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A CONVERSE TO A THEOREM OF STEINHAUS

Abstract

A result of H. Steinhaus states that any Lebesgue measurable set $X \subseteq \mathbb{R}$ with the positive Lebesgue measure has a property that its difference set contains an open interval around zero. In this note we will prove a statement, which, in a sense, complements it.

Let m denote the Lebesgue measure on the real line. In 1920, H.Steinhaus proved the following fact [2]:

Theorem 0.1 (H. Steinhaus.)**.** *For every Lebesgue measurable set $X \subseteq \mathbb{R}$ with $mX > 0$, its difference set; i.e., the set $X - X = \{x - y : x, y \in X\}$ contains an open interval around zero.*

A natural question, which does not seem to be studied in the literature, is, informally, what are, if any, other Borel probability measures on the real line that would share with the Lebesgue measure the property described in the Steinhaus theorem? To make this question more precise, we introduce

Definition 0.2. A Borel measure μ on $X \subseteq \mathbb{R}$ is said to *have Steinhaus Property* (abbreviated $\mathcal{SP}$), if for every μ-measurable set X with $\mu X > 0$, the difference set $X - X$ contains an open interval around zero.

Observe that every measure that has $\mathcal{SP}$ must be non-atomic. Suppose, for example, μ has an atom $\{x\}$. Then the set $\{x\} - \{x\} = \{0\}$. Thus μ cannot have $\mathcal{SP}$.

Remark 1. Instead of the measures on $\mathbb{R}$, we can, without loss of generality, consider measures on $[0, 1)$, or on the circle $S^1 = \mathbb{R}/\mathbb{Z}$. Indeed, for any measure μ on $\mathbb{R}$, we have $\mu = \sum_{n\in\mathbb{Z}} \mu_n$, where μ_n are measures on $[n, n+1), n \in \mathbb{Z}$, obtained from μ by restricting it to $[n, n+1)$. So a measure μ has $\mathcal{SP}$ if and only if the measure μ_n has $\mathcal{SP}$ for every $n \in \mathbb{Z}$.

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Remark 2. It is easy to see that a measure μ has $\mathcal{SP}$ if and only if for every measurable set X with $\mu X > 0$, there exists a number $0 < t < 1$ such that

$$X \bigcap (X + x) \neq \varnothing, \text{ for all } x \in [-t, t].$$

Proposition 0.3. *For a measure μ on $[0,1)$ not to have $\mathcal{SP}$ is sufficient to satisfy the following condition: There exist a $\mu-$measurable set $X \subset [0,1)$ of positive measure and a sequence of real numbers $\{t_n\}_{n=1}^{\infty}$, $t_n \to 0$, as $n \to \infty$ with the property:*

$$\mu(X + t_n) \to 0, \text{ as } n \to \infty.$$

PROOF. Without loss of generality, we can assume there exist a set X and $\varepsilon > 0$ such that $\mu(X+t_n) < \varepsilon/2^n$. In that case it would be enough to introduce a new set $X' = X \setminus \bigcap_{n>1}(X + t_n)$ for which we get $X' \bigcap (X' + t_n) = \varnothing$ and $\mu(X') > \mu(X)/2$. By Remark 2, this implies that μ does not have $\mathcal{SP}$. □

Proposition 0.4. *Any absolutely continuous measure $\mu \ll m$ has $\mathcal{SP}$.*

PROOF. Indeed, if not, then there is a measurable set X with a positive measure $\mu X > 0$ such that the zero is contained in $X - X$ together with some open neighborhood, but then due to Steinhaus Theorem 0.1 $mX = 0 \Rightarrow \mu X = 0$, which contradicts $\mu \ll m$.

□

Lemma 0.5. *Let μ be a non-atomic Borel measure on $[0,1)$. Then $[0,1)$ can be written as a disjoint union $X_a \bigcup X_s$ of Borel sets X_a and X_s, so that we have*

(i) $\mu(X_a) = 0$, $m(X_s) = 0$ *and* $\mu \ll m$ *on* X_a.
Moreover, $\mu(X_s) = 0$ *iff* $\mu \ll m$.

(ii) $\mu = \mu_a + \mu_s$, *where* $\mu_a = \mu_{|X_a}, \mu_s = \mu_{|X_s}$ *are uniquely defined non-atomic Borel measures, such that* $\mu_a \ll m$ *and* $\mu_s \perp m$.

This fact is a corollary of the Lebesgue Decomposition Theorem, (see, e.g., [1], p. 278.)

Lemma 0.6. *Let μ be a non-atomic measure, suppose that X_s, μ_s are defined as in Lemma 0.5. Then the function*

$$f(t) = \mu_s(X_s + t)$$

vanishes on $[0,1)$ Lebesgue almost everywhere.

Proof. Let $\chi_A(u)$ be the characteristic function of a set A. The integral of f with respect to Lebesgue measure can be written as follows:

$$\int_{[0,1)} f\,dm = \int_{[0,1)} \mu_s(X_s+t)\,dm(t) = \int_{[0,1)}\int_{[0,1)} \chi_{X_s+t}(u)\,d\mu_s(u)\,dm(t).$$

The latter is an iterated integral. Fubini's Theorem enables us to switch the order of integration, because the integrand $\chi_{X_s+t}(u)$ is a bounded function (for this fact see [1], p. 307). Note that for any $A \subseteq \mathbb{R},\ t,u \in \mathbb{R}$ our characteristic function can be written as $\chi_{A+t}(u) = \chi_{u-A}(t)$. At the same time, recall that the Lebesgue measure is invariant under translation and inversion (*mod* 1); i.e., for all $t \in \mathbb{R}$ and any measurable $A \subset [0,1)$, one has

$$\int_{[0,1)} \chi_{A+t}(u)\,dm(u) = m(A+t) = m(A) = \int_{[0,1)} \chi_A(u)\,dm(u)$$

$$\int_{[0,1)} \chi_{-A}(u)\,dm(u) = m(-A) = m(A) = \int_{[0,1)} \chi_A(u)\,dm(u).$$

Thus, we obtain

$$\int_{[0,1)} \mu_s(X_s+t)\,dm(t) = \int_{[0,1)}\int_{[0,1)} \chi_{X_s+t}(u)\,dm(t)\,d\mu_s(u) =$$

$$\int_{[0,1)}\int_{[0,1)} \chi_{u-X_s}(t)\,dm(t)\,d\mu_s(u) = \int_{[0,1)}\int_{[0,1)} \chi_{-X_s}(t)\,dm(t)\,d\mu_s(u) =$$

$$\int_{[0,1)}\int_{[0,1)} \chi_{X_s}(t)\,dm(t)\,d\mu_s(u) = \int_{[0,1)} m(X_s)\,d\mu(u).$$

Recall the definition of X_s and μ_s where $mX_s = 0$. The last integral is equal to zero. This causes $\int_{[0,1)} \chi_{X_s+t}\,dm$ to be zero, which implies that the function $f(t) = \mu_s(X_s+t)$ must be zero for Lebesgue almost all $t \in [0,1)$. $\square$

The above Lemma finally makes it possible to answer the question posed at the beginning:

Theorem 0.7 (A Converse to Steinhaus Theorem)**.** *A Borel measure on* $\mathbb{R}$ *has* $\mathcal{SP}$ *if and only if it is absolutely continuous.*

Proof. The "if" part is Theorem 0.1. Show the "only if" part. First, recall that due to Remark 1, without loss of generality we may assume that μ is defined on $[0,1)$. Let μ be a singular measure on $[0,1)$. Apply Lemma 0.5.

We will get a set M_s with $\mu M_s > 0$ and the measure $\mu_s = \mu_{|M_s}$ for which, due to Lemma 0.6, the function $f(t) = \mu_s(M_s + t)$ is zero a.e. on $[0, 1)$. In particular, $f(t_n) = 0$ for some sequence $\{t_n\}$ converging to 0.

Consider the set $Y = \bigcap_k \big(M_s \setminus (M_s + t_k)\big)$. Since $f(t_n) = 0$ for all n, we have $\mu_s Y = \mu_s M_s > 0$. Therefore, $\mu Y \geq \mu_s Y$. The set Y, by definition, has also the property that $Y \bigcap (Y + t_n) = \varnothing$ for all n. But this, according to Proposition 0.3, immediately implies that μ does not have $\mathcal{SP}$. $\square$

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References

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[2] H. Steinhaus, *Sur les distances des points dans les ensembles de mesure positive*, Fund. Math., **1** (1920), 94–104.