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SOME REMARKS ON VARIETIES IN POLYDISCS AND BOUNDED HOLOMORPHIC FUNCTIONS

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This note deals with certain questions which arise in connection with the extension problem for bounded holomorphic functions of several complex variables.

An analytic variety V in the unit polydisc

$$U^N = \{(z_1, \cdots, z_N) \in \mathbf{C}^N : |z_1|, \cdots, |z_N| < 1\}$$

is said to have the *H^∞-extension property* if for every $f \in H^\infty(V)$, the space of bounded holomorphic functions on V, there is $F \in H^\infty(U^N)$ such that $F \mid V = f$. A related property which V may possess is that of being *defined as a set by bounded functions* in the sense that

$$V = \{\mathfrak{z} \in U^N : f(\mathfrak{z}) = 0 \text{ for all } f \in H^\infty(U^N) \text{ which vanish on } V\}\ .$$

We will begin with an example of a remarkably well behaved variety $V \subset U^N$ which fails to have the H^∞-extension property, and after this we give an example of a one dimensional disc D embedded as a submanifold of U^N which not only fails to be defined as a set by bounded functions but is, in fact, a *determining set* for $H^\infty(U^N)$ in that if $f \in H^\infty(U^N)$ vanishes on D, then f is the zero function. Positive results obtained include a geometric condition for a U^k embedded in a U^N to be defined as a set by bounded functions and a result to the effect that if a variety V has the H^∞-extension property and if it satisfies another, possibly redundant, condition, then V is defined as a set by bounded functions.

The general problem of determining which subvarieties of U^N possess the H^∞-extension property seems to be difficult, but some results in this direction are contained in the papers [1] and [9].

We begin with an example of a disc contained in U^2 which does not have the H^∞-extension property.

EXAMPLE 1. Denote by $\mathbf{C}^*$ the Riemann sphere and in $\mathbf{C}^*$ let

$$\varDelta = \{\zeta : |1 - \zeta^2|^{-1} < 1\}\ .$$

The set $\varDelta$ is conformally equivalent to U. If we define $\Phi: \mathbf{C}^* \backslash \{1, -1\} \rightarrow \mathbf{C}^2$ by $\Phi(\zeta) = ((1 - \zeta^2)^{-1}, \varepsilon(1 - \zeta)^{-1})$, then for all choices of $\varepsilon > 0$, the mapping Φ carries $\mathbf{C}^* \backslash \{1, -1\}$ biholomorphically onto a closed, algebraic submanifold M of C^2, and if ε is small enough, then $M \cap U^2 = \Phi(\varDelta)$ so that $\Phi(\varDelta)$ is a disc embedded in U^2. Assume that $\Phi(\varDelta)$ has the

H^∞-extension property so that if $f \in H^\infty(\Delta)$, there is $F \in H^2(U^2)$ such that $f = F \circ \Phi$. Let $h: \Delta \to U$ be a conformal homeomorphism chosen so that $\lim_{y \to 0^+} h(iy) = 1$ and $\lim_{y \to 0^-} h(iy) = -1$. Our assumption on $\Phi(\Delta)$ implies that if $\delta > 0$ is small enough, then there exists $F \in H^\infty(U^2)$ such that $\|F\|_{U^2} \leqq 1$ and $F(\Phi(\zeta)) = \delta h(\zeta)$ for all $\zeta \in \Delta$. In particular

$$\lim_{y \to 0^+} F(\Phi(\pm iy)) = \pm\delta \,. \tag{1}$$

If z and w are points of U, set

$$[z, w] = \left|\frac{z - w}{1 - \bar{w}_z}\right| .$$

By the invariant form of Schwarz's lemma, we know that if g lies in the unit ball of $H^\infty(U)$, then

$$[g(z), g(w)] \leqq [z, w] \,. \tag{2}$$

If we apply (2) to the function h_η given for fixed $\eta \in U$ by $h_\eta(\zeta) = F(\eta, \zeta)$, then for $\eta = (1 - (iy)^2)^{-1}$, $z = \varepsilon/(1 - iy)$, $w = \varepsilon/(1 + iy)$, we are led to

$$\lim_{y \to 0^+} [F(\Phi(iy)), F(\Phi(-iy))] \leqq \lim_{y \to 0^+} \left[\frac{\varepsilon}{1 - iy}, \frac{\varepsilon}{1 + iy}\right] = 0 \,.$$

However, (1) implies that

$$\lim_{y \to 0^+} [F(\Phi(iy)), F(\Phi(-iy))] = \frac{2\delta}{1 + \delta^2} \,.$$

This contradiction shows that the disc $\Phi(\Delta)$ does not have the H^∞-extension property.

The present example is not the first known instance of this phenomenon; another, more involved example was given in [9, Example II. 7]. It can be show, at least for certain choices of the Blaschke product involved, that example has the additional property of being a determining set for $H^\infty(U^2)$. The variety $\Phi(\Delta)$ of the present example is not nearly so pathological, for it is the intersection of U^2 with the zero set of a certain polynomial in two variables. Alexander [1] has also given an example of a variety in U^2 which lacks the H^∞-extension property. His example is the intersection of U^2 with a certain algebraic curve, but it is not irreducible. In [8] Rudin has also given an example.

In connection with Example 1, it is interesting to consider the composition $\Phi \circ \psi$ where ψ is a conformal homeomorphism from the unit disc to Δ such that $\psi(0) = \infty$. If we let $\varphi_1(\zeta) = (1 - \zeta^2)^{-1}$, then $\varphi_1 \circ \psi$ is a two-to-one map from the disc onto itself, and it can be verified without difficulty that $\varphi_1 \circ \psi(z) = \alpha z^2$ for some α of modulus one. Also, if $\varphi_2(\zeta) = \varepsilon(1-\zeta)^{-1}$, then $\varphi_2 \circ \psi$ is one-to-one from the disc into itself, and it is not hard to see that if z_0 and z_1 are the two points in the unit circle carried onto 0 by ψ, then $\varphi_2 \circ \psi$ continues across the two arcs of the unit circle determined by z_0 and z_1. These two points are certain algebraic singularities of the function $\varphi_2 \circ \psi$. These remarks should be compared with the extension theorems for bounded holomorphic functions proved in [9] and [10]; they show that those extension theorems are essentially the best of their kind.

EXAMPLE 2. In this example we will construct in U^N, $N \geqq 2$, a disc which is a determining set for $H^\infty(U^N)$.

Let $\Omega = U \backslash [0, 1)$, and let $h: U \to \Omega$ be a conformal homeomorphism which takes 1 to 1, i to 0 and which has the property that $\operatorname{Im} h(\zeta) \downarrow 0$ as $\zeta \to e^{i\theta}$ if $\theta \in (0, \pi/2)$. The function h admits a unique extension to a continuous function from $\bar{U}$ to $\bar{U}$.

Let $r_k = 1 - k^{-1}$, and let $s_1 > 0$ be very small so that $\{r_k + is_k\}$ does not satisfy the Blaschke condition, i.e., this sequence is not the zero set of a function bounded and holomorphic in U. If $\{s_k\}$ is chosen properly and if $\alpha_k = h^{-1}(r_k + is_k)$, the sequence $\{\alpha_k\}$ will satisfy the Blaschke condition. Let B be the Blaschke product with $\{\alpha_k\}$ as its zero set, and define Φ by $\Phi(\zeta) = (h(\zeta), B(\zeta))$. The sequence $\{\alpha_k\}$ converges to the point 1, so it follows that at every point of ∂U, either $|B|$ or $|h|$ assumes continuously the value 1. Since h' is zero-free and h is one-to-one, it follows that $\Delta = \Phi(U)$ is an analytic submanifold of U^2.

We will prove that if $F \in H^\infty(U^2)$ and $F \circ \Phi = 0$, then F is the zero function, i.e., that Δ is a determining set for $H^\infty(U^2)$. If $F \in H^\infty(U^2)$ vanishes on $\Phi(U)$, then $F(r_k + is_k, 0) = 0$ for all k, so since $\{r_k + is_k\}$ does not satisfy the Blaschke condition, F must vanish identically on the disc $D = \{(z, 0): |z| < 1\}$. If F does not vanish identically, there is a factorization $F(z, w) = w^p G(z, w)$ where p is a positive integer and G a bounded holomorphic function which does not vanish identically on D. As F vanishes on $\Phi(U)$, G must also. This implies, as we have just seen, that $G(r_k + is_k, 0) = 0$ whence G vanishes on the disc D, contrary to hypothesis.

Thus we have a disc in U^2 which is a determining set for $H^\infty(U^2)$. It is quite simple, using the existence of this disc, to find a disc in U^N, $N \geqq 2$, which is a determining set for $H^\infty(U^N)$. We proceed inductively. Suppose that $\Delta \subset U^k$ is a disc which is a determining set for $H^\infty(U^k)$. Then $U^{k+1} = U^k \times U \supset \Delta \times U$. The set $\Delta \times U$ is biholomorphically equivalent to U^2 so there is a one dimensional disc Δ' which is a determining set for $\Delta \times U$. Suppose that $F \in H^\infty(U^{k+1})$ vanishes on Δ'. If we take on U^{k+1} the coordinates $(\mathfrak{z}, \zeta)$, $\mathfrak{z} \in U^k$, $\zeta \in U$, then since Δ' is a determining set for $\Delta \times U$, it follows that for each

$\zeta \in U$, $F(\cdot, \zeta)$ vanishes identically. As this holds for every $\zeta \in U$, F must be the zero function.

Our next example is a direct consequence of the construction given in Example 2.

EXAMPLE 3. In U^N, $N \geqq 3$, there exist irreducible varieties V which are at positive distance, in the sense of the usual metric on C^N, from the distinguished boundary, T^N, of U^N and yet which are not defined, as sets, by bounded functions. To optain such an example, let $\varDelta$ be an irreducible variety, e.g., a disc, which is a determing set for $H^\infty(U^2)$. The set $\varDelta \times \{0\} \subset U^2 \times U^{N-2} = U^N$ is an example of a variety of the desired kind.

This example is of interest because it contracts markedly with a theorem of Rudin [7] according to which *if $V \subset U^N$ is a variety of codimension 1 which is at positive distance from T^N, then not only is V defined as a set by a single bounded function, but, in addition, there is an $F \in H^\infty(U^N)$ with the property that every function holomorphic in U^N and vanishing on V admits a factorization $G = FH$, H holomorphic in U^N.*[1]

Our next result gives a sufficient condition for a disc or polydisc contained in U^N to be defined, as a set, by bounded holomorphic functions.

THEOREM 4. *Let $\Phi\colon U^k \to U^N$ be a proper, holomorphic map, $k \leqq N$, say $\Phi(\mathfrak{z}) = (\varphi_1(\mathfrak{z}), \cdots, \varphi_N(\mathfrak{z}))$. If there is a $\delta > 0$ with the property that for each $\mathfrak{z} \in U^k$ at least $N - k$ of $|\varphi_1(\mathfrak{z})|, \cdots, |\varphi_N(\mathfrak{z})|$ are no more than $1 - \delta$, then the variety $\Phi(U^k)$ is defined as a set by bounded holomorphic functions.*

Let us remark that since Φ is proper, $\Phi(U^k)$ is a variety by [4, Th. V.C. 5].

Proof. Consider first the case that $k = 1$. Let

$$K = \{(z_1, \cdots, z_N) \in U^N\colon |z_1|, \cdots, |z_N| \leqq 1 - \delta\} .$$

The set $\Phi^{-1}(K)$ is compact, and by the maximum modulus theorem no component of the set $\Sigma = U \backslash \Phi^{-1}(K)$ can be bounded away from ∂U, so Σ is connected. If $\zeta \in \Sigma$, then for some j, $|\varphi_j(\zeta)| > 1 - \delta$. Let

$$\Sigma_j = \{\zeta \in \Sigma\colon |\varphi_j(\zeta)| > 1 - \delta\} .$$

[1] *Added in proof.* Y.-T. Siu in his paper Sheaf cohomology with bounds and bounded holomorphic functions, Proc. Amer. Math. Soc. **21** (1969), 226–229, has given a cohomological proof of this and a related result.

The sets Σ_j are all open, they are pairwise disjoint, and their union is the connected set Σ. Thus one of them, say Σ_1, is the whole of Σ and all the other Σ_j are empty. The map Φ is proper so it follows that $|\varphi_1(\zeta)| \to 1$ as $|\zeta| \to 1$. (Although we do not need this fact, it follows that φ_1 is a finite Blaschke product.)

Now consider the case of general k. By [8, Th. II. 3], we may reindex the functions $\varphi_1, \cdots, \varphi_N$, so that if $\mathfrak{z} = (z_1, \cdots, z_N)$, then for $1 \leqq j \leqq k$, $\varphi_j(\mathfrak{z})$ depends only on z_j and so that if

$$\varphi_j^*(e^{i\theta}) = \lim_{r \to 1^-} \varphi_j(re^{i\theta}) ,$$

then $|\varphi_j^*(e^{i\theta})| = 1$ on a set of θ's of positive measure. It follows that we can choose $z_2^0, \cdots, z_k^0$, of modulus less than one so that for some η, $1 > \eta > |\varphi_j(z_j^0)| > 1 - \delta$. Define $\psi\colon U \to U^k$ by

$$\psi(\zeta) = (\zeta, z_2^0, \cdots, z_k^0) .$$

The map ψ is proper, so $\Phi \circ \psi$ is a proper map from U into U^N. We have that of the N coordinates of $\Phi(\psi(\zeta))$, $k-1$, viz., $\varphi_2(z_2^0), \cdots, \varphi_k(z_k^0)$, exceed $1-\delta$, so by our hypothesis on Φ, at least $N-k$ of $|\varphi_1(\zeta)|$, $|\varphi_{k+1}(\psi(\zeta))|, \cdots, |\varphi_N(\psi(\zeta))|$ are less than $1-\delta$. Thus the map $\Phi \circ \psi$ has the property that if $\Phi \circ \psi(\zeta) = (w_1, \cdots, w_N)$, then at least $N-1$ of $|w_1|, \cdots, |w_N|$ are less than η. By our consideration of the case $k=1$, it follows that one of $|\varphi_1|, |\varphi_{k+1} \circ \psi|, \cdots, |\varphi_N \circ \psi|$, tends to one at the boundary of the unit disc while the others remain bounded away from one. As $|\varphi_1^*| = 1$ on a set of positive measure, we may conclude that $|\varphi_1(z_1)| \to 1$ as $|z_1| \to 1$. In the same way we can show that $|\varphi_j(z_j)| \to 1$ as $|z_j| \to 1$, $2 \leqq j \leqq k$.

Let $\pi\colon U^N \to U^k$ be the natural projection onto the first k coordinates, and set $V = \Phi(U^k)$. We know that V is a variety, and what we have done implies that π carries V properly onto U^k. Thus the triple $(V, \pi \mid V, U^k)$ is an analytic cover so our result is a consequence of the following general fact.

LEMMA 5. *Let $\Omega \subset \mathbf{C}^m$ and $\Omega' \subset \mathbf{C}^n$ be bounded domains, let $V \subset \Omega \times \Omega'$ be a purely m dimensional variety, and let $\pi\colon \Omega \times \Omega' \to \Omega$ be the natural projection. If $(V, \pi | V, \Omega')$ is an analytic cover, then V is defined as a set by bounded holomorphic functions on $\Omega \times \Omega'$.*

This lemma is contained in the proof of [4, III. B. 19].

We finish with a result which partially-only partially-answers an obvious question: If the variety $V \subset U^N$ has the H^∞-extension property, does it necessarily follow that V is defined as a set by bounded holomorphic functions? It seems probable that this question has an affirmative answer without qualification on the variety V, but we are able to prove a result in this direction only by making an additional assumption.

THEOREM 6. *If $V \subset U^N$ is a variety with the H^∞-extension property and if V is open in the spectrum of $H^\infty(V)$, then V is defined as a set by bounded holomorphic functions.*

We understand by the *spectrum* of a commutative Banach algebra A the space consisting of the nonzero complex homomorphisms of A taken with the weak* topology. We denote the spectrum of A by $\Sigma(A)$.

Proof. We define an ideal $I^\infty(V)$ and a variety $\tilde{V}$ by

$$I^\infty(V) = \{f \in H^\infty(U^N): f \quad \text{vanishes on} \quad V\}$$

and

$$\tilde{V} = \{\mathfrak{z} \in U^N: f(\mathfrak{z}) = 0 \quad \text{for all} \quad f \in I^\infty(V)\}\,.$$

The variety $\tilde{V}$ evidently contains V and we will prove, under the hypotheses of the theorem, that $\tilde{V} = V$. The restriction map ρ from $H^\infty(U^N)$ to $H^\infty(V)$ is onto and consequently $\Sigma(H^\infty(V))$ can be identified with the set

$$\{\varphi \in \Sigma(H^\infty(U^N)): \varphi f = 0 \quad \text{if} \quad f \in I^\infty(V)\}\,.$$

This set contains $\tilde{V}$ in a natural way and as V is assumed to be open in $\Sigma(H^\infty(V))$, it follows that V is an open subset of $\tilde{V}$. Plainly, V is closed in $\tilde{V}$.

As $\tilde{V} \supset V$, the hypotheses of the theorem imply that the restriction map ρ' from $H^\infty(\tilde{V})$ to $H^\infty(V)$ is onto so we can identify $\Sigma(H^\infty(V))$ with

$$\{\varphi \in \Sigma(H^\infty(\tilde{V})); \varphi f = 0 \quad \text{if} \quad f \in \ker \rho'\}\,.$$

The characteristic function χ of $\tilde{V}\backslash V$ lies in $H^\infty(\tilde{V})$ since V is open and closed in $\tilde{V}$. Since $\chi \in \ker \rho'$, it follows that $\tilde{V}\backslash V$ cannot meet $\Sigma(H^\infty(V))$. We know that $\tilde{V} \subset \Sigma(H^\infty(V))$ so we conclude that $\tilde{V} = V$ as was to be proved.

Our formulation of Theorem 6 suggests another question: *If V is an analytic variety, is it open in $\Sigma(H^\infty(V))$?* This question does not seem to have an obvious answer even for subvarieties of a polydisc though it does seem likely that generally V is open in $\Sigma(H^\infty(V))$. The following remarks are relevant.

REMARKS 7. (a) It is well known that the unit disc is open in $\Sigma(H^\infty(U))$. (See [5].) Similarly, U^N is open in $\Sigma(H^\infty(U^N))$.

(b) It is not hard to see that for many familiar open sets S in C^N, e.g., balls, special analytic polyhedra [4], S is open in $\Sigma(H^\infty(S))$.

(c) For general one dimensional varieties V, we do not know that V is open in $\Sigma(H^\infty(V))$, but the following rather ad hoc argument settles the question for certain Riemann surfaces. Let R be an open connected Riemann surface of finite genus so that $R = R_1 \backslash E$, R_1 a compact Riemann surface and E a closed subset thereof. *If $H^\infty(R)$ contains a nonconstant function, then R is open in $\Sigma(H^\infty(R))$.* Since R is contained in a compact surface and $H^\infty(R)$ contains a nonconstant function, it follows easily from the Riemann-Roch theorem that $H^\infty(R)$ separates points on R. (In the case that R_1 is of genus zero, this sort of result is in papers of Rudin [6] and Wermer [11]; the case of general, finite, genus follows in an analogous way.)

Let $\zeta_0 \in R$. By the Riemann-Roch theorem there exists a function h meromorphic on the ambient surface R_1 which has only one pole, that at ζ_0 and of assigned order p if p is large enough. Thus, for a suitable function $h_1 \in H^\infty(R)$, the function $H = hh_1$ will have at ζ_0 a simple pole, it will be holomorphic on $R \backslash \{\zeta_0\}$, and it will be bounded off a neighborhood of ζ_0. Define an operator $T: H^\infty(R) \to H^\infty(R)$ by

$$T(f) = (f - f(\zeta_0))H .$$

The properties of the function H show that T is a bounded linear operator on $H^\infty(R)$ and that

$$T(fg) = gT(f) + f(\zeta_0)T(g) .$$

Thus in the terminology of Banaschewki [3], T is a *bounded derivation of type* (I, ζ_0). By Proposition 1 of [3], a result previously obtained by Bishop [2], there is a homeomorphism Φ from the open unit disc U onto an *open* set in $\Sigma(H^\infty(R))$ such that $\Phi(0) = \zeta_0$ and such that if $f \in H^\infty(R)$, then $\hat{f} \circ \Phi$ is holomorphic on U. Since there is a disc in R through ζ_0, it follows from the openness of $\Phi(U)$ in $\Sigma(H^\infty(R))$ that some neighborhood of ζ_0 in R is at the same time a neighborhood of ζ_0 in $\Sigma(H^\infty(R))$. It follows that R is open in $\Sigma(H^\infty(R))$ as was to be proved.

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