y]) = \frac{\lambda([x,y])}{L^k}$, where $L = \max\{|x_1 - y_1|, \ldots, |x_k - y_k|\}$. If $r([x, y]) = 1$, we call $[x, y]$ a *cube*.

If X is a subset of $\mathbb{R}^k$ and $x \in \mathbb{R}^k$, $d(x, X)$ is the distance of x from X and $\delta(X)$ is the diameter of X.

2 Almost Everywhere Pointwise Hölder Functions

Definition 2.1. A function $f : X \subseteq \mathbb{R}^k \to \mathbb{R}$ is said to be *locally Hölder* with exponent $\gamma \in]0, 1]$ in a set $Y \subseteq X$ if there is a positive number L with the property that for every $x \in Y$ there exists a positive number δ such that

$$|f(x) - f(y)| \leq L|x - y|^\gamma$$

whenever $y \in X$ and $|x - y| < \delta$.
A function $f : X \subseteq \mathbb{R}^k \to \mathbb{R}$ is said to be *pointwise Hölder* with exponent $\gamma \in]0, 1]$ in a set $Y \subseteq X$ if for every $x \in Y$ there exist a positive number L and a positive number δ such that

$$|f(x) - f(y)| \leq L|x - y|^\gamma$$

whenever $y \in X$ and $|x - y| < \delta$.
We recall that in the case when γ equals 1 the function f is said to be locally Lipschitz ([7]: page 64) and pointwise Lipschitz, respectively.

In order to obtain a characterization of almost everywhere pointwise Hölder functions we intend to investigate the ratio $\frac{|f(x)-f(y)|}{|x-y|^\gamma}$ requiring the interval $[x, y]$ be non-empty and have an opportune parameter of regularity. To this aim, given $X \subseteq \mathbb{R}^k$, $f : X \to \mathbb{R}$, $\alpha \in]0, 1[$, $\gamma \in]0, 1]$ and $n \in \mathbb{N}$, by $X_{n,\alpha,\gamma}$ we denote the subset of all points x in X such that

$$(y \in X, x < y, |x - y| < \frac{1}{n}, r([x, y]) > \alpha)$$

implies $|f(x) - f(y)| \leq n|x - y|^\gamma$.

In order to prove Theorem 2.8 we need to recall two classical results.

Lemma 2.2. *([15]: Theorem 35.2) If X is any set (measurable or not) in $\mathbb{R}^k$, then x is a point of density for X for almost all $x \in X$. A necessary and sufficient condition that X be measurable is that x be a point of dispersion for X for almost all $x \in \mathbb{R}^k \setminus X$.*

Lemma 2.3. *([20]: Lemma, page 233) Let X be a subset of $\mathbb{R}^k$ and let x_0 be a density point of X. Then there exists $\theta > 0$ such that for every cube Q satisfying $\delta(Q) \leq d(x_0, Q) \leq \theta$ it is $\lambda^\star(X \cap Q) > 0$.*

We also recall two results from [6].

Lemma 2.4. *([6]: 1.1) Let X be a subset of $\mathbb{R}^k$, x_0 a density point of X and $0 < \alpha < 1$. Then there exist two positive numbers δ and η (the second one does not depend on x_0) such that for every $x \in \mathbb{R}^k$ if $0 < |x - x_0| < \delta$, then there exists $y \in X$ such that*

$$(y < x, y < x_0, r([y, x]) > \alpha, r([y, x_0]) > \alpha, |y - x_0| \leq \eta |x - x_0|).$$

Lemma 2.5. *([6]: 1.2) Let X be a subset of $\mathbb{R}^k$, x_0 a density point of X and $0 < \alpha < 1$. If x is a point of $\mathbb{R}^k$ satisfying $x_0 < x$ and $r([x_0, x]) > \alpha$, then for each $\eta > 0$ there exists $y \in X$ such that*

$$(y < x_0, |y - x_0| < \eta, r([y, x]) > \alpha, r([y, x_0]) > \alpha).$$

Proposition 2.6. *Let $X \subseteq \mathbb{R}^k$, $f : X \to \mathbb{R}$, $\alpha \in]0, 1[$, $\gamma \in]0, 1]$ and $n \in \mathbb{N}$. If X is measurable, then $X_{n,\alpha,\gamma}$ is also measurable.*

Proof. By Lemma 2.2 it is enough to show that if x_0 is a point of X which is a density point of $X_{n,\alpha,\gamma}$, then $x_0 \in X_{n,\alpha,\gamma}$. Let $x \in X$ be a density point of $X_{n,\alpha,\gamma}$ satisfying

$$(x_0 < x, |x - x_0| < \frac{1}{n}, r([x_0, x]) > \alpha),$$

by Lemma 2.5 there exists a sequence $\{y_p\}_{p \in \mathbb{N}} \subseteq X_{n,\alpha,\gamma}$ such that

$$y_p < x_0, |y_p - x_0| < \frac{1}{p}, r([y_p, x_0]) > \alpha \text{ and } r([y_p, x]) > \alpha, \forall p \in \mathbb{N}.$$

Therefore, for every $p > \frac{n}{1 - n|x_0 - x|}$

$$|f(x) - f(x_0)| \leq |f(x) - f(y_p)| + |f(y_p) - f(x_0)| \leq n(|x - y_p|^\gamma + |y_p - x_0|^\gamma),$$

so, letting p tend to $+\infty$, we have $|f(x) - f(x_0)| \leq |x - x_0|^\gamma$. □

Proposition 2.7. *Let $X \subseteq \mathbb{R}^k$, $f : X \to \mathbb{R}$, $\alpha \in]0,1[$, $\gamma \in]0,1]$ and $n \in \mathbb{N}$. Then f is locally Hölder with exponent γ at all points of X which are density points of $X_{n,\alpha,\gamma}$.*

Proof. Let $x_0 \in X$ be a density point of $X_{n,\alpha,\gamma}$. By Lemma 2.4 there exist two positive numbers δ and η (the second one does not depend on x_0) such that if x is a point of $\mathbb{R}^k$ whose distance from x_0 is positive and less than δ, then there exists $y \in X_{n,\alpha,\gamma}$ satisfying:

1. $y < x, y < x_0, r([y,x]) > \alpha, r([y,x_0]) > \alpha$, and
2. $|y - x_0| \leq \eta |x - x_0|$.

Hence, if $x \in X$ is such that $|x - x_0| < \min\{\frac{1}{(1+\eta)n}, \delta\}$, then there exists $y \in X_{n,\alpha,\gamma}$ satisfying (2) and, therefore, such that

$$|y - x_0| < \frac{1}{n}, |y - x| \leq (1+\eta)|x - x_0| < \frac{1}{n}.$$

From the last inequality and from (1) and (2) it follows that

$$\begin{aligned} |f(x) - f(x_0)| \leq & n(|x-y|^\gamma + |y - x_0|^\gamma) \\ \leq & n[(1+\eta)^\gamma |x - x_0|^\gamma + \eta^\gamma |x - x_0|^\gamma] \\ = & n[(1+\eta)^\gamma + \eta^\gamma]|x - x_0|^\gamma \end{aligned}$$

The number L we are looking for is $L = n[(1+\eta)^\gamma + \eta^\gamma]$. □

The following theorem is a characterization of almost everywhere pointwise Hölder functions. This result essentially points out that in order for f be almost everywhere pointwise Hölder it is enough that f have the Hölder condition inside certain angles of a fixed width.

Theorem 2.8. *Let X be a bounded measurable subset of $\mathbb{R}^k$, let $f : X \to \mathbb{R}$ and let $\gamma \in]0,1]$. Then the following are equivalent:*

1. *f is almost everywhere pointwise Hölder with exponent γ in X,*
2. *for every $\epsilon > 0$ there exist $n \in \mathbb{N}$ and $\alpha \in]0,1[$ such that $\lambda(X \setminus X_{n,\alpha,\gamma}) < \epsilon$.*

Proof. (1) $\Leftarrow$ (2) It is enough to observe that, by Proposition 2.6 and by Lemma 2.2, almost every point of $X_{n,\alpha,\gamma}$ is a density point of $X_{n,\alpha,\gamma}$. It follows from Proposition 2.7 that f is almost everywhere locally Hölder with exponent γ in $X_{n,\alpha,\gamma}$.

(2) $\Rightarrow$ (1) If f is almost everywhere pointwise Hölder with exponent γ, then, denoting by $X_{n,\gamma}$ the set of all $x \in X$ such that

$$(y \in X, |y-x| < \frac{1}{n}) \Rightarrow (|f(x) - f(y)| \leq n|y-x|^{\gamma}),$$

we have $\lambda(X \setminus \cup_{n\in\mathbb{N}} X_{n,\gamma}) = 0$ and, as for any $\alpha \in]0,1[$,

$$X_{n,\gamma} \subseteq X_{n,\alpha,\gamma}, \forall n \in \mathbb{N},$$

it follows that $\lambda(X \setminus \cup_{n\in N} X_{n,\alpha,\gamma}) = 0$. Since $\{X_{n,\alpha,\gamma}\}_{n\in\mathbb{N}}$ is an increasing sequence of measurable sets, the assertion follows. □

3 Absolute Continuity-Type and Bounded Variation-Type Conditions

We observe that in the literature there exist characterizations of almost everywhere differentiability, among others ([6], [10], [16]), involving properties analogous to the classical notions of absolute continuity and bounded variation. In order to show that opportune characterizations of the same type also hold for almost everywhere pointwise Hölder functions we introduce the following definitions.

Let $f : X \subseteq \mathbb{R}^k \to \mathbb{R}$, $\alpha \in]0,1[$ and $\gamma \in]0,1]$. We say that f *has property* $\mathcal{P}_{1,\alpha,\gamma}$ *in* $Y \subseteq X$ if for every $\sigma > 0$ there exists $\delta > 0$ such that

$$\sum_{i=1}^{p} |f(b_i) - f(a_i)|^{\frac{k}{\gamma}} < \sigma$$

for each finite sequence of pairwise non-overlapping intervals $\{[a_1, b_1], \ldots, [a_t, b_t]\}$ satisfying

$$(a_i, b_i \in X, r([a_i, b_i]) > \alpha, [a_i, b_i] \cap Y \neq \emptyset, \forall i \in \{1, \ldots, t\}, \sum_{i=1}^{t} \lambda([a_i, b_i]) < \delta).$$

We say that f *satisfies* $\mathcal{P}_{2,\alpha,\gamma}$ *in* $Y \subseteq X$ if there exist two positive numbers K and δ such that $\sum_{i=1}^{t} |f(b_i) - f(a_i)|^{\frac{k}{\gamma}} < K$ for every finite sequence of pairwise non-overlapping intervals $\{[a_1, b_1], \ldots, [a_t, b_t]\}$ satisfying

$$(a_i, b_i \in X, r([a_i, b_i]) > \alpha, [a_i, b_i] \cap Y \neq \emptyset, |a_i - b_i| < \delta, \forall i \in \{1, \ldots, t\}).$$

Theorem 3.1. *Let X be a bounded measurable subset of $\mathbb{R}^k$, $f : X \to \mathbb{R}$ and $\gamma \in]0,1]$. The following are equivalent:*

(a) *The function f is almost everywhere pointwise Hölder with exponent γ in X.*

(b) *For every $\epsilon > 0$ there exist a measurable subset Y of X and $\alpha \in]0,1[$ such that f has property $\mathcal{P}_{1,\alpha,\gamma}$ in Y and $\lambda(X \setminus Y) < \epsilon$.*

(c) *For every $\epsilon > 0$ there exist a measurable subset Y of X and $\alpha \in]0,1[$ such that f has property $\mathcal{P}_{2,\alpha,\gamma}$ in Y and $\lambda(X \setminus Y) < \epsilon$.*

Proof. $(a) \Rightarrow (b)$ As (a) holds, by Theorem 2.8, for every $\epsilon > 0$ there exist $n \in \mathbb{N}$ and $\alpha \in]0,1[$ such that $\lambda(X \setminus X_{n,\alpha,\gamma}) < \epsilon$. Denoted by $X'_{n,\alpha,\gamma}$ the set of all points of $X_{n,\alpha,\gamma}$ which are density points of $X_{n,\alpha,\gamma}$, for every $p \in \mathbb{N}$ we call Y_p the set of all $x \in X'_{n,\alpha,\gamma}$ with the property that

$$(a, b \in X, x \in [a,b], r([a,b]) > \alpha, |b-a| < \frac{1}{p})$$

implies

$$(|f(b) - f(a)| \le 4n[(1+\eta)^\gamma + \eta^\gamma]|b-a|^\gamma),$$

where η is the same as in Lemma 2.4. The sequence $\{Y_p\}_{p\in\mathbb{N}}$ is an increasing sequence of measurable sets. In fact if $x \in X'_{n,\alpha,\gamma} \setminus Y_p$, then there exists an interval $[a,b]$ such that

$$(a, b \in X, x \in [a,b], r([a,b]) > \alpha, |b-a| < \frac{1}{p})$$

and

$$(|f(b) - f(a)| > 4n[(1+\eta)^\gamma + \eta^\gamma]|b-a|^\gamma).$$

Since $[a,b]$ has empty intersection with Y_p, there exists a set Y disjoint from Y_p and containing $X'_{n,\alpha,\gamma} \setminus Y_p$. (The set Y is measurable since it is union of non-degenerate intervals ([21]: Lemma 4.1, page 112.) Because $X'_{n,\alpha,\gamma} \setminus Y_p = X'_{n,\alpha,\gamma} \cap Y$ the measurability of Y_p follows. Moreover $X'_{n,\alpha,\gamma} = \cup_{p\in\mathbb{N}} Y_p$. In fact if $x_0 \in X'_{n,\alpha,\gamma}$, by Lemma 2.4, there exists $\delta > 0$ such that for every $x \in \mathbb{R}^k$ with $0 < |x - x_0| < \delta$ there exists $y \in X$ satisfying

$$(y < x, y < x_0, r([y,x]) > \alpha, r([y,x_0]) > \alpha, |y - x_0| \le \eta|x - x_0|).$$

Let $p > \max\{\frac{1}{\delta}, n(\eta+1)\}$ and let $[a,b]$ be an interval such that

$$(a, b \in X, x_0 \in [a,b], |b-a| < \frac{1}{p}, r([a,b]) > \alpha).$$

By Lemma 2.5 there exist two points $y_1, y_2 \in X^{'}_{n,\alpha,\gamma}$ such that

$$y_1 < a, y_1 < x_0, y_2 < b, y_2 < x_0$$

$$|y_1 - x_0| \leq \eta |a - x_0| < \frac{1}{n}, |y_2 - x_0| \leq \eta |b - x_0| < \frac{1}{n}$$

and

$$r([y_1, a]) > \alpha, r([y_1, x_0]) > \alpha, r([y_2, b]) > \alpha, r([y_2, x_0]) > \alpha.$$

Therefore

$$|f(b) - f(a)| \leq |f(b) - f(y_2)| + |f(y_2) - f(x_0)| + |f(x_0) - f(y_1)| + |f(y_1) - f(a)|$$

and, as

$$|a - y_1| \leq |a - x_0|(1 + \eta) < \frac{1}{n} \text{ and } |b - y_2| \leq |b - x_0|(1 + \eta) < \frac{1}{n},$$

it follows that

$$\begin{aligned}
|f(b) - f(a)| \leq& n(|b - y_2|^\gamma + |y_2 - x_0|^\gamma + |x_0 - y_1|^\gamma + |y_1 - a|^\gamma) \\
\leq& n[(1 + \eta)^\gamma |b - x_0|^\gamma + \eta^\gamma |b - x_0|^\gamma \\
& + \eta^\gamma |a - x_0|^\gamma + (1 + \eta)^\gamma |a - x_0|^\gamma] \\
\leq& n[(1 + \eta)^\gamma + \eta^\gamma](|b - x_0|^\gamma + |b - x_0|^\gamma + |a - x_0|^\gamma + |a - x_0|^\gamma) \\
\leq& 4n[(1 + \eta)^\gamma + \eta^\gamma]|b - a|^\gamma.
\end{aligned}$$

In order to prove that (b) holds it is enough to show that f has property $\mathcal{P}_{1,\alpha,\gamma}$ in Y_p. To this aim let us observe that if I is an interval with parameter of regularity greater than α then there exists K_α such that $(\delta(I))^k \leq K_\alpha \lambda(I)$. Hence, if $[a, b]$ is an interval such that

$$(a, b \in X, [a, b] \cap Y_p \neq \emptyset, r([a, b]) > \alpha, |b - a| < \frac{1}{p}),$$

this yields

$$|f(b) - f(a)|^{\frac{k}{\gamma}} \leq \{4n[(1 + \eta)^\gamma + \eta^\gamma]\}^{\frac{k}{\gamma}} |b - a|^k \leq H_{\alpha,\gamma} \lambda([a, b]),$$

where

$$H_{\alpha,\gamma} = K_\alpha \{4n[(1 + \eta)^\gamma + \eta^\gamma]\}^{\frac{k}{\gamma}}.$$

Let $\sigma > 0$. If $\{[a_1, b_1], \dots, [a_t, b_t]\}$ is a finite sequence of non-overlapping intervals such that

$$([a_i, b_i] \cap Y_{p,\gamma} \neq \emptyset, r([a_i, b_i]) > \alpha, \forall i \in \{1, \dots, t\}, \sum_{i=1}^{t} \lambda([a_i, b_i]) < \frac{\sigma}{H_{\alpha,\gamma}}),$$

then

$$\sum_{i=1}^{t} |f(b_i) - f(a_i)|^{\frac{k}{\gamma}} < H_{\alpha,\gamma} \sum_{i=1}^{t} \lambda([a_i, b_i]) < \sigma.$$

$(b) \Rightarrow (c)$. It is enough to show that there exists $\alpha \in]0, 1[$ such that

$$(f \text{ has property } \mathcal{P}_{1,\alpha,\gamma} \text{ in the measurable subset } Y \text{ of } X) \Rightarrow (f \text{ has property } \mathcal{P}_{2,\alpha,\gamma} \text{ in } Y).$$

Let f have property $\mathcal{P}_{1,\alpha,\gamma}$ in Y. Then there exists $\delta > 0$ such that for every finite sequence of pairwise non-overlapping intervals $\{[a_1, b_1], \dots, [a_t, b_t]\}$ with

$$(a_i, b_i \in X, r([a_i, b_i]) > \alpha, [a_i, b_i] \cap Y \neq \emptyset, \forall i \in \{1, \dots, t\}, \sum_{i=1}^{t} \lambda([a_i, b_i]) < \delta)$$

we have $\sum_{i=1}^{t} |f(b_i) - f(a_i)|^{\frac{k}{\gamma}} < 1$. On the other hand, we can find a finite cover of Y, $\{C_1, \dots, C_s\}$, with C_i $(1 \leq i \leq s)$ open sphere having volume less than δ, with the following property.

> There exists $\tau > 0$ such that every interval having measure less than τ, with parameter of regularity greater than α and having empty intersection with Y, is contained in at least one sphere from the cover.

If $\{[a_1, b_1], \dots, [a_t, b_t]\}$ is a finite sequence of pairwise non-overlapping intervals such that

$$([a_i, b_i] \cap Y \neq \emptyset, r([a_i, b_i]) > \alpha, |a_i - b_i| < \tau^{\frac{1}{k\gamma}}, \forall i \in \{1, \dots, t\})$$

and, for every $j \in \{1, \dots, s\}$, we denote by N_j the set of $i \in \{1, \dots, t\}$ such that $[a_i, b_i] \subseteq C_j$ then

$$\sum_{i=1}^{t} |f(b_i) - f(a_i)|^{\frac{k}{\gamma}} \leq \sum_{j=1}^{s} \sum_{i \in N_j} |f(b_i) - f(a_i)|^{\frac{k}{\gamma}}$$

and, because $\sum_{i\in N_j} \lambda([a_i,b_i]) \le \lambda(C_j) < \delta$, it follows that

$$\sum_{i=1}^{t} |f(b_i)-f(a_i)|^{\frac{k}{\gamma}} < r.$$

$(c) \Rightarrow (a)$ By Theorem 2.8, it is enough to show that, if Y denotes a measurable subset of X where f satisfies property $\mathcal{P}_{2,\alpha,\gamma}$, then the set

$$Y_0 = Y \setminus \cup_{n\in\mathbb{N}} X_{n,\alpha,\gamma}$$

has measure zero. To this aim let us observe that if x is a point of Y_0, then for every $n \in \mathbb{N}$, there exists $y \in X$ such that

$$x < y, |x-y| < \frac{1}{n}, r([x,y]) > \alpha \text{ and } |f(x)-f(y)| > n|x-y|^{\gamma}.$$

Hence, for any $p \in \mathbb{N}$, we have that for every $q \ge p$ there exists $y \in X$ such that

$$x < y, |x-y| < \frac{1}{q}, r([x,y]) > \alpha \text{ and } p|y-x|^{\gamma} \le q|y-x|^{\gamma} < |f(x)-f(y)|.$$

Denoted by $\mathcal{F}_p$ the set of the intervals $[x,y]$ such that

$$(x \in Y_0, y \in X, x < y, r([x,y]) > \alpha, p|y-x|^{\gamma} < |f(x)-f(y)|),$$

$\mathcal{F}_p$ is a Vitali-covering of Y_0. Hence there exists an at most countable subset of pairwise non-overlapping intervals, $\{[x_i,y_i]\}_{i\in\mathbb{N}'}(\mathbb{N}' \subseteq \mathbb{N})$, of $\mathcal{F}_p$ such that

$$\lambda(Y_0 \setminus \cup_{i\in\mathbb{N}'}[x_i - y_i]) = 0.$$

Since f has property $\mathcal{P}_{2,\alpha,\gamma}$ in Y, there exist two positive numbers K and δ such that $\sum_{i=1}^{t} |f(b_i)-f(a_i)|^{\frac{k}{\gamma}} < K$ for every finite sequence $\{[a_1,b_1],\ldots,[a_t,b_t]\}$ of pairwise non-overlapping intervals such that

$$a_i, b_i \in X, r([a_i,b_i]) > \alpha, [a_i,b_i] \cap Y \ne \emptyset, |a_i - b_i| < \delta, \forall i \in \{1,\ldots,t\}.$$

Therefore, fixed $p > \frac{1}{\delta}$, we have $\sum_{i\in\mathbb{N}'} |f(y_i)-f(x_i)| \le K$. Because

$$p|y_i - x_i|^{\gamma} < |f(y_i)-f(x_i)| \; \forall i \in \mathbb{N}',$$

we have

$$\begin{aligned}
p\lambda(Y_0) \leq & p\lambda(Y_0 \setminus \cup_{i\in\mathbb{N}'}[x_i, y_i]) + \sum_{i\in\mathbb{N}'} p\lambda([x_i, y_i]) \\
\leq & \sum_{i\in\mathbb{N}'} p|x_i - y_i|^k \\
\leq & \frac{1}{p^{\frac{k}{\gamma}-1}} \sum_{i\in\mathbb{N}'} |f(x_i) - f(y_i)|^{\frac{k}{\gamma}} \leq \frac{K}{p^{\frac{k}{\gamma}-1}},
\end{aligned}$$

for every $p > \frac{1}{\delta}$. And so $\lambda(Y_0) \leq \frac{K}{p^{\frac{k}{\gamma}}}, \forall p > \frac{1}{\delta}$, implies $\lambda(Y_0) = 0$. $\square$

4 Summable Functions Generated by Almost Everywhere Pointwise Hölder Functions

Let X be a bounded measurable subset of $\mathbb{R}^k$. The aim of this section is to characterize functions $f : X \to \mathbb{R}$ satisfying the property

"$M_{(f,\gamma)}$ is summable",

where $\gamma \in]0, 1]$ and $M_{(f,\gamma)}$ is defined as

$$M_{(f,\gamma)}(x) = \limsup_{y\to x} \frac{|f(x) - f(y)|}{|x - y|^\gamma}.$$

Clearly, if X is a bounded measurable subset of $\mathbb{R}^k$ and $f : X \to \mathbb{R}$, then

> $M_{(f,\gamma)}$ summable $\Rightarrow$ $M_{(f,\gamma)}$ finite almost everywhere $\Rightarrow$ f almost everywhere pointwise Hölder with exponent γ.

Proposition 4.1. *Let $f : X \to \mathbb{R}$, where X is a bounded measurable subset of $\mathbb{R}^k$. The following are equivalent:*

1. *The function $M_{(f,\gamma)}$ is summable in X.*
2. *There exist an increasing sequence $\{X_n\}_{n\in\mathbb{N}}$ of closed subsets of X and an increasing sequence $\{L_n\}_{n\in\mathbb{N}}$ of positive numbers such that, for every $n \in \mathbb{N}$,*

$$\limsup_{y \to x} \frac{|f(y) - f(x)|}{|y - x|^\gamma} \leq L_n \ \text{ for every } x \text{ in } X_n,$$

$$\lambda(X \setminus \cup_{n\in\mathbb{N}} X_n) = 0 \;\; \textit{and} \;\; \sum_{n\in\mathbb{N}} L_n \lambda(X_n \setminus X_{n-1}) < +\infty$$

(set $X_0 = \emptyset$).

PROOF. (1) $\Rightarrow$ (2). Let X' be a subset of X of measure zero such that, for every $x \in X \setminus X'$, $M_{(f,\gamma)}(x) < +\infty$. For every $n \in \mathbb{N}$, we denote by Y_n the set

$$\{x \in X \setminus X' : M_{(f,\gamma)}(x) < n\}.$$

Set $L_n = n$. The sequence $\{Y_n\}_{n\in\mathbb{N}}$ is an increasing sequence of measurable sets such that $\lambda(X \setminus \cup_{n\in\mathbb{N}} Y_n) = 0$, and

$$n - 1 \leq M_{(f,\gamma)}(x) < n, \forall x \in Y_n \setminus Y_{n-1}$$

(set $Y_0 = \emptyset$).
Then

$$\begin{aligned}
\sum_{n\in\mathbb{N}} n\lambda(Y_n \setminus Y_{n-1}) &= \sum_{n\in\mathbb{N}} (1 + n - 1)\lambda(Y_n \setminus Y_{n-1}) \\
&= \sum_{n\in\mathbb{N}} \lambda(Y_n \setminus Y_{n-1}) + \sum_{n\in\mathbb{N}} (n-1)\lambda(Y_n \setminus Y_{n-1}) \\
&\leq \lambda(X) + \sum_{n\in\mathbb{N}} (n-1)\lambda(Y_n \setminus Y_{n-1}) \\
&\leq \lambda(X) + \sum_{n\in\mathbb{N}} \int_{Y_n \setminus Y_{n-1}} M_{(f,\gamma)}(x)dx \\
&\leq \lambda(X) + \int_X M_{(f,\gamma)}(x)dx < +\infty.
\end{aligned}$$

Set $X_0 = \emptyset$ and let $\{X_n\}_{n\in\mathbb{N}}$ be an increasing sequence of closed sets such that

$$X_n \subseteq Y_n \text{ and } \lambda(Y_n \setminus X_n) < \frac{1}{(n+1)2^n}, \forall n \in \mathbb{N}.$$

Clearly, $\lambda(X \setminus \cup_{n\in\mathbb{N}} X_n) = 0$ and for every $n \in \mathbb{N}$,

$$M_{(f,\gamma)}(x) \leq L_n \; \forall x \in X_n.$$

Moreover, because

$$X_n \setminus X_{n-1} \subseteq (Y_n \setminus Y_{n-1}) \cup (Y_{n-1} \setminus X_{n-1}),$$

we have

$$\sum_{n\in\mathbb{N}} L_n\lambda(X_n\setminus X_{n-1}) = \sum_{n\in\mathbb{N}} n\lambda(X_n\setminus X_{n-1}) \\ \leq \sum_{n\in\mathbb{N}} n\lambda(Y_n\setminus Y_{n-1}) + \sum_{n\in\mathbb{N}} \frac{1}{2^{n-1}} < +\infty.$$

(2) $\Rightarrow$ (1). Clearly f is almost everywhere pointwise Hölder, so it is continuous almost everywhere and hence it is measurable.

Let us now show that $M_{(f,\gamma)}$ is measurable. We proceed in a way similar to the one in Theorem 4.3, page 113 of [21]. Let $h \in \mathbb{N}$ and let

$$D_h(f,x) = \limsup_{|x-y|<\frac{1}{h}} \frac{|f(y)-f(x)|}{|y-x|^{\gamma}.}$$

Clearly,

$$M_{(f,\gamma)}(x) = \lim_{h\to+\infty} D_h(f,x). \tag{$\star$}$$

Now let a be any finite number and let E be a subset of X. Consider the set

$$E_x = \{x \in E : D_h(f,x) > a\}.$$

We see at once that if f is constant on E, the set of the points $x \in E$ at which $D_h(f,x) > a$ is open in E. Thus the set X_x and consequently the expression $D_h(f,x)$ as a function of x is measurable whenever the function f is finite, measurable and assumes at most countably many distinct values.

This being so, let f be any finite measurable function. We can represent it as the limit of a uniformly convergent sequence $\{f_n\}_{n\in\mathbb{N}}$ of measurable functions each of which assumes at most a countable number of distinct values. For instance we may write $f_n(x) = \frac{i}{n}$, when $\frac{i}{n} \leq f(x) < \frac{i+1}{n}$ for $i = \ldots, -2, -1, 0, 1, 2, \ldots$. We then have $D_h(f,x) = \lim_{n\to+\infty} D_h(f_n,x)$, and since by the above the functions $D_h(f_n,x)$ are measurable in x, so is $D_h(f,x)$. It follows at once from ($\star$) that $M_{(f,\gamma)}$ is also measurable. Since for every $n \in \mathbb{N}$ and for every $x \in X_n$ we have $M_{(f,\gamma)}(x) \leq L_n$, it follows that

$$0 \leq \int_X M_{(f,\gamma)}(x)\,dx = \sum_{n\in\mathbb{N}} \int_{X_n\setminus X_{n-1}} M_{(f,\gamma)}(x)\,dx \leq \sum_{n\in\mathbb{N}} L_n\lambda(X_n\setminus X_{n-1}) < +\infty.$$

□

Next follows a theorem in the same spirit as Theorem 2.8.

Theorem 4.2. *Let $f : X \to \mathbb{R}$, where X is a bounded measurable subset of $\mathbb{R}^k$. The following are equivalent:*

1. *The function $M_{(f,\gamma)}$ is summable in X.*
2. *There exist $\alpha \in]0,1[$, an increasing sequence $\{X_n\}_{n\in\mathbb{N}}$ of closed subsets of X and an increasing sequence $\{L_n\}_{n\in\mathbb{N}}$ of positive numbers such that, for every $n \in \mathbb{N}$,*

$$\limsup_{\substack{y \to x \\ y > x \\ r([x,y]) > \alpha}} \frac{|f(y) - f(x)|}{|y - x|^\gamma} \leq L_n, \quad \text{for every } x \text{ in } X_n,$$

$$\lambda(X \setminus \cup_{n\in\mathbb{N}} X_n) = 0 \quad \text{and} \quad \sum_{n\in\mathbb{N}} L_n \lambda(X_n \setminus X_{n+1}) < +\infty$$

(set $X_0 = \emptyset$).

Proof. (1) $\Rightarrow$ (2). Let $\alpha \in]0,1[$. By Proposition 4.1 there exist an increasing sequence $\{X_n\}_{n\in\mathbb{N}}$ of closed subsets of X and an increasing sequence $\{L_n\}_{n\in\mathbb{N}}$ of positive numbers such that, for every $n \in \mathbb{N}$,

$$\limsup_{y \to x} \frac{|f(y) - f(x)|}{|y - x|^\gamma} \leq L_n, \ \forall x \in X_n,$$

$$\lambda(X \setminus \cup_{n\in\mathbb{N}} X_n) = 0 \ \text{ and } \ \sum_{n\in\mathbb{N}} L_n \lambda(X_n \setminus X_{n-1}) < +\infty.$$

By considering all density points of X_n contained in X_n and by replacing X_n with a suitable closed subset of the collection of these density points and by calling it still X_n for notational convenience, we can write, without loss of generality,

$$\limsup_{\substack{y \to x \\ y > x \\ r([x,y]) > \alpha}} \frac{|f(y) - f(x)|}{|y - x|^\gamma} \leq \limsup_{y\to x} \frac{|f(y) - f(x)|}{|y - x|^\gamma} \leq L_n, \ \forall x \in X_n.$$

Hence (2) follows.

(2) $\Rightarrow$ (1) For every $n \in \mathbb{N}$, let $T_n = 2L_n$. For every $n \in \mathbb{N}$, $X_n = \cup_{k\in\mathbb{N}} X_{n,k}$, where $X_{n,k}$ is the set of all points in X_n with the property that

$$\frac{|f(y) - f(x)|}{|y - x|^\gamma} \leq T_n$$

whenever y is an element of X satisfying $x < y, |x-y| < \frac{1}{k}$ and $r([x,y]) > \alpha$. Each $X_{n,k}$ is measurable. It is enough to show that if $x_0 \in X$ is a density point of $X_{n,k}$, then $x_0 \in X_{n,k}$. Let $y \in X$ such that

$$x_0 < y, |y-x_0| < \frac{1}{k} \text{ and } r([y,x_0]) > \alpha.$$

By Lemma 2.5, there exists a sequence $\{y_p\}_{p\in\mathbb{N}} \subseteq X_{n,k}$ such that

$$y_p < x_0, |y_p - x_0| < \frac{1}{p}, r([y_p,x_0]) > \alpha \text{ and } r([y_p,y]) > \alpha, \forall p \in \mathbb{N}.$$

Then, for all $p > \frac{k}{1-k|x_0-y|}$,

$$|f(y)-f(x_0)| \le |f(y)-f(y_p)| + |f(y_p)-f(x_0)| \le T_n(|y-y_p|^\gamma + |y_p-x_0|^\gamma),$$

and so, letting p tend to $+\infty$,

$$|f(y)-f(x_0)| \le T_n|y-x_0|^\gamma.$$

The function f is locally Hölder at all density points of $X_{n,k}$. By Lemma 2.4 there exist two positive numbers δ and η (the second does not depend on x_0) such that if $y \in \mathbb{R}^k$ has distance from x_0 positive and less than δ, then there exist $x \in X_{n,k}$ such that

1. $x < y, x < x_0, r([x,y]) > \alpha, r([x,x_0]) > \alpha$, and
2. $|x-x_0| \le \eta|y-x_0|$.

Hence, if $y \in X$ satisfies $|y-x_0| < \min\{\frac{1}{(1+\eta)k}, \delta\}$, then there exists x satisfying (2) and hence such that

$$|x-x_0| < \frac{1}{k} \text{ and } |x-y| \le (1+\eta)|y-x_0| < \frac{1}{k}.$$

From the last inequality and from (1) and (2) it follows that

$$\begin{aligned}|f(y)-f(x_0)| \le& |f(y)-f(x)| + |f(x)-f(x_0)| \\ \le& T_n(|y-x|^\gamma + |x-x_0|^\gamma) \\ \le& T_n[(1+\eta)^\gamma + \eta^\gamma]|y-x_0|^\gamma\end{aligned}$$

Therefore, because f is almost everywhere locally Hölder in $X_{n,k}$, f is almost everywhere pointwise Hölder in X. Moreover, for almost every $x \in X_n$,

$$0 \le M_{(f,\gamma)}(x) \le T_n[(1+\eta)^\gamma + \eta^\gamma],$$

and so

$$\begin{aligned}
0 \leq & \int_X M_{(f,\gamma)}(x)dx \\
= & \sum_{n\in\mathbb{N}} \int_{X_n \setminus X_{n-1}} M_{(f,\gamma)}(x)dx \\
\leq & \sum_{n\in\mathbb{N}} T_n[(1+\eta)^\gamma + \eta^\gamma]\lambda(X_n \setminus X_{n-1}) \\
= & 2[(1+\eta)^\gamma + \eta^\gamma] \sum_{n\in\mathbb{N}} L_n \lambda(X_n \setminus X_{n-1}) < +\infty. \qquad \square
\end{aligned}$$

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