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THE NUMERICAL RANGE OF AN OPERATOR

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Let A be a continuous linear operator on a complex Hilbert space X with inner product $<, >$ and associated norm $\|\ \|$. Let $W(A) = \{\langle Ax, x\rangle \mid \|x\| = 1\}$ be the numerical range of A and for each complex number z let $M_z = \{x \mid \langle Ax, x\rangle = z\|x\|^2\}$. Let $\curlyvee M_z$ be the linear span of M_z and $M_z \oplus M_z = \{x + y \mid x \in M_z$ and $y \in M_z\}$. An element z of $W(A)$ is characterized in terms of the set M_z as follows:

THEOREM 1. If $z \in W(A)$, then $\curlyvee M_z = M_z \oplus M_z$ and

(i) z is an extreme point of $W(A)$ if and only if M_z is linear;

(ii) if z is a nonextreme boundary point of $W(A)$, then $\curlyvee M_z$ is a closed linear subspace of X and $\curlyvee M_z = \cup\{M_w \mid w \in L\}$, where L is the line of support of $W(A)$, passing through z. In this case $\curlyvee M_z = X$ if and only if $W(A) \subset L$.

(iii) if $W(A)$ is a convex body, then x is an interior point of $W(A)$ if and only if $\curlyvee M_z = X$.

It is well-known that $W(A)$ is a convex subset of the complex plane. Thus if $z \in W(A)$, either z is an *extreme point* (not in the interior of any line segment with endpoints in $W(A)$), a nonextreme boundary point, or an interior point (with respect to the usual plane topology) of $W(A)$. Thus Theorem 1 characterizes every point of $W(A)$.

The following additional notation and terminology are used. If $K \subset X$, then $K^\perp$ denotes the orthogonal complement of K. An operator A is *normal* if and only if $AA^* = A^*A$ and *hyponormal* only if $AA^* \ll A^*A$. A line L is a *line of support* for $W(A)$ if and only if $W(A)$ lies in one of the closed half-planes determined by L and $L \cap \overline{W(A)} \neq \varnothing$.

In the last section of the paper consideration is given to $\bigcap\{$maximal linear subspaces of $M_z\}$. One result is that if A is hyponormal and z a boundary point of $W(A)$, then $\bigcap\{$maximal linear subspaces of $M_z\} = \{x \mid Ax = zx$ and $A^*x = z^*x\}$. This generalizes Stampfli's result in [3]: if A is hyponormal and z is an extreme point of $W(A)$, then z is an eigenvalue of A. In [2] MacCluer proved this theorem for A normal.

2. A proof of Theorem 1.

Lemmas 1 and 2 provide the core of the proof of Theorem 1.

LEMMA 1. *Let z be in the interior of a line segment with endpoints a and b in $W(A)$, $x \in M_a$, $y \in M_b$, $\|x\| = \|y\| = 1$. There exist* *real numbers s and t in* $(0, 1)$ *and a complex number* λ, $|\lambda| = 1$, *such that* $tx + (1 - t)\lambda y \in M_z$ *and* $sx - (1 - s)\lambda y \in M_z$. *Consequently,*

$$M_a \subset M_z \oplus M_z \, .$$

Proof. In proof of the convexity of $W(A)$ given in [1], pp. 317–318, it is shown that $tx + (1 - t)\lambda y \in M_z$ for some real number t in $(0, 1)$ and some complex λ, $|\lambda| = 1$. A slight modification of the argument shows that $sx - (1 - s)\lambda y \in M_z$ for some real number s in $(0, 1)$. Therefore, since M_z is homogeneous and $s, t \in (0, 1)$, $x \in M_z \oplus M_z$, proving the last assertion.

LEMMA 2. *Let L be a line of support of $W(A)$ and $N = \bigcup \{M_w \mid w \in L\}$.*

(i) *There exists a real number θ such that $N = \{x \mid e^{i\theta}(A - z)x = e^{-i\theta}(A^* - z^*)x\}$ for all z in L.*

(ii) *N is a closed linear subspace of X.*

(iii) *$N = X$ if and only if $W(A) \subset L$.*

Proof. (i) Let θ be such that $e^{i\theta}(w - z)$ is real for all w and z in L. Then $N = \{x \mid < e^{i\theta}(A - z)x, x > \text{ is real}\}$. Therefore since L is a line of support of $W(A)$, Im $e^{i\theta}(A - z) \gg 0$ or $\ll 0$ and thus $N = \{x \mid e^{i\theta}(A - z)x = e^{-i\theta}(A^* - z^*)x\}$. Conclusion (ii) follows immediately from (i), and (iii) follows from the definition of N.

Proof of Theorem 1. Let $z \in W(A)$. (i) In Lemma 2 of [3] it is proven that M_z is linear if z is an extreme point of $W(A)$. If z is not an extreme point of $W(A)$, z is in the interior of a line segment with end points a and b in $W(A)$. By Lemma 1, $M_a \subset M_z \oplus M_z$. Since $a \neq z$, $M_a \cap M_z = \{0\}$. Therefore M_z cannot be linear. (ii) Assume now that z is a nonextreme boundary point of $W(A)$. Let L be the line of support of $W(A)$, passing through z, and let $N = \bigcup \{M_w \mid w \in L\}$. Lemma 1 implies that $M_w \subset M_z \oplus M_z$ whenever $w \in L$; consequently, $N \subset M_z \oplus M_z$. Lemma 2 (ii) implies that $\vee M_z \subset N$. Therefore, $M_z \oplus M_z = \vee M_z = N$ and thus by Lemma 2 (iii) $\vee M_z = X$ if and only if $W(A) \subset L$. (iii) Assume now that $W(A)$ is a convex body. If z is an interior point of $W(A)$, Lemma 1 implies that $M_a \subset M_z \oplus M_z$ for each a in $W(A)$. Therefore

$$X = \bigcup \{M_a \mid a \in W(A)\} \subset M_z \oplus M_z \subset \vee M_z = X \, .$$

On the other hand if z is a boundary point of $W(A)$ either $\vee M_z = M_z$ or $\vee M_z = N$ and in either case $\vee M_z \neq X$ since $W(A)$ is a convex body.

3. $\bigcap$ {Maximal linear subspaces of M_z}. Although M_z may

not be linear, it is homogeneous and closed. Therfore if $M_z \neq \{0\}$ and $x \in M_z$, there exists a nonzero maximal linear subspace of M_z, containing x. Consideration of the intersection of these maximal linear subspaces yields information about eigenvalues and eigenvectors of A.

THEOREM 2. *Let $z \in W(A)$ and $K_z = \bigcap \{$maximal linear subspaces of $M_z\}$. If z is a boundary point of $W(A)$, let $N = \bigcup \{M_w \mid w \in L\}$, where L is a line of support for $W(A)$, passing through z.*

(i) *If z is a boundary point of $W(A)$, $x \in K_z$, and $Ax \in N$, then $Ax = zx$ and $A^*x = z^*x$. Conversely, if $Ax = zx$ and $A^*x = z^*x$, then $x \in K_z$.*

(ii) *If $W(A)$ is a convex body and z is in the interior of $W(A)$, $K_z = \{x \mid Ax = zx$ and $A^*x = z^*x\}$.*

Proof. By elementary techniques it can be shown that for each complex z

(1) $K_z = M_z \bigcap [(A - z)(\vee M_z)]^\perp \bigcap [(A^* - z^*)(\vee M_z)]^\perp$ and that if z is extreme,

(2) $M_z \subset [(A - z)N]^\perp \bigcap [(A^* - z^*)N]^\perp$.

(The proof of (2) depends upon the fact that M_z is linear if z is extreme.) (i) Let z be a boundary point of $W(A)$. By Theorem 1, $K_z = M_z$ if z is extreme and $\vee M_z = N$ if z is nonextreme. Moreover, if $x \in K_z$ and $Ax \in N$, Lemma 2 implies that

$$(A - z)x \in N \quad \text{and} \quad (A^* - z^*)x \in N .$$

It now follows from (1) and (2) that $Ax = zx$ and $A^*x = z^*x$. The converse follows immediately from (1). (ii) If $W(A)$ is a convex body and z is in the interior of $W(A)$, $\vee M_z = X$ by Theorem 1 and (1) implies that $K_z = \{x \mid Ax = zx$ and $A^*x = z^*x\}$.

COROLLARY 1. *If A is hyponormal and z is a boundary point of $W(A)$, $\bigcap \{$maximal linear subspaces of $M_z\} = \{x \mid Ax = zx$ and $A^*x = z^*x\}$. In particular, if z is an extreme point of $W(A)$, z is an eigenvalue of A.*

Proof. Again let $N = \bigcup \{M_w \mid w \in L\}$, where L is a line of support for $W(A)$, passing through z. In Lemma 3 of [3] Stampfli proves that $A(N) \subset N$. Thus by Theorem 2, (i) $K_z = \{x \mid Ax = zx$ and $A^*x = z^*x\}$. Moreover, if z is extreme, $K_z = M_z \neq \{0\}$.

One last remark about potential eigenvalues and eigenvectors: it is immediate from Lemma 2 (i) that if z is a boundary point of $W(A)$, $Ax = zx$ if and only if $A^*x = z^*x$.

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