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POLYNOMIAL RINGS AND H_i-LOCAL RINGS

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Three theorems concerning when certain localities L of $R[X_1, \cdots, X_k]$ are H_i-local rings (that is, for every height i prime ideal p in L, depth p = altitude $L - i$) are proved, where R is a local ring. A number of known corollaries, as well as some new ones, easily follow.

All rings in this paper are assumed to be commutative rings with an identity element, and the undefined terminology is, in general, the same as that in [3].

In this paper we prove three theorems concerning when certain localities of $R[X_1, \cdots, X_k]$ are H_i-local rings (see (2.1.1) for the definition), where (R, M) is a local ring. The reason the results are of interest and importance is that H_i-local rings were introduced to enable one to closely examine some of the chain conditions for prime ideals and some of the chain conjectures (for example, Chain Conjecture, H-Conjecture, Catenary Chain Conjecture, etc. (see [8, (1.6), (4.1), and (4.2)])). These conjectures have been open problems for quite some time, so it is hoped that by studying the condition that a local ring is H_i (which is a weaker condition than, for example, that it satisfy the f.c.c. (see (2.10.1))), some new insight on chains of prime ideals in local rings will be gained which will help settle the chain conjectures.

The first theorem (2.6) shows that $D_k = R[X_1, \cdots, X_k]_{(M, X_1, \cdots, X_k)}$ is C_i (2.1.2) if and only if it is H_i and H_{i+1}. This is an important result, and it lends support to the Catenary Chain Conjecture, as is explained after (2.18). Quite a few corollaries of (2.6) are then proved, some of which are new, and some of which are previously known results which easily follows from (2.6) and which partly indicate its importance.

In §3 the second theorem is proved. It shows that if D_k is H_i and $i \leqq k$, then D_k is $H_0, H_1, \cdots, H_i$ (3.1). From this it follows that, for all $j > 0$, D_{k+j} is $H_0, \cdots, H_i$ (3.2). Also, some other related results are given in §3.

In §4 it is proved that R is C_i if and only if $R(X) = R[X]_{MR[X]}$ is C_i (4.1). This answers a question asked in [12]. It follows from (4.1) that a local domain is C_i if and only if, for all analytically independent elements b, c in R, $R[c/b]_{MR[c/b]}$ is C_{i-1} (4.2).

Many of the references in this paper are very new, and a number of the results in this paper probably couldn't have been proved till now, since the needed preliminary results have only very recently been proved. Because so many of the references haven't as yet appeared in print, to facilitate the readibility of this paper, a number of the needed results from these references are explicitely stated before their first use in this paper.

2. First theorem. In this section we prove the first main theorem in this paper in (2.6). Up till then, a number of preliminary definitions and lemmas are given. After proving (2.6), quite a few corollaries are given.

We begin with the following definition.

(2.1) DEFINITION. Let A be a ring and let i be a nonnegative integer.

(2.1.1) A is said to be an *H_i-ring* (or, A is said to be H_i) in case, for all height i prime ideals p in A, depth $p =$ altitude $A - i$ (that is, height $p +$ depth $p =$ altitude A).

(2.1.2) A is said to be a *C_i-ring* (or, A is said to be C_i) in case A is H_i, H_{i+1}, and, for all height i prime ideals p in A, all maximal ideals in the integral closure of A/p have the same height (= altitude A/p = altitude $A - i$).

Numerous properties of H_i-local domains and of C_i-local domains are given in **[4]** and **[5]**, and most of these results have been generalized to local rings in **[12]**. Most of the results on these rings which we need in this article are summarized in the following remark.

(2.2) REMARK. Let (R, M) be a local ring. Then the following statements hold:

(2.2.1) Clearly R is H_i and C_i, for all $i \geqq$ altitude $R - 1$ (vacuously, for $i >$ altitude R).

(2.2.2) R is H_i (respectively, C_i) if and only if, for all height j ($j \leqq i$) prime ideals p in R, R/p is H_{i-j} (respectively, C_{i-j}) and either depth $p =$ altitude $R - j$ or depth $p \leqq i - j$ [**12**, (2.4) and (3.3)].

(2.2.3) R is H_i if and only if $R(X) = R[X]_{MR[X]}$ is H_i [**12**, (2.7)].

(2.2.4) R is C_i if and only if $D = R[X]_{(M,X)}$ is H_{i+1} [**12**, (3.7)].

We will show in (2.6) below that D in (2.2.4) is C_i if and only if D is H_i and H_{i+1}. To prove this result, we need two more definitions and lemmas. (For (2.3.1), recall that a *maximal chain of prime ideals in a ring A* is a chain of prime ideals $p_0 \subset \cdots \subset p_n$ in A such that p_0 is minimal, p_n is maximal, and height $p_i/p_{i-1} = 1$, $i = 1, \cdots, n$. The *length* of the chain is n.)

(2.3) DEFINITION. Let (R, M) be a local ring.

(2.3.1) $c(R) = \{n$; there exists a minimal prime ideal z in R such that there exists a maximal chain of prime ideals of length n in some integral extension domain of $R/z\}$.

(2.3.2) $D_k = R[X_1, \cdots, X_k]_{(M, X_1, \cdots, X_k)}$, $k = 1, 2, \cdots$.

We emphasize that $k > 0$ in (2.3.2). We *do not* define $D_0 = R$, since many of the statements below which hold for D_k $(k > 0)$ do not hold for R.

The following lemma is a new and important result. It is mainly because of this lemma that we are able to prove the first theorem in this paper. Most of the lemma was proved for local domains only very recently [**10**, (2.14)] and [**11**, (5.2)]. To prove the lemma, we use, besides the domain case, the following result [**2**, Lemma 1]: If $P \subset P_1 \subset \cdots \subset P_{n-1} \subset Q$ is a saturated chain of prime ideals in a Noetherian ring, then there exists such a chain, say $P \subset P'_1 \subset \cdots \subset P'_{n-1} \subset Q$, such that height $P'_i =$ height $P + i$ $(i = 1, \cdots, n-1)$.

(2.4) LEMMA. *The following statements are equivalent for a local ring R*:

(2.4.1) $n \in c(R)$ (2.3.1).

(2.4.2) $n + k \in c(D_k)$ (2.3.2).

(2.4.3) *There exists a prime ideal p in D_k such that height $p = n + k - 1$ and depth $p = 1$.*

Proof. By [**10**, (2.14.1) $\Leftrightarrow$ (2.14.5)], (2.4.1) $\Leftrightarrow$ (2.4.2) when R is a local domain. Therefore, since the minimal prime ideals in D_k are the ideals zD_k with z a minimal prime ideal in R and $D_k/zD_k \cong (R/z)[X_1, \cdots, X_k]_{(M/z, X_1, \cdots, X_k)}$, (2.4.1) $\Leftrightarrow$ (2.4.2) for local rings.

(2.4.3) implies (2.4.2), since if w is a minimal prime ideal in D_k such that $w \subseteq p$, height $p/w = n + k - 1$, and depth $p = 1$, then in D_k/w there clearly exists a maximal chain of prime ideals of length $n + k$ and D_k/w is integral over D_k/w.

Finally, if (2.4.2) holds, then, by [**10**, (2.14.4) $\Leftrightarrow$ (2.14.5)], there exists a minimal prime ideal w in D_k such that there exists a maximal chain of prime ideals of length $n + k$ in D_k/w. Therefore, since w is minimal, there exists a maximal chain of prime ideals of length $n + k$ in D_k. Hence, by [**2**, Lemma 1] (see the paragraph preceeding this lemma), there exists a prime ideal p in D_k such that height $p = n + k - 1$ and depth $p = 1$, so (2.4.2) implies (2.4.3).

To prove (2.5) below, we need to recall three further results. (a) If $A \subseteq B$ are integral domains such that A is Noetherian and B is integral over A, and if P is a prime ideal in B such that $1 <$ height $P = h <$ height $P \cap A$, then there exist infinitely many prime ideals p in A such that $p \subset P \cap A$, height $p = 1$, and height $(P \cap A)/p = h - 1$ [**7**, Proposition 2.10]. (b) If p is a prime ideal in a Noetherian ring A, then at most a finite number of prime ideals P in A are such that $p \subset P$, height $P/p = 1$, and height $P >$ height $p + 1$ [**1**, Theorem 1]. (c) If p is a prime ideal in a local ring R and $n \in c(R/p)$, then $n +$ height $p \in c(R)$ [**10**, (2.25.1)].

(2.5) LEMMA. *Let (R, M) be a local ring, let a = altitude R, and assume that R is H_i and H_{i+1}. Then either D_1 (2.3.2) is H_{i+1} or there exists a prime ideal p in D_1 such that height $p = i+1$ and depth $p = 1$.*

Proof. Assume that D_1 isn't H_{i+1}. Then R isn't C_i (2.2.4), so, by hypothesis, there exists a height i prime ideal p in R such that there exists a maximal ideal N in the integral closure of R/p such that $a - i >$ height N = (say) h. If $h > 1$, then there exist infinitely many prime ideals q' in R/p such that height $q' = 1$ and depth $q' = h - 1$ [7, Proposition 2.10] (see (a) in the paragraph preceeding this lemma). Therefore there exist infinitely many prime ideals q in R such that $p \subset q$, height $q/p = 1$, and depth q = depth $q/p = h - 1$. Thus, by [1, Theorem 1] (see (b) above), there exist such q such that height q = height $p + 1 = i + 1$. Then height q + depth $q = i + h < a$; but this contradicts that R is H_{i+1}. Hence $h = 1$, so $1 \in c(R/p)$. Therefore $i + 1 \in c(R)$ [10, (2.25.1)] (see (c) above), so there exists, in D, a height $i+1$ and depth one prime ideal (2.4).

Before proving the first theorem, it should be noted that D_1 need not be H_{i+1} in (2.5). For example, let R be a local domain such that altitude $R = 2$ and such that there exists a height one maximal ideal in the integral closure of R (for example, let R be as in [3, Example 2, pp. 203–205] in the case $m = 0$ and $r = 1$). Then R is H_0 and H_1, but D_1 isn't H_1, by (2.2.4), since R isn't C_0.

We will now prove the first main result in this paper. It will be seen after its proof that a number of corollaries easily follow.

(2.6) THEOREM. *Let (R, M) be a local ring, and let D_k be as in (2.3.2). Then the following statements are equivalent:*

(2.6.1) D_k is C_i.

(2.6.2) D_k is H_i and H_{i+1}.

(2.6.3) D_{k+1} is H_{i+1}.

Proof. By (2.2.4), (2.6.1) $\Leftrightarrow$ (2.6.3); and clearly (2.6.1) implies (2.6.2). To show that (2.6.2) implies (2.6.3), suppose that D_{k+1} isn't H_{i+1}. Then $i + 1 <$ altitude $D_{k+1} - 1$ (2.2.1). Also, by (2.5), there exists in D_{k+1} a height $i+1$ and depth one prime ideal. Therefore, by (2.4), in D_k there exists a height i and depth one prime ideal. Thus, since D_k is H_i, $i + 1$ = altitude D_k = altitude $D_{k+1} - 1$; contradiction. Therefore D_{k+1} is H_{i+1}.

The following corollary can be extended (with suitable assumptions) to local rings, much as in [12, (3.14)], but we content ourselves with the domain case here. The corollary will be used in (2.9) below to give a big improvement to [**4**, (4.14)]. To prove (2.7), we need the following result [**4**, (4.7)]: A local domain (R, M) is C_i $(i > 0)$ if and only if, for all x in the quotient field of R such that $N = (M, x)R[x]$ is proper, $R[x]_N$ is H_i.

(2.7) COROLLARY. *Let F be the quotient field of a local domain (R, M). Then the following statements are equivalent, for $i > 0$:*

(2.7.1) *R is C_i and C_{i+1}.*

(2.7.2) *For all $x \in F$ such that $N = (M, x)R[x]$ is proper, $R[x]_N$ is C_i.*

(2.7.3) *For all $x \in F$ such that $N = (M, x)R[x]$ is proper, $R[x]_N$ is H_i and H_{i+1}.*

Proof. If (2.7.1) holds, then D_1 is H_{i+1} and H_{i+2} (2.2.4). Therefore D_1 is C_{i+1} (2.6). Hence, since $R[x]_N = D_1/K$, for some height one prime ideal K in D_1, (2.7.2) holds, by (2.2.2).

It is clear that (2.7.2) implies (2.7.3).

Finally, if (2.7.3) holds, then, by [**4**, (4.7)], R is C_i and C_{i+1}. Therefore (2.7.1) holds.

If we let $i = 0$ in (2.7), then (2.7.1) implies (2.7.2), and (2.7.2) implies (2.7.3), by the proof of (2.7). Also, (2.7.2) implies (2.7.1), since (2.7.2) implies that R is C_0 (take $x = 0$), and (2.7.2) implies that R is C_1 (since all $R[x]_N$ are H_1). Finally, (2.7.3) does not imply (2.7.1) for $i = 0$, as can be seen by [**3**, Example 2, pp. 203–205] in the case $m = 0$ and $r > 0$.

In the following corollary, some of the subscripts may be negative. Therefore, the following convention will be adopted: The statement that a ring A is H_i (respectively, C_i) with $i < 0$ says nothing about the ring (it is vacuously true).

Since the case $k = 1$ of the following corollary is given in (2.2.4), we restrict attention to the case $k > 1$.

(2.8) COROLLARY. *With the notation of* (2.6), *the following statements are equivalent, for $k > 1$:*

(2.8.1) *D_k is H_i.*

(2.8.2) *D_j is $H_{i-(k-j)}, \cdots, H_i$, for some j $(1 \leqq j \leqq k)$.*

(2.8.3) *D_j is $C_{i-(k-j)}, \cdots, C_{i-1}$, for all j $(1 \leqq j < k)$.*

(2.8.4) *R is $C_{i-k}, \cdots, C_{i-1}$.*

Proof. By (2.2.4), (2.8.3) (for $j = 1$) implies (2.8.4) and (2.8.4) implies (2.8.2) (for $j = 1$). Further, by successive applications of (2.6), (2.8.2) implies (2.8.1) and (2.8.1) implies (2.8.3).

The following corollary is a considerable sharpening of [**4**, (4.14)]. As with (2.7), the result can be extended to local rings, but we content ourselves with the local domain case here.

(2.9) COROLLARY. *Let F be the quotient field of a local domain (R, M), and let i and k be integers such that $1 < k < i$. Then the following statements are equivalent*:

(2.9.1) *R is $C_{i-k}, \cdots, C_{i-1}$.*

(2.9.2) *For all j $(1 \leqq j < k)$ and for all $x_1, \cdots, x_j$ in F such that $N = (M, x_1, \cdots, x_j)A$ is proper, where $A = R[x_1, \cdots, x_j]$, A_N is $C_{i-k}, \cdots, C_{i-1-j}$.*

(2.9.3) *There exists j $(1 \leqq j \leqq k)$ such that for all $x_1, \cdots, x_j$ in F such that $N = (M, x_1, \cdots, x_j)A$ is proper, where $A = R[x_1, \cdots, x_j]$, A_N is $H_{i-k}, \cdots, H_{i-j}$.*

(2.9.4) *For all $x_1, \cdots, x_k$ in F such that $N = (M, x_1, \cdots, x_k)A$ is proper, where $A = R[x_1, \cdots, x_k]$, A_N is H_{i-k}.*

Proof. If (2.9.1) holds, then, for $j = 1, \cdots, k$, D_j is $C_{i-k+j}, \cdots, C_{i-1}$ (2.8). Hence, since $A_N = D_j/K$, for some height j prime ideal K in D_j (where A_N is as in (2.9.2)), (2.9.2) follows from (2.2.2).

If (2.9.2) holds, then, for $j = k - 1$, all A_N (of (2.9.2)) are C_{i-k}. Therefore, by [**4**, (4.7)] (see the paragraph preceeding (2.7)), (2.9.4) holds.

It is clear that (2.9.4) implies (2.9.3), for $j = k$.

Finally, if (2.9.3) holds, then, for $A = R[x_1, \cdots, x_j]$, let $A_h = R[x_1, \cdots, x_h]$ $(h = 1, \cdots, j)$ and let $L_h = (A_h)_{(M, x_1, \cdots, x_h)}$. Then it follows from (2.7) that all the rings L_j are $C_{i-k}, \cdots, C_{i-j-1}$. Therefore, by (2.7), all the rings L_{j-1} are $C_{i-k}, \cdots, C_{i-j}$. Repeating this, it follows that (2.9.1) holds.

If we let $k = i$ in (2.9), then, by the proof of (2.9), (2.9.1) implies (2.9.2). Also, if we let $k = 1$, then (2.9.1) and (2.9.4) are equivalent, by [**4**, (4.7)].

To derive some further results, the following definitions are needed.

(2.10) DEFINITION. Let A be a ring.

(2.10.1) A satisfies the *first chain condition for prime ideals* (*f.c.c.*) in case every maximal chain of prime ideals in A has length equal to altitude A.

(2.10.2) A satisfies the *second chain condition for prime ideals* (*s.c.c.*) in case, for each minimal prime ideal z in A, depth z = altitude A and every integral extension domain of A/z satisfies the f.c.c.

(2.10.3) A satisfies the *chain condition for prime ideals* (*c.c.*) in case, for each pair of prime ideals $P \subset Q$ in A, $(A/P)_{Q/P}$ satisfies the s.c.c.

Many properties of rings which satisfy one of these chain conditions are known. Many of these are summarized in [**7**, Remarks 2.22–2.25].

In much of what follows we shall assume that a = altitude $R > 1$

definition, if $1 \in c(R)$, then there exists a height one maximal ideal, say N, in some integral extension domain, say S, of R. Then, if N' is a prime ideal in the integral closure of S such that $N' \cap S = N$, then height $N' = 1$. Also, by [3, (10.14)], $1 = \text{height } N' = \text{height } N' \cap R'$, so there exists a height one maximal ideal in R'. Finally, if there exists such a maximal ideal, then clearly $1 \in c(R)$.

The following corollary is a somewhat surprising result.

(2.15) COROLLARY. *Assume that a local ring R is H_i and H_{i+1}. Then either R is C_i, or there exists a height i prime ideal p in R such that there exists a height one maximal ideal in the integral closure of R/p.*

Proof. Assume that R isn't C_i, so $i < \text{altitude } R - 1$ (2.2.1). Also, by (2.2.4) and (2.5), there exists a height $i+1$ and depth one prime ideal, say P, in D_1. Let $p = P \cap R$. Then $pD_1 \subset P$ (otherwise, $1 = \text{depth } P = \text{depth } p + 1$, so $p = M$, hence height $M = \text{height } p = \text{height } P = i + 1 < \text{altitude } R$; contradiction). Therefore height $p = i$ and, in $(R/p)[X]_{(M/p,X)} \cong D_1/pD_1$, there exists a height one and depth one prime ideal. Therefore, by (2.14), there exists a height one maximal ideal in the integral closure of R/p.

The following corollary gives an equivalence of a local ring being C_i.

(2.16) COROLLARY. *Let R be a local ring, and let $i < \text{altitude } R - 1$. Then R is C_i if and only if R is H_i, H_{i+1}, and for each height i prime ideal p in R, there are no height one maximal ideals in the integral closure of R/p.*

Proof. If R is C_i, then, since $i < \text{altitude } R - 1$, the conclusion holds. The converse follows from (2.15).

The following known result is an immediate corollary to (2.15).

(2.17) COROLLARY. (cf. [8, (3.2)].) *Assume that R is an H_1-local domain. Then either D_1 is H_1, or there exists a height one maximal ideal in the integral closure of R.*

Proof. R is H_0 and H_1, so either R is C_1 (hence D_1 is H_1 (2.2.4)), or there exists a height one maximal ideal in the integral closure of R (2.15).

If, in (2.17), D_1 is H_1, then since D_1 is clearly H_0, it follows from (2.6) that D_2 is H_1. Repeating this, D_i is H_1, for all $i \geqq 1$. This will be generalized in (3.2) below.

(2.18) REMARK. In (2.6)–(2.17), attention has been directed at D_k. It might be thought that if, instead of $(M, X_1, \cdots, X_k)$, some other maximal ideal, say N, in $R_k = R[X_1, \cdots, X_k]$ such that $N \cap R = M$, had been singled out, then perhaps different results would have been obtained. This isn't true. For, it is known [**13**, (5.5)] that D_k is H_i (respectively, C_i) if and only if $(R_k)_N$ is H_i (respectively, C_i), for all maximal ideals N in R_k such that $N \cap R = M$.

(2.6) lends support to the Catenary Chain Conjecture (that is, if a local domain R is H_i ($i = 0, 1, \cdots, a$ = altitude R), then R is $C_1, \cdots, C_a$ (see [**8**, (4.2)] and [**12**, (3.13)]). (2.6) shows that this holds for the local domains D_k (and even more, namely, D_k is also C_0). If this could be proved for all local domains, then the Catenary Chain Conjecture holds. Even though we aren't now able to show this for all local domains, there is an important class of local rings which have this property, as is shown in (2.19) below. The following paragraph gives some information needed for (2.19).

In (2.6) it was seen that the rings D_k are C_i if and only if they are H_i and H_{i+1}. There is another class of rings which have this property, namely, Henselian local rings [**9**, (2.12)]. It is also true for a Henselian local ring R, as for D_k (see (2.4)), that $n \in c(R)$ if and only if there exists a height $n-1$ and depth one prime ideal in R [**11**, (4.2.2) and (a) $\Leftrightarrow$ (b)]. In this regard, the class C of local rings R with this property ($n \in c(R)$ if and only if there exists a height $n-1$ and depth one prime ideal in R) was introduced and studied in [**11**, § 4]. Because of (2.15), we can prove the following important property of the rings in C.

(2.19) PROPOSITION. *Assume that $R \in C$ (of the preceeding paragraph). If R is H_i and H_{i+1}, then R is C_i.*

Proof. Suppose that R isn't C_i. Then, by (2.15), there exists a height i prime ideal p in R such that $1 \in c(R/p)$. Thus, by (c) in the paragraph preceeding (2.5), $i + 1 \in c(R)$. Hence, since $R \in C$, there exists a height i and depth one prime ideal in R. Therefore, by hypothesis, i = altitude $R - 1$, so R is C_i (2.2.1); contradiction. Therefore R is C_i.

3. Second theorem. In this section we prove the second main theorem in this paper (3.1). The theorem shows that if D_k is H_i and $i \leqq k$, then D_k is $H_0, H_1, \cdots, H_i$. Then some related results are proved.

(3.1) THEOREM. *Let (R, M) be a local ring, let a = altitude $R \geqq 1$, and let D_k be as in (2.3.2). Assume that there exists a positive integer $i \leqq k$ such that D_k is H_i. Then D_k is $H_0, \cdots, H_i$.*

Proof. Suppose that D_k isn't H_j, for some $0 \leqq j < i$. Then there exists a height j prime ideal p in D_k such that depth $p < a + k - j$, so $d = \text{depth } p \leqq i - j$ (2.2.2). Therefore $j + d \in c(D_k)$ and $j + d \leqq i \leqq k$. Hence $1 \in c(D_{k-j-d+1})$ (2.4) and $k - j - d + 1 \geqq 1$. Therefore, necessarily altitude $D_{k-j-d+1} = 1$ (2.4), so $k = j + d$ and $a = 0$; contradiction. Therefore D_k is H_j, for $0 \leqq j \leqq i$.

Before giving some corollaries to (3.1), it should be noted that if D_k is H_i and $i > k$, then D need not be H_{i-1}. For example, let R be as in [3, Example 2, pp. 203–205] in the case $m = 0$ and $r = 2$. Then R is a local domain of altitude 3 which satisfies the f.c.c and there exists a height one maximal ideal in the integral closure R' of R, so R is H_i $(i = 0, 1, 2, 3)$ and R isn't C_0. Also, R is C_1, since, for each height one prime ideal p in R, there exists exactly one prime ideal p' in R' such that $p' \cap R = p$, and then $R'/p' = R/p$, hence, since R'/p' is a homomorphic image of the regular local ring $R'_{NR'}$ (see [3]), R/p satisfies the s.c.c. Therefore D_1 is H_2 and isn't H_1 (2.2.4).

(3.2) COROLLARY. *With (R, M) and $a > 0$ as in (3.1) assume that there exist positive integers $i \leqq k$ such that D_k is H_i. Then, for all $j \geqq 0$, D_{k+j} is H_i.*

Proof. Clearly it suffices to prove that D_{k+1} is H_i. For this, since D_k is H_i, D_k is H_i and H_{i-1} (3.1). Therefore D_{k+1} is H_i (2.6).

We next give two results which are closely related to (3.1).

(3.3) PROPOSITION. *Assume that R is an integrally closed local domain and that D_1 is H_2. Then D_1 is H_1.*

Proof. Since D_1 is H_2, R is H_1 (2.2.4). Therefore, by hypothesis, R is C_0, so D_1 is H_1 (2.2.4).

(3.4) PROPOSITION. *Let (R, M) be a local ring, and assume that there exist integers $k > 0$ and $j \geqq 0$ such that D_k is H_{k+j}. If there exists $h > 0$ such that D_{k+h} is $H_{k+h+j+1}$, then D_{k+h} is H_{k+h+j}.*

Proof. Suppose that D_{k+h} isn't H_{k+h+j}, so there exists a height $k + h + j$ and depth one prime ideal in D_{k+h} (2.2.2). Therefore, by (2.4), there exists a height $k + j$ and depth one prime ideal in D_k. Hence, by hypothesis, $k + j + 1 = \text{altitude } D_k$, so $k + j + h + 1 = \text{altitude } D_{k+h}$, hence D_{k+h} is H_{k+j+h} (2.2.1); contradiction. Therefore D_{k+h} is H_{k+h+j}.

By combining (3.1) and (3.4), we have the following result.

(3.5) COROLLARY. *Let* (R, M) *be a local ring, and assume that* D_k *is* H_k, *for some* $k > 0$. *If there exists* $h > 0$ *such that* D_{k+h} *is* H_{k+h+1}, *then* D_{k+h} *is* H_i, *for all* $i = 0, 1, \cdots, k + h + 1$.

Proof. Clear by (3.4) and (3.1).

4. Third theorem. In this section it is shown in (4.1) that R is C_i if and only if $R(X) = R[X]_{MR[X]}$ is C_i, where (R, M) is a local ring. Then a number of corollaries of (4.1) are given.

In [**12**, (3.19)] the following question was asked: If (R, M) is a C_i-local ring, is $R(X) = R[X]_{MR[X]}$ C_i? The answer is yes, as is shown in the following theorem.

(4.1) THEOREM. *Let* (R, M) *be a local ring. Then* R *is* C_i *if and only if* $R(X) = R[X]_{MR[X]}$ *is* C_i.

Proof. Assume first that R is C_i. Then $R(X)$ is H_i and H_{i+1} (2.2.3). Also, $L = R[Y]_{(M,Y)}$ is H_{i+1} (2.2.4). Let $N = (M, Y)L$. Then $L(X) = L[X]_{NL[X]}$ is H_{i+1} (2.2.3). Therefore $R(X)$ $[Y]_{(MR(X),Y)}$ is H_{i+1}, hence $R(X)$ is C_i (2.2.4).

Conversely, if $R(X)$ is C_i, then R is H_i and H_{i+1} (2.2.3). Also, if p is a height i prime ideal in R, then all maximal ideals in the integral closure of R/p have the same height, since, by hypothesis, all maximal ideals in the integral closure of $(R/p)(X) \cong R(X)/pR(X)$ have the same height. Therefore R is C_i.

The following corollary can be extended to local rings and to the case where b is a zero-divisor in R, much as in [**12**, (2.11)]. However, we content ourselves with the domain case here. The corresponding result for H_i-local domains is given in [**4**, (4.3)], and this fact will be used in the proof of (4.2).

(4.2) COROLLARY. *Let* (R, M) *be a local domain, and let* $i > 1$. *Then the following statements are equivalent*:

(4.2.1) R *is* C_i.

(4.2.2) *For all* k $(1 \leqq k \leqq i)$ *and for all analytically independent elements* $b, c_1, \cdots, c_k$ *in* R, A_{MA} *is* C_{i-k}, *where* $A = R[c_1/b, \cdots, c_k/b]$.

(4.2.3) *There exists* k $(1 \leqq k < i)$ *such that for all analytically independent elements* $b, c_1, \cdots, c_k$ *in* R, A_{MA} *is* C_{i-k}, *where* $A = R[c_1/b, \cdots, c_k/b]$.

Proof. It is known [**6**, Lemma 4.3] that, if $b, c_1, \cdots, c_k$ are analytically independent in R, then MA is a depth k prime ideal, where $A = R[c_1/b, \cdots, c_k/b]$, so $A_{MA} = R(X_1, \cdots, X_k)/K$, for some height k

prime ideal K in $R(X_1, \cdots, X_k)$. Therefore, if (4.2.1) holds, then, since $R(X_1, \cdots, X_k)$ is C_i, by (4.1), A_{MA} is C_{i-k} (2.2.2), so (4.2.2) holds.

It is clear that (4.2.2) implies (4.2.3).

Finally, if (4.2.3) holds, then, since $i > 1$, it is known that R is H_i and H_{i+1} [**4**, (4.3)] (since, for this fixed k, all A_{MA} are H_{i-k} and H_{i-k+1}). Also, if p is a height i prime ideal in R, then let $b, c_1, \cdots, c_k$ in p such that height $(b, c_1, \cdots, x_k)R = k + 1 \leqq i$. Then, by [**6**, Lemmas 4.3 and 4.2] and with $A = R[c_1/b, \cdots, c_k/b]$, $p^* = pA_{MA}$ is a height $i - k$ prime ideal, depth $p^* =$ depth p, and $A_{MA}/p^* \cong (R/p)(X_1, \cdots, X_k)$. Therefore all maximal ideals in the integral closure of R/p have the same height, since this holds for A_{MA}/p^*, by hypothesis. Therefore R is C_i.

It is clear from the proof of (4.2), that if we let $i = 1$, then (4.2.1) implies (4.2.2) and (4.2.2) implies (4.2.3). Does (4.2.2) imply (4.2.1) if $i = 1$? Does (4.2.3) imply (4.2.1) if $i > 1$ and $k = i$? The author doesn't know the answer to either question. However, if R is as in Nagata's examples [**3**, Example 2, pp. 203–205], and if all A_{MA} are C_0 (in either case), then R is C_i.

We will now extend (4.1) to quasi-local rings integral over a local ring. The corresponding result for H_i-quasi-local rings is given in [**12**, (2.21)].

(4.3) COROLLARY. *Let $(R, M) \subseteq (S, N)$ be quasi-local rings such that R is Noetherian, S is integral over R, and minimal prime ideals in S contract in R to minimal prime ideals. Then the following statements are equivalent*:

(4.3.1) *R is C_i.*

(4.3.2) *S is C_i.*

(4.3.3) *$S(X) = S[X]_{NS[X]}$ is C_i.*

Proof. By [**12**, (3.18)] (4.3.1) $\Leftrightarrow$ (4.3.2). Also, R is C_i if and only if $R(X)$ is C_i (4.1). Further, since $S(X)$ is integral over $R(X)$, $R(X)$ is C_i if and only if $S(X)$ is C_i [**12**, (3.18)]. Therefore (4.3.1) $\Leftrightarrow$ (4.3.3).

We also can extend (4.2) to rings S as in (4.3), as will now be shown. We again content ourselves with the integral domain case. Again the corresponding result for H_i-quasi-local rings is given in [**12**, (2.22)].

(4.4) COROLLARY. *Let $(R, M) \subseteq (S, N)$ be as in* (4.3), *assume that S is an integral domain, and let $i > 1$. Then the following statements are equivalent*:

(4.4.1) *S is C_i.*

(4.4.2) *For all k $(1 \leqq k \leqq i)$ and for all analytically independent elements $b, c_1, \cdots, c_k$ in S, B_{NB} is C_{i-k}, where $B = S[c_1/b, \cdots, c_k/b]$.*

and/or that D_k is H_i, for some $i > 0$. The following remark gives justification for not considering the case $a \leqq 1$ and/or $i = 0$.

(2.11) REMARK. Let (R, M) be a local ring. Then the following statements hold:

(2.11.1) If R is H_0, then, for all $k \geqq 1$, D_k is H_0 (and conversely).

(2.11.2) If altitude $R \leqq 1$, then R satisfies the s.c.c., so for all $k \geqq 1$, D_k is H_i, for all $i \geqq 0$.

Proof. (2.11.1) is clear, since the minimal prime ideals in D_k are the ideals zD_k with z a minimal prime ideal in R, and depth zD_k = depth $z + k$.

(2.11.2) R is clearly C_0, so D_1 is H_1 (2.2.4). Also, D_1 is H_0 (2.11.1) and H_2 (2.2.1). Therefore D_2 is H_1 and H_2 (2.6). Also D_2 is H_0 (2.11.1) and H_3 (2.2.1). Therefore, the conclusion follows by repeating this.

The following two known results are special cases of (2.8).

(2.12) COROLLARY. (cf. [**12**, (3.10)].) *Let (R, M) be a local ring, and let a = altitude $R > 1$. Then D_{a-1} is H_{a-1} if and only if R satisfies the s.c.c.*

Proof. D_{a-1} is H_{a-1} if and only if R is $C_0, \cdots, C_{a-2}$, by (2.8), if and only if R satisfies the s.c.c. [**12**, (3.5.1)].

Since every local ring of altitude $\leqq 2$ satisfies the conclusion of (2.13), we restrict attention to the case $a > 2$ in (2.13).

(2.13) COROLLARY. (cf. [**12**, (3.12)].) *Let (R, N) be a local ring, and let a = altitude $R > 2$. Then D_{a-2} is H_{a-1} if and only if R is $H_1, \cdots, H_a$ and, for all minimal prime ideals z in R, the integral closure of R/z satisfies the c.c.*

Proof. D_{a-2} is H_{a-1} if and only if R is $C_1, \cdots, C_{a-2}$, by (2.8), if and only if R is $H_1, \cdots, H_a$ and, for each minimal prime ideal z in R, the integral closure of R/z satisfies the c.c. [**12**, (3.5.2)].

To prove another corollary to (2.6), we need the following lemma.

(2.14) LEMMA. *Assume that R is a local domain. Then $2 \in c(D_1)$ if and only if there exists a height one maximal ideal in the integral closure R' of R.*

Proof. By (2.4), $2 \in c(D_1)$ if and only if $1 \in c(R)$. Now, by

(4.4.3) *There exists* k $(1 \leqq k < i)$ *such that for all analytically independent elements* $b, c_1, \cdots, c_k$ *in* S, B_{NB} *is* C_{i-k}, *where* $B = S[c_1/b, \cdots, c_k/b]$.

Proof. Assume that (4.4.1) holds and let $b, c_1, \cdots, c_k$ $(1 \leqq k \leqq i)$ be analytically independent elements in S. Then $b, c_1, \cdots, c_k$ are analytically independent in the local domain $L = R[b, c_1, \cdots, c_k]$ [**7**, Remark 4.4 (iii)] and L is C_i [**12**, (3.18)]. Therefore I_{PI} is C_{i-k} (4.2), where $I = L[c_1/b, \cdots, c_k/b]$ and P is the maximal ideal in L. Also, B_{NB} is integral over I_{PI}, where $B = S[c_1/b, \cdots, c_k/b]$, hence B_{NB} is C_{i-k} [**12**, (3.18)].

It is clear that (4.4.2) implies (4.4.3).

Finally, assume that (4.4.3) holds and let $b, c_1, \cdots, c_k$ be analytically independent elements in R. Then $b, c_1, \cdots, c_k$ are analytically independent in S [**7**, Remark 4.4 (iii)], and so, by hypothesis, B_{NB} is C_{i-k}, where $B = S[c_1/b, \cdots, c_k/b]$. Then B_{NB} is integral over A_{MA}, where $A = R[c_1/b, \cdots, c_k/b]$, so A_{MA} is C_{i-k} [**12**, (3.18)]. Therefore, by (4.2), R is C_i, hence S is C_i [**12**, (3.18)].

Again, if $i = 1$ in (4.4), then (4.4.1) implies (4.4.2) and (4.4.2) implies (4.4.3), by the proof of (4.4).

We close this paper with the following remark.

(4.5) Remark. Let $(R, M) \subseteq (S, N)$ be as in (4.3). Then the statements analogous to (2.6)–(2.9) and (3.1)–(3.5) hold for S.

Proof. It is known [**12**, (2.17) and (3.18)] that, with $R \subseteq S$ as in (4.3), R is H_i (respectively, C_i) if and only if S is H_i (respectively, C_i). Using this and the respective statements (2.6)–(2.9) and (3.1)–(3.5), the details needed to prove this remark are, in all but two cases, easily supplied. Only (2.7) and (2.9) could cause any problem, and both can be handled in essentially the same way, which we indicate for (2.7). Namely, if x is an element in the quotient field of S and Q is a maximal ideal in $S[x]$ such that $Q \cap S = N$, then let $x = c/b$ $(b, c \in S)$ and let $f(X) = \Sigma_0^k a_j X^j$ be a monic polynomial such that $f = f(x) \in Q$ and $Q = (N, f)S[x]$. Then $L = R[b, c, a_0, \cdots, a_k]$ is a local domain with maximal ideal $P = N \cap L$, $Q' = Q \cap L[x] = (P, f)L[x]$ is a maximal ideal, and $S[x]_Q$ is integral over $L[x]_{Q'}$ (since every maximal ideal in $S[x]$ which lies over Q' must contain N and f, hence must be Q). From here, the other details are easily supplied, using the cited references.

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