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EIGENVALUES ASSOCIATED WITH BOREL SETS

Abstract

Every Borel subset K of an interval $[c,d]$ induces a sequence of eigenvalues. If K is closed, the asymptotic behavior of the eigenvalues is related to the positions and lengths of its complementary intervals. The rate of growth becomes "lowest possible" if K has self-similarity properties. Eigenvalues of a vibrating string with singular mass distribution are eigenvalues associated with a set K.

1 Introduction.

Let K be a given Borel subset of the interval $[c,d]$, and let χ_K denote its characteristic function. We call $\lambda \in \mathbb{C}$ an eigenvalue associated with K if there exist absolutely continuous functions $u,v : [c,d] \to \mathbb{C}$, not both identically zero, that solve the system of differential equations

$$u' = (1-\chi_K(x))v, \; v' = -\lambda\chi_K(x)u \quad \text{for } x \in [c,d] \text{ a.e.,} \tag{1}$$

and the boundary conditions

$$u(c) = u(d) = 0. \tag{2}$$

This eigenvalue problem is a special case of a problem introduced and investigated by Atkinson [1, Chapter 8]. Some basic properties of the sequence of eigenvalues associated with K are mentioned in Section 2.

How does the structure of K influence the asymptotic behavior of its sequence of eigenvalues? In Section 3 we show that knowledge of the lengths of

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the complementary intervals of a closed set K provides a lower bound on the growth rate of the eigenvalues. For example, if the complementary intervals of K have lengths $1/m^2$, $m \in \mathbb{N}$, then the eigenvalues λ_n must grow at least as fast as n^3. Of course, the true growth rate of the eigenvalues depends not only on the lengths of the complementary intervals but also on their location. We show that the lower bound on the growth rate of eigenvalues is achieved when K is a symmetric perfect set of positive measure. This indicates that the growth rate of eigenvalues is "lowest possible" when K has fractal properties.

In Section 4 we show that the eigenvalue problem of a vibrating string whose mass distribution is singular to Lebesgue measure can be transformed to a problem of the form (1), (2) when K is properly chosen. This allows us to relate results of this paper to those of Fujita [3], McKean and Ray [6], and Uno and Hong [7].

2 Eigenvalues of Sets.

Let K be a Borel subset of $[c, d]$. We will assume that the Lebesgue measure $\nu(K)$ of K is less than $d - c$, and that

$$\nu(K \cap [c, e)) > 0,\ \nu(K \cap (e, d]) > 0 \text{ for all } e \in (c, d). \tag{3}$$

We consider the eigenvalue problem consisting of the system (1) and the boundary conditions (somewhat more general than (2))

$$\cos\alpha\, u(c) = \sin\alpha\, v(c),\ \cos\beta\, u(d) = \sin\beta\, v(d), \tag{4}$$

where $\alpha \in [0, \pi)$, $\beta \in (0, \pi]$. This eigenvalue problem satisfies all of Atkinson's assumptions (i), (ii), (iii), (iv); see [1, pages 203–204].

We mention a few consequences. By [1, Theorem 8.3.1], the eigenvalues are real. By [1, Theorem 8.4.5], the eigenvalues can be arranged as an increasing sequence

$$\lambda_0 < \lambda_1 < \lambda_2 < \dots$$

If (u, v) is an eigenfunction corresponding to the eigenvalue λ_n, then the set of zeros of u within $[c, d]$ consists of n disjoint intervals if we do not count those intervals that contain c or d. The number of eigenvalues may be finite. If there are infinitely many eigenvalues, the sequence of eigenvalues tends to infinity. We know from [2, Theorem 4.3] that the number of eigenvalues of K is finite if and only if there is a finite union L of intervals such that $\nu(K \Delta L) = 0$.

In Atkinson's analysis as in our own the Prüfer angle plays a crucial role. For real λ let $(u(x, \lambda), v(x, \lambda))$ be the solution of (1) determined by the initial values

$$u(c, \lambda) = \sin\alpha,\ v(c, \lambda) = \cos\alpha. \tag{5}$$

Let $\theta(x) = \theta(x, \lambda)$ be the absolutely continuous Prüfer angle defined by

$$\theta(x) = \arg(v(x, \lambda) + iu(x, \lambda)),\ \theta(c) = \alpha.$$

The Prüfer angle satisfies the first order differential equation

$$\theta' = (1 - \chi_K(x)) \cos^2 \theta + \lambda \chi_K(x) \sin^2 \theta. \tag{6}$$

The function $\theta(d, \lambda)$ is increasing and Atkinson showed that the eigenvalue λ_n is the solution of

$$\theta(d, \lambda_n) = \beta + n\pi. \tag{7}$$

Therefore, the behavior of $\theta(d, \lambda)$ as $\lambda \to +\infty$ determines the behavior of the eigenvalues λ_n as $n \to \infty$.

If $\lambda = 0$, then (6) has constant solutions $\theta(x) = \frac{\pi}{2} + k\pi$, $k \in \mathbb{Z}$. Therefore $\theta(d, 0) < \frac{3}{2}\pi$ which shows that the eigenvalues λ_n with $n \geq 2$ are positive. The eigenvalues of the Dirichlet problem ($\alpha = 0$, $\beta = \pi$) are all positive while the eigenvalues of the Neumann problem ($\alpha = \beta = \pi/2$) satisfy $\lambda_0 = 0 < \lambda_n$ for all $n \geq 1$.

Sometimes it is useful to consider a modified Prüfer angle $\phi(x)$. For given $\lambda, \gamma > 0$ the absolutely continuous function $\phi(x)$ is determined by

$$\phi(x) = \arg(v(x, \lambda) + i\gamma u(x, \lambda)),\ \phi(c) \in [0, \pi).$$

The modified Prüfer angle satisfies the differential equation

$$\phi' = \gamma(1 - \chi_K(x)) \cos^2 \phi + \gamma^{-1} \lambda \chi_K \sin^2 \phi. \tag{8}$$

The values of $\phi(x)$ and $\theta(x, \lambda)$ agree when $u(x, \lambda) = 0$ or $v(x, \lambda) = 0$. Therefore,

$$|\phi(x) - \theta(x, \lambda)| < \frac{\pi}{2} \quad \text{for all } x \in [c, d]. \tag{9}$$

3 Eigenvalue Estimates.

If K is a closed set, then its complement $\tilde{K} = [c, d] \setminus K$ is open (relative to $[c, d]$) and thus $\tilde{K}$ is a union of disjoint open intervals, the complementary intervals of K.

Theorem 1. *Let K be a closed subset of $[c, d]$ with infinitely many complementary intervals I_m, $m \in \mathbb{N}$. For $k \in \mathbb{N}$ let*

$$\delta_k := \sum_{m=k}^{\infty} \nu(I_m).$$

Then, for all $\lambda > 0$ and $k \in \mathbb{N}$,

$$\theta(d,\lambda) - \theta(c,\lambda) \ \le 2\lambda^{1/2}\delta_k^{1/2}\nu(K)^{1/2} + k\pi, \tag{10}$$

where $\theta(x,\lambda)$ is any solution of (6).

PROOF. If $\nu(K) = 0$, then (6) has constant solutions $(q+\frac{1}{2})\pi$ for each integer q. Therefore, $\theta(d,\lambda) - \theta(c,\lambda) < \pi$ which implies (10). Let us assume that $\nu(K) > 0$. Then we define a positive number γ by

$$\lambda\nu(K) = \delta_k\gamma^2. \tag{11}$$

Let $\phi(x)$ be the modified Prüfer angle satisfying (8). We write $[c,d]$ as the disjoint union of $k-1$ complementary intervals I_m, $m = 1, 2, \ldots, k-1$, and (at most) k closed intervals J_m, $m = 1, 2, \ldots, k$. For a subinterval I of $[c,d]$ let $\phi|_I$ denote the increase of ϕ over I. Then, by integrating (8) over J_m, we obtain

$$\phi|_{J_m} \le \gamma\nu(\tilde{K} \cap J_m) + \gamma^{-1}\lambda\nu(K \cap J_m).$$

We add these inequalities and use (11) to find

$$\sum_{m=1}^{k} \phi|_{J_m} \le \gamma\delta_k + \gamma^{-1}\lambda\nu(K) = 2\lambda^{1/2}\delta_k^{1/2}\nu(K)^{1/2}.$$

On the complementary intervals I_m the function χ_K vanishes. Therefore, we have $\phi|_{I_m} \le \pi$. Adding our inequalities we obtain

$$\begin{aligned}\phi(d) - \phi(c) &= \sum_{m=1}^{k} \phi|_{J_m} + \sum_{m=1}^{k-1} \phi|_{I_m} \\ &\le 2\lambda^{1/2}\delta_k^{1/2}\nu(K)^{1/2} + (k-1)\pi.\end{aligned}$$

Using (9) we obtain (10). $\square$

Of course, inequality (10) should be used with the complementary intervals ordered by decreasing length; $\nu(I_1) \ge \nu(I_2) \ge \ldots$. In (10) we will choose k such that k and $\lambda^{1/2}\delta_k^{1/2}$ are approximately of the same size. We may select an increasing function $g : (0,\infty) \to (0,\infty)$ such that

$$g(k^2\delta_k^{-1}) = k,\ k \in \mathbb{N}. \tag{12}$$

Then, for given $\lambda > 0$, we choose a positive integer k such that

$$k - 1 < g(\lambda) \le k. \tag{13}$$

It follows that $\lambda\delta_k \le k^2$, and (10) implies

$$\theta(d,\lambda) - \theta(c,\lambda) \le (2\nu(K)^{1/2} + \pi)(g(\lambda)+1) \text{ for } \lambda > 0. \tag{14}$$

For example, consider a closed set K whose complementary intervals I_m have length $\frac{1}{m(m+1)}$, $m = 1,2,3,\dots$. Then $\delta_k = \frac{1}{k}$. We may choose $g(\lambda) = \lambda^{1/3}$ to obtain

$$\theta(d,\lambda) - \theta(c,\lambda) \le (2\nu(K)^{1/2} + \pi)(\lambda^{1/3}+1).$$

From equation (7) we obtain the following lower bound for eigenvalues.

Corollary 2. *Let K be a closed subset of $[c,d]$ with infinitely many complementary intervals and satisfying* (3)*. Let g be chosen according to* (12)*. Then the positive eigenvalues λ_n associated with K for given $\alpha \in [0,\pi)$, $\beta \in (0,\pi]$ satisfy*

$$g(\lambda_n) \ge \frac{\beta - \alpha + n\pi}{2\nu(K)^{1/2} + \pi} - 1.$$

For example, for a closed set K with complementary intervals I_m of length $\frac{1}{m(m+1)}$, $m = 1,2,3,\dots$, the eigenvalues λ_n of the Dirichlet problem satisfy

$$\lambda_n^{1/3} \ge \frac{(n+1)\pi}{2\nu(K)^{1/2} + \pi} - 1.$$

If we know the positions of the complementary intervals we may estimate better. For instance, consider a sequence

$$c = 0 = \xi_0 < \xi_1 < \xi_2 < \cdots < 2 = d$$

with

$$\xi_{2k} - \xi_{2k-1} = \xi_{2k-1} - \xi_{2k-2} = \frac{1}{k(k+1)} \text{ for } k \ge 1.$$

Take

$$K = \{2\} \cup \bigcup_{k=0}^{\infty} [\xi_{2k}, \xi_{2k+1}].$$

The complementary intervals I_m of K have length $\frac{1}{m(m+1)}$. When we write $[0,2]$ as a disjoint union of the intervals I_m, $m = 1,2,\dots,k-1$ and closed intervals J_m, $m = 1,2\dots,k$, then

$$\phi|_{I_m} \le \pi,\ \phi|_{J_m} \le \pi \text{ for } m = 1,2,\dots,k-1.$$

When $\gamma = \lambda^{1/2}$ we find

$$\phi|_{J_k} \leq 2\lambda^{1/2}\delta_k = 2\lambda^{1/2}\frac{1}{k}$$

which leads to

$$\theta(2,\lambda) - \theta(0,\lambda) \leq (2k-1)\pi + 2\lambda^{1/2}\frac{1}{k}.$$

If we select k such that $(k-1)^4 < \lambda \leq k^4$, it follows that

$$\theta(2,\lambda) - \theta(0,\lambda) \leq 2(\pi+1)\lambda^{1/4} + \pi \quad \text{for } \lambda > 0,$$

and, for the eigenvalues λ_n of the Dirichlet problem,

$$\lambda_n^{1/4} \geq \frac{n\pi}{2\pi+2}.$$

This demonstrates that the actual growth rate of eigenvalues may be larger than that enforced by Corollary 2.

We now consider sets for which the lower bound of Corollary 2 gives us the correct growth rate. Consider a symmetric perfect set $K \subset [c,d]$ determined by a sequence $\{\epsilon_q\}_{q\in\mathbb{N}}$ of parameters in $(0,1)$. This set is defined by a construction similar to that which leads to the Cantor set. In the first step we remove from $[c,d]$ a central open interval of fractional length ϵ_1. In the second step we remove a central open interval of fractional length ϵ_2 from each of the two remaining closed intervals and so on. The set of points that is not removed is the symmetric perfect set K. If $[c,d] = [0,1]$ and $\epsilon_q = \frac{1}{3}$ for all q, we obtain the Cantor set. The measure of K is given by

$$\nu(K) = (d-c)\prod_{q=1}^{\infty}(1-\epsilon_q).$$

In the qth step we remove 2^{q-1} complementary intervals I_m, $m = 2^{q-1}, 2^{q-1}+1, \ldots, 2^q - 1$, each with length

$$\nu(I_m) = (d-c)2^{-q+1}\epsilon_q\prod_{j=1}^{q-1}(1-\epsilon_j).$$

Then, if $k = 2^q$,

$$\delta_k = \sum_{m=k}^{\infty}\nu(I_m) = (d-c)\left(\prod_{i=1}^{q}(1-\epsilon_i) - \prod_{i=1}^{\infty}(1-\epsilon_i)\right). \tag{15}$$

Theorem 3. *Let $K \subset [c,d]$ be a symmetric perfect set of positive measure. Choose an increasing function $g : (0,\infty) \to (0,\infty)$ with*

$$g(k^2\delta_k^{-1}) = k \text{ for } k = 2^q,\ q = 0,1,2,\dots,$$

where δ_k is given by (15)*. Then, for every solution $\theta(x,\lambda)$ of* (6)*,*

$$\theta(d,\lambda) - \theta(c,\lambda) \geq \frac{1}{2}Cg(\lambda) - \pi \quad \text{for } \lambda > 0, \tag{16}$$

where

$$C := \min\left\{\frac{\pi}{6}, \frac{1}{2}\nu(K)^{1/2}\right\}.$$

Proof. Inequality (16) certainly holds when $g(\lambda) < 1$. Therefore let $\lambda > 0$ be such that $g(\lambda) \geq 1$. We choose a nonnegative integer q such that

$$k := 2^q \leq g(\lambda) < 2^{q+1}. \tag{17}$$

We write $[c,d]$ as the disjoint union of the complementary intervals I_m, $m = 1,2,\dots,k-1$, and closed intervals J_m, $m = 1,2,\dots,k$. All intervals J_m have the same length. By construction of K, the sets $K \cap J_m$ are translates of each other. Thus $\nu(K \cap J_m)$ and $\nu(\tilde{K} \cap J_m)$ do not depend on m. We use the modified Prüfer angle ϕ depending on the positive number γ defined by

$$\lambda\nu(K \cap J_m) = \gamma^2\nu(\tilde{K} \cap J_m).$$

Note that γ is independent of m. Suppose that the increase $\phi|_{J_m}$ is at most $\pi/6$. Then there is an interval H of length at most $\pi/6$ such that $\phi(x) \in H$ for all $x \in J_m$. There are $r, s \geq 0$ with $r + s = \frac{1}{2}$ such that $\cos^2 t \geq r$ and $\sin^2 t \geq s$ for all $t \in H$. Integrating (8) over J_m we obtain

$$\phi|_{J_m} \geq r\gamma\nu(\tilde{K} \cap J_m) + s\gamma^{-1}\lambda\,\nu(K \cap J_m).$$

By definition of γ this gives

$$\phi|_{J_m} \geq (r+s)\lambda^{1/2}\nu(K \cap J_m)^{1/2}\nu(\tilde{K} \cap J_m)^{1/2}.$$

Hence we obtain, for all m,

$$\phi|_{J_m} \geq \min\left\{\frac{\pi}{6}, \frac{1}{2}\lambda^{1/2}\nu(K \cap J_m)^{1/2}\nu(\tilde{K} \cap J_m)^{1/2}\right\}. \tag{18}$$

We have
$$\frac{\nu(\tilde{K}\cap J_m)}{\nu(K\cap J_m)}=\frac{\delta_k}{\nu(K)}.$$
It follows from (17) that $\lambda\delta_k \ge k^2$. Hence
$$\begin{aligned}\lambda\nu(K\cap J_m)\nu(\tilde{K}\cap J_m) &= \lambda\delta_k\nu(K)^{-1}\nu(K\cap J_m)^2\\ &\ge k^2\nu(K)^{-1}k^{-2}\nu(K)^2=\nu(K).\end{aligned}$$
Therefore, (18) gives
$$\phi|_{J_m}\ge C \text{ for } m=1,2,\dots,k.$$
Since $\phi|_{I_m}\ge 0$, we find that
$$\phi(d)-\phi(c)\ge\sum_{m=1}^{k}\phi|_{J_m}\ge 2^qC\ge\frac{1}{2}Cg(\lambda).$$
Replacing ϕ by θ we obtain (16). □

Corollary 4. *Let K be as in Theorem 3. Then the positive eigenvalues λ_n associated with K satisfy*
$$g(\lambda_n)\le\frac{2}{C}(\beta-\alpha+(n+1)\pi).$$

If K is a symmetric perfect set with positive measure and we choose g as in (12) then Corollaries 2 and 4 show that there are positive constants A, B such that $An \le g(\lambda_n)\le Bn$ for sufficiently large n.

4 Singular Vibrating Strings.

Consider a given finite measure $\rho:\mathcal{B}\to[0,\infty)$, where $\mathcal{B}$ denotes the σ-algebra of Borel subsets of the interval $[a,b]$. For simplicity, we will assume that ρ has no atoms. Measures with atoms can be treated along similar lines; see [8]. We also assume that
$$\rho([a,e))>0,\ \ \rho((e,b])>0 \text{ for all } e\in(a,b).$$
Consider the integral equations (ν denotes Lebesgue measure)
$$U(t)-U(a)=\int_{[a,t)}V(\tau)\,d\nu(\tau),\quad t\in[a,b], \tag{19}$$
$$V(t)-V(a)=-\,\lambda\int_{[a,t)}U(\tau)\,d\rho(\tau),\quad t\in[a,b] \tag{20}$$

subject to the boundary conditions

$$\cos\alpha\, U(a) = \sin\alpha\, V(a),\ \cos\beta\, U(b) = \sin\beta\, V(b), \tag{21}$$

where $\alpha \in [0,\pi)$, $\beta \in (0,\pi]$. This is Krein's eigenvalue problem for a vibrating string whose mass distribution is given by ρ; see [5]. A complex number λ is called an eigenvalue if there exists a nontrivial continuous solution $(U,V) : [a,b] \to \mathbb{C}^2$ of bounded variation to the system (19), (20) satisfying (21). In the remainder of this section we will assume that our string is singular; that is, the measure ρ is singular with respect to Lebesgue measure ν. Eigenvalue problems of this sort were considered first by McKean and Ray [6] and Uno and Hong [7].

We transform the eigenvalue problem for a singular string to an eigenvalue problem of the type considered in Section 2. To this end consider the measure $\omega := \nu + \rho$, and let

$$h(t) := \omega([a,t)) = t - a + \rho([a,t))$$

be its distribution function. Let $d := \omega([a,b])$. Then $h : [a,b] \to [0,d]$ is continuous, strictly increasing and onto. There is $G \in \mathcal{B}$ such that $\nu(G) = \rho([a,b]\setminus G) = 0$. Let $K := h(G) \subset [0,d]$. Note that K is a Borel set, $\nu(K) < d$ and (3) holds. Consider the solution $u(x,\lambda)$, $v(x,\lambda)$ of system (1) with $c = 0$ determined by the initial values (5). It is an exercise in Real Analysis to show that

$$U(t,\lambda) := u(h(t),\lambda),\ V(t,\lambda) := v(h(t),\lambda) \tag{22}$$

solve the integral equations (19), (20) and have initial values

$$U(a,\lambda) = \sin\alpha,\ V(a,\lambda) = \cos\alpha.$$

For the proof of this and more general statements see [8]. Consequently, the eigenvalues of a singular vibrating string agree with the eigenvalues associated with K. Note that the numbers α, β appearing in the boundary conditions are the same for both problems. We see from (22) that $(U(t,\lambda), V(t,\lambda))$, $t \in [a,b]$, describes the same continuous curve in the plane as $(u(x,\lambda), v(x,\lambda))$, $x \in [0,d]$. If we define an absolutely continuous Prüfer angle $\theta(x,\lambda)$ as in Section 2, then $\theta(h(t),\lambda)$ will be a corresponding Prüfer angle for the eigenvalue problem of a vibrating string; that is,

$$\theta(h(t),\lambda) = \arg(V(t,\lambda) + iU(t,\lambda)).$$

We apply this transformation to obtain the following result.

Theorem 5. *Consider a vibrating string whose defining measure ρ is supported on the Cantor set $G \subset [0,1]$. Then its positive eigenvalues λ_n satisfy*

$$\frac{\beta-\alpha+n\pi}{\sqrt{6M}+\pi}-1 \le \lambda_n^\tau, \tag{23}$$

where $\tau := \frac{\ln 2}{\ln 6}$, and $M := \rho(G)$ is the total mass of the string.

PROOF. Since the complementary intervals of $K = h(G)$ have the same lengths as those of the Cantor set, we have $\delta_k = (\frac{2}{3})^q$ when $k = 2^q$. From Theorem 1 we know that

$$\theta(d,\lambda)-\theta(0,\lambda) \le 2\lambda^{1/2}\left(\tfrac{2}{3}\right)^{q/2} M^{1/2} + 2^q\pi. \tag{24}$$

For $\lambda \ge 1$ we choose $q = 0,1,2,\dots$ such that $6^q \le \lambda < 6^{q+1}$. We note that $6^\tau = 2$ which implies $2^q \le \lambda^\tau$ and

$$(\tfrac{2}{3})^{q/2} = 6^{q(\tau-1/2)} < (\tfrac{\lambda}{6})^{\tau-1/2}.$$

Then we obtain

$$\theta(d,\lambda)-\theta(0,\lambda) \le (\sqrt{6M}+\pi)\lambda^\tau \text{ for } \lambda \ge 1.$$

Inequality (24) with $q = 0$ shows that

$$\theta(d,\lambda)-\theta(0,\lambda) \le (\sqrt{6M}+\pi)(\lambda^\tau+1) \text{ for } \lambda > 0.$$

This implies (23). □

It should be noted that the measure ρ in Theorem 5 can be any finite (atomless) measure supported on the Cantor set. We now consider the special case where ρ is the Cantor measure. One may define this measure through its distribution function $g(t) = \rho([0,t))$, the Cantor ternary function. Then $h(t) = t + g(t)$. It is easy to verify that $K = h(G) \subset [0,2]$ is a symmetric perfect set determined by the parameters

$$\epsilon_q = \frac{\frac{1}{3}}{1+(\frac{3}{2})^{q-1}}. \tag{25}$$

Its measure is $\nu(K) = \rho([0,1]) = 1$.

Theorem 6. *The positive eigenvalues λ_n of the vibrating string whose mass distribution is given by the Cantor measure satisfy*

$$\frac{\beta-\alpha+n\pi}{\sqrt{6}+\pi}-1 \le \lambda_n^\tau \le 4(\beta-\alpha+(n+1)\pi). \tag{26}$$

Proof. The lower bound for λ_n follows from Theorem 5. Theorem 3 with $g(\lambda) = \lambda^\tau$ yields

$$\theta(2,\lambda) - \theta(0,\lambda) \geq \tfrac{1}{4}\lambda^\tau - \pi \quad \text{for } \lambda > 0.$$

Therefore, using (7) we arrive at (26). □

By different methods, Theorem 6 has been proved in [6], [7] in the form: There are positive constants C_1 and C_2 such that $C_1 \leq \frac{\lambda_n^\tau}{n} \leq C_2$. Our method of proof has the advantage to lead to explicit estimates not containing unknown constant.

The eigenvalues of the Neumann problem reflect the self-similarity of the Cantor measure in the following striking way.

Theorem 7. *The eigenvalues λ_n of the Neumann problem for the vibrating string whose mass distribution is given by the Cantor measure satisfy*

$$\lambda_{2n} = 6\lambda_n,\ n = 0, 1, 2, \ldots \tag{27}$$

Proof. By making the substitution $t = 3s$ in (19), (20) we find that

$$U(t, 6\lambda) = U(3t, \lambda),\ V(t, 6\lambda) = 3V(3t, \lambda) \text{ for } 0 \leq t \leq \tfrac{1}{3}.$$

Since $V(0, \lambda_n) = V(1, \lambda_n) = 0$, the Prüfer angle satisfies

$$\theta(h(\tfrac{1}{3}), 6\lambda_n) = \theta(2, \lambda_n) = \frac{\pi}{2} + n\pi.$$

Since $\chi_K(x) = 0$ for $\frac{5}{6} = h(\frac{1}{3}) \leq x \leq h(\frac{2}{3}) = \frac{7}{6}$, it follows from (6) that

$$\theta(x, 6\lambda_n) = \frac{\pi}{2} + n\pi \quad \text{if } \tfrac{5}{6} \leq x \leq \tfrac{7}{6}.$$

Since $\chi_K(x + \frac{7}{6}) = \chi_K(x)$, we obtain

$$\theta(x + \tfrac{7}{6}, 6\lambda_n) = n\pi + \theta(x, 6\lambda_n)$$

which leads to

$$\theta(2, 6\lambda_n) = n\pi + \theta(\tfrac{5}{6}, 6\lambda_n) = \frac{\pi}{2} + 2n\pi = \theta(2, \lambda_{2n}).$$

This proves (27). □

Consider the sequence

$$\frac{\lambda_n^\tau}{n},\ n = 1, 2, 3, \ldots \tag{28}$$

with the eigenvalues λ_n from Theorem 7. For every $k = 1, 2, 3, \ldots$ the subsequence

$$\frac{(\lambda_{k2^n})^\tau}{k2^n} = \frac{(6^n \lambda_k)^\tau}{k2^n} = \frac{\lambda_k^\tau}{k} \tag{29}$$

is constant. By approximating K by finite unions of intervals it is possible to compute the eigenvalues λ_k for small k. For example, $\lambda_1 = 7.09\ldots$ and $\lambda_3 = 61.26\ldots$. It follows that the values of (29) for $k = 1$, $k = 3$ are different. This shows that the sequence (28) does not converge. Hence the factors $\pi/(\sqrt{6} + \pi)$ and 4 of n appearing in the lower and upper bounds in (26) cannot be made equal.

Fujita [3] found the growth rate of eigenvalues of a more general family of self-similar singular strings. The method of proof depends on the trace formula for the Green's function and a Tauberian theorem for the Stieltjes transform. These results may be derived more directly using the Prüfer angle as follows. We consider only an example since the general case treated in [3] can be dealt with in a very similar way.

Consider a singular string over $[a, b] = [0, 1]$ whose mass distribution is defined as follows; see Hutchinson [4]. Given $\rho_1, \rho_2 > 0$ with $\rho_1 + \rho_2 = 1$ there is a unique probability measure $\rho : \mathcal{B} \to [0, 1]$ such that

$$\rho(A) = \rho_1 \rho(S_1^{-1}(A)) + \rho_2 \rho(S_2^{-1}(A)) \text{ for } A \in \mathcal{B},$$

where

$$S_1(t) := \frac{t}{3},\ S_2(t) := \frac{2}{3} + \frac{t}{3}.$$

The measure ρ is supported on the Cantor set G. If $\rho_1 = \rho_2 = \frac{1}{2}$ we obtain again the Cantor measure. Let $K = h(G)$ be the corresponding subset of $[0, 2]$. The set K may be constructed in the same way as a symmetric perfect set but, in general, the complementary intervals are not removed from the middle of the remaining intervals. Define

$$f(\lambda) := \theta(2, \lambda) - \theta(0, \lambda).$$

The self-similarity of the measure ρ implies that $U(S_j(t), \lambda)$, $\frac{1}{3}V(S_j(t), \lambda)$ solve system (19), (20) with λ replaced by $\frac{1}{3}\rho_j\lambda$. Therefore, $\theta(h(S_j(t)), \lambda)$ is a modified Prüfer angle for a solution of system (19), (20) with λ replaced by $\frac{1}{3}\rho_j\lambda$. We note that if θ_1, θ_2 are any solutions of (6), then

$$\theta_1(d) - \theta_1(c) \le \theta_2(d) - \theta_2(c) + \pi.$$

We conclude that

$$f(\tfrac{1}{3}\rho_1\lambda) - 2\pi \le \theta(h(\tfrac{1}{3}),\lambda) - \theta(0,\lambda) \le f(\tfrac{1}{3}\rho_1\lambda) + 2\pi$$

and

$$f(\tfrac{1}{3}\rho_2\lambda) - 2\pi \le \theta(2,\lambda) - \theta(h(\tfrac{2}{3}),\lambda) \le f(\tfrac{1}{3}\rho_2\lambda) + 2\pi.$$

Adding these inequalities to

$$0 \le \theta(h(\tfrac{2}{3}),\lambda) - \theta(h(\tfrac{1}{3}),\lambda) \le \pi,$$

we arrive at the functional inequalities

$$f(\tfrac{1}{3}\rho_1\lambda) + f(\tfrac{1}{3}\rho_2\lambda) - 4\pi \le f(\lambda) \le f(\tfrac{1}{3}\rho_1\lambda) + f(\tfrac{1}{3}\rho_2\lambda) + 5\pi. \tag{30}$$

If we set $f_1(\lambda) = f(\lambda) + 5\pi$, we obtain

$$f_1(\lambda) \le f_1(\tfrac{1}{3}\rho_1\lambda) + f_1(\tfrac{1}{3}\rho_2\lambda). \tag{31}$$

Choose $\mu_1 > 0$ and set $\mu := 3(\min\{\rho_1,\rho_2\})^{-1}$. Then (31) implies that

$$f_1(\lambda) \le C_1\lambda^\eta \text{ for } \lambda \ge \mu_1,$$

where $\eta > 0$ is determined by $\rho_1^\eta + \rho_2^\eta = 3^\eta$ and

$$C_1 := \max_{\mu_1 \le \lambda \le \mu\mu_1} \frac{f_1(\lambda)}{\lambda^\eta}.$$

Hence

$$\theta(2,\lambda) - \theta(0,\lambda) \le C_1\lambda^\eta - 5\pi$$

which leads to the eigenvalue estimate

$$C_1\lambda_n^\eta \ge \beta - \alpha + (n+5)\pi \tag{32}$$

provided that $\lambda_n \ge \mu_1$. In a similar way we find that $f_2(\lambda) := f(\lambda) - 4\pi$ satisfies

$$f_2(\lambda) \ge f_2(\tfrac{1}{3}\rho_1\lambda) + f_2(\tfrac{1}{3}\rho_2\lambda)$$

which implies that

$$f_2(\lambda) \ge C_2\lambda^\eta \text{ for } \lambda \ge \mu_2,$$

where $\mu_2 > 0$ is chosen so large that

$$C_2 := \min_{\mu_2 \le \lambda \le \mu\mu_2} \frac{f_2(\lambda)}{\lambda^\eta} > 0.$$

Then we obtain the upper bound

$$C_2\lambda_n^\eta \le \beta - \alpha + (n-4)\pi. \tag{33}$$

We mention that using some simple estimates for the solutions of (6) with λ between μ_j and $\mu\mu_j$ it is again possible to replace C_1 and C_2 by explicit values.

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