ON ISOMETRIC ISOMORPHISM OF GROUP ALGEBRAS

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1. Introduction. Let G be a locally compact group with right invariant Haar measure m [2, Chapter XI]. The class $L(G)$ of integrable functions on G forms a Banach algebra, with norm and product defined respectively by

$$\|x\| = \int |x(g)| \, m(dg) ,$$

$$(xy)(g) = \int x(gh^{-1}) \, y(h) \, m(dh) .$$

The algebra is called real or complex according as the functions $x(g)$ and the scalar multipliers take real or complex values.

Suppose that τ is an isomorphism (algebraic and homeomorphic) of the group G onto a second locally compact group Γ having right invariant Haar measure μ; let c be the constant value of the ratio $m(E)/\mu(\tau E)$, and let χ be a continuous character on G. If T is the mapping of $L(G)$ onto $L(\Gamma)$ defined by

$$(Tx)(\tau g) = c\,\chi(g)\; x(g), \qquad x \in L(G),$$

then it is easily verified that T is a linear map preserving products and norms; for short, T is an *isometric isomorphism* of $L(G)$ onto $L(\Gamma)$.

It is the purpose of the present note to show that, conversely, *any isometric isomorphism of* $L(G)$ *onto* $L(\Gamma)$ *has the above form,* in both the real and complex cases.

We mention in passing that if T is merely required to be a *topological* isomorphism then G and Γ need not even be algebraically isomorphic. In fact, let G and Γ be any two finite abelian groups each having n elements, of which k are of order 2. Then the complex group algebras of G and Γ are topologically isomorphic to the direct sum of n complex fields, and the real algebras are topologically isomorphic to the direct sum of $k + 1$ real fields and $(n - k - 1)/2$ two-dimensional algebras equivalent to the complex field. The algebraic content of this statement

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follows from a theorem of Perlis and Walker [4], but for the sake of completeness we sketch a direct proof.

Since the character group of G is isomorphic to G there are exactly k characters $\chi_1, \chi_2, \cdots, \chi_k$ on G of order 2. Together with the identity character χ_0 these are all of the characters on G which take only real values. The remaining characters $\chi_{k+1}, \cdots, \chi_{n-1}$ fall into complex-conjugate pairs, $\bar{\chi}_{2m} = \chi_{2m+1}$, $m = (k+1)/2, (k+3)/2, \cdots, (n-2)/2$. For $0 \leq j \leq n-1$ let $x_j \in L(G)$ (complex) be the vector with components $(1/n)\chi_j(g)$. It is readily verified that the x_j are orthogonal idempotents, so that $L(G)$ can be written as the sum of n complex fields, and the same holds for the complex algebra $L(\Gamma)$. In the real case we retain the vectors x_j for $0 \leq j \leq k$, and replace the remaining ones by the (real) vectors $y_m = x_{2m} + x_{2m+1}$, $z_m = ix_{2m} - ix_{2m+1}$, whose law of multiplication is easily seen to be $y_m^2 = y_m$, $z_m^2 = -y_m$, $y_m z_m = z_m y_m = z_m$, while all other products vanish. Since the vectors x_j, y_m, z_m span $L(G)$ we see that $L(G)$ is represented as the sum of $k+1$ real fields and $(n-k-1)/2$ complex fields; the same representation is obtained for the real algebra $L(\Gamma)$; this completes the proof of the algebraic part of the assertion. The fact that these algebras are also homeomorphic follows from the fact that all norms in a finite dimensional Banach space are equivalent.

2. Statement of results. For any fixed $g_0 \in G$ let us denote the translation operator $x(g) \longrightarrow x(g_0^{-1}g)$, $x \in L(G)$, by S_{g_0}; operators Σ_γ are defined similarly for $L(\Gamma)$. In this notation our precise result is:

THEOREM 1. *Let T be an isometric isomorphism of the (real, complex) algebra $L(G)$ onto the (real, complex) algebra $L(\Gamma)$. There is an isomorphism τ of G onto Γ, and a (real, complex) continuous character χ on G such that*

$$TS_g T^{-1} = \chi(g)\Sigma_{\tau g}, \qquad g \in G, \tag{1A}$$

$$(Tx)(\tau g) = c\,\chi(g)\,x(g), \qquad g \in G, \quad x \in L(G), \tag{1B*}$$

where c is the constant value of the ratio $m(E)/\mu(\tau E)$.

For the proof we make use of a theorem due to Kawada [3] concerning *positive*

*I am obliged to Professor C. E. Rickart for suggesting the probable existence of a formula of this kind.

isomorphisms of $L(G)$ onto $L(\Gamma)$ in the real case; a mapping $P: L(G) \longrightarrow L(\Gamma)$ is called positive in case $x(g) \geq 0$ a.e. in G if and only if $(Px)(\gamma) \geq 0$ a.e. in Γ. Kawada's result reads:

THEOREM K. *Let P be a positive isomorphism of $L(G)$ onto $L(\Gamma)$, both algebras real. There is an isomorphism τ of G onto Γ such that $PS_gP^{-1} = k_g\Sigma_{\tau g}$, $g \in G$, where k_g is positive for each g.*

In order to deduce Theorem 1 from Theorem K we need two intermediate results, of which the first is a sharpening of Kawada's theorem, while the second reveals the close connection which holds between isometric and positive isomorphisms.

THEOREM 2. *Let P be a positive isomorphism of real $L(G)$ onto $L(\Gamma)$. Then:*

(2A) *P is an isometry;*

(2B) *$k_g = 1$ for all $g \in G$;*

(2C) *P is given by the formula $(Px)(\tau g) = cx(g)$, where c is the constant value of the ratio $m(E)/\mu(\tau E)$.*

THEOREM 3. *Let T be an isometric isomorphism of $L(G)$ onto $L(\Gamma)$. There is a continuous character $\chi(\gamma)$ on Γ such that if the mapping $P: L(G) \longrightarrow L(\Gamma)$ is defined by $(Px)(\gamma) = \chi(\gamma)(Tx)(\gamma)$, $x \in L(G)$, $\gamma \in \Gamma$, then P is a positive isomorphism of the real subalgebra of $L(G)$ onto the real subalgebra of $L(\Gamma)$. The character χ is real or complex with $L(G)$ and $L(\Gamma)$.*

3. Proof of Theorem 2. P and its inverse are both order-preserving operators, and therefore are bounded [1, p.249]. Consequently the ratio $\|Px\|/\|x\|$ is bounded away from zero and infinity as x varies over $L(G)$, $x \neq 0$. If x is a positive element of $L(G)$ it follows by repeated application of Fubini's theorem that $\|x^n\| = \|x\|^n$; since Px is also positive, and $P(x^n) = (Px)^n$, we have the result that for fixed positive $x \neq 0$ the quantity $\{\|Px\|/\|x\|\}^n$ is bounded above and below for $n = 1, 2, \cdots$. Hence P is isometric at least for the positive elements of $L(G)$. But now for any $x \in L(G)$ we may write $x = x^+ + x^-$, where x^+ and x^- denote respectively the positive and negative parts of x. Then

$$\|x\| = \|x^+ + x^-\| = \|x^+\| + \|x^-\| = \|Px^+\| + \|Px^-\| \geq \|Px^+ + Px^-\| = \|Px\|.$$

Applying the argument to P^{-1} we obtain the result

$$\|x\| = \|P^{-1}Px\| \leq \|Px\| \leq \|x\|,$$

which is the statement (2A).

Theorem (2B) follows at once from this and Theorem K. For if $x \in L(G)$ then $\|S_g x\| = m_g \|x\|$, where m_g is the constant value of the ratio $m(gE)/m(E)$. Similarly, $\|\Sigma_{\tau g} \xi\| = \mu_{\tau g} \|\xi\|$. Since τ is a homeomorphism, $\mu_{\tau g} = m_g$. The constant k_g may now be evaluated by taking norms on both sides of the equation $PS_gP^{-1} = k_g \Sigma_{\tau g}$, and must therefore have the value unity.

To prove part (2C) of the theorem we observe that the operator Q defined by $(Qx)(\tau g) = cx(g)$ satisfies the relation $QS_gQ^{-1} = \Sigma_{\tau g}$, and is an isomorphism of $L(G)$ onto $L(\Gamma)$. Then $QS_gQ^{-1} = PS_gP^{-1}$, $g \in G$, and consequently $R = P^{-1}Q$ is a continuous automorphism of $L(G)$ which commutes with every S_g. We shall show that R must be the identity mapping.

Segal [5, p. 84] has shown that the product xy of two elements x, y belonging to $L(G)$ may be written as a Bochner integral, which in our notation takes the form

$$xy = \int x(h)\, m_h^{-1} \{S_h y\}\, m(dh),$$

where the quantity in braces is a vector-valued function of $h \in G$, and the function m_g was defined above. Applying the operator R we obtain

$$R(xy) = \int x(h)\, m_h^{-1} \{RS_h y\}\, m(dh) = \int x(h)\, m_h^{-1} \{S_h R y\}\, m(dh) = x R y .$$

But R is an automorphism, and so also $R(xy) = (Rx)(Ry)$. Thus $x = Rx$, all $x \in L(G)$, which shows that $P = Q$, as was to be proved.

4. Proof of Theorem 3. We first require several lemmas, all of which share the hypothesis: T is an isometric isomorphism of $L(G)$ onto $L(\Gamma)$, indifferently real or complex. For x, $y \in L(G)$ we write ξ for Tx, η for Ty. We denote by $E(x)$ the set $\{g \mid g \in G, x(g) \neq 0\}$, which is regarded as being determined only up to a null-set; $E(\xi)$ in Γ is defined in the same fashion. (Although we make no use of this fact, the first three lemmas below actually hold in case T is an isometry between two arbitrary L-spaces.)

LEMMA 1. *If $E(x) \cap E(y) = \Lambda$ then $E(\xi) \cap E(\eta) = \Lambda$, and conversely.*

Proof. The hypotheses imply that for all scalars A we have $\|x + Ay\| = \|x\| + |A|\,\|y\|$. Then for all A we have $\|\xi + A\eta\| = \|\xi\| + |A|\,\|\eta\|$, which implies that $E(\xi)$ and $E(\eta)$ are disjoint. For the converse we need only replace T by T^{-1}.

LEMMA 2. *If $E(x) \subseteq E(y)$ then $E(\xi) \subseteq E(\eta)$, and conversely.*

Proof. Suppose that $E(x) \subseteq E(y)$, but that $E(\xi) \nsubseteq E(\eta)$. Then we may write $\xi = \xi_1 + \xi_2$, with $E(\xi_1) \subseteq E(\eta)$, $E(\xi_2) \cap E(\eta) = \Lambda = E(\xi_1) \cap E(\xi_2)$. Let $T^{-1}\xi_i = x_i$; then from Lemma 1 it follows that $E(x_1) \cap E(x_2) = \Lambda = E(x_2) \cap E(y)$. But $E(x_1) \cup E(x_2) = E(x) \subseteq E(y)$; this contradiction yields the result.

LEMMA 3. *Let B in Γ be a σ-finite measurable set (that is, the sum of a countable number of sets of finite measure). Then there is a positive $x \in L(G)$ such that $E(\xi) = B$.*

Proof. Let $\eta \in L(\Gamma)$ be chosen so that $E(\eta) = B$. Let $y = T^{-1}\eta$, and set $x(g) = |y(g)|$, $g \in G$. Then $x \in L(G)$, $E(x) = E(y)$, and therefore from Lemma 2 it follows that $E(\xi) = B$.

LEMMA 4. *Let x and y be positive elements of $L(G)$. For $\gamma \in E(\xi)$ let $K_\xi(\gamma) = \xi(\gamma)/|\xi(\gamma)|$, and define $K_\eta(\gamma)$ in similar fashion. Then $K_\xi(\gamma) = K_\eta(\gamma)$ almost everywhere on $E(\xi) \cap E(\eta)$.*

Proof. Since x and y were taken to be positive we have $\|x + y\| = \|x\| + \|y\|$. Therefore $\|\xi + \eta\| = \|\xi\| + \|\eta\|$. Then $|\xi(\gamma) + \eta(\gamma)| = |\xi(\gamma)| + |\eta(\gamma)|$ a.e. in Γ. Hence, since the functions K have modulus 1,

$$\left|K_\xi(\gamma) K_\eta(\gamma)^{-1} |\xi(\gamma)| + |\eta(\gamma)|\right| = |\xi(\gamma)| + |\eta(\gamma)|$$

a.e. in $E(\xi) \cap E(\eta)$. But then $K_\xi(\gamma) K_\eta(\gamma)^{-1} = 1$ a.e. on $E(\xi) \cap E(\eta)$, as was to be proved.

LEMMA 5. *There is a unique continuous character χ on Γ with the property that for all positive $x \in L(G)$ we have $\xi(\gamma) = \chi(\gamma)|\xi(\gamma)|$ a.e.; χ is real or complex with $L(G)$ and $L(\Gamma)$.*

Proof. Let Γ_0 be the open-closed invariant subgroup of Γ generated by a compact neighborhood of the identity. Since Γ_0 is σ-finite we may apply Lemma 3 to obtain a positive $x \in L(G)$ such that $E(\xi) = \Gamma_0$. Now $x \geq 0$ implies that $\|x^2\| = \|x\|^2$; then also $\|\xi^2\| = \|\xi\|^2$. The element ξ^2 is given by the formula

$$\xi^2(\gamma) = \int_\Gamma \xi(\gamma\delta^{-1})\, \xi(\delta)\, \mu(d\delta) = \int_{\Gamma_0} \xi(\gamma\delta^{-1})\, \xi(\delta)\, \mu(d\delta).$$

Since x^2 is also positive we have from Lemma 4 that $K_{\xi^2}(\gamma) = K_\xi(\gamma)$ a.e. on $E(\xi^2) \cap E(\xi) \subseteq \Gamma_0 = E(\xi)$. Writing simply $K(\gamma)$ for the common value, we see that the relation $\xi^2(\gamma) = K(\gamma)|\xi^2(\gamma)|$ therefore holds in Γ_0 even outside of $E(\xi^2)$. Then

$$|\xi^2(\gamma)| = K(\gamma)^{-1} \int_{\Gamma_0} \xi(\gamma\delta^{-1})\,\xi(\delta)\,\mu(d\delta)$$

$$= \int_{\Gamma_0} K(\gamma)^{-1} K(\gamma\delta^{-1})\,K(\delta)\,|\xi(\gamma\delta^{-1})\,\xi(\delta)|\,\mu(d\delta).$$

Integrating over Γ_0 again we obtain

$$\|\xi^2\| = \int \mu(d\gamma) \int K(\gamma)^{-1} K(\gamma\delta^{-1})\,K(\delta)\,|\xi(\gamma\delta^{-1})\,\xi(\delta)|\,\mu(d\delta)$$

$$= \|\xi\|^2 = \int \mu(d\gamma) \int |\xi(\gamma\delta^{-1})\,\xi(\delta)|\,\mu(d\delta).$$

Therefore $K(\gamma)^{-1}K(\gamma\delta^{-1})K(\delta) = 1$ a.e. on $\Gamma_0 \times \Gamma_0$. Then there is a null-set $N \subset \Gamma_0$ such that $\gamma \notin N$ implies $K(\gamma\delta^{-1})\,K(\delta) = K(\gamma)$ for almost all $\delta \in \Gamma_0$. We integrate this equation over a set M of finite positive measure and obtain

$$K(\gamma)\,\mu(M) = \int_{\Gamma_0} K(\gamma\delta^{-1})\,K(\delta)\,\phi_M(\delta)\,\mu(d\delta)$$

$$= \int_{\Gamma_0} K(\delta^{-1})\,K(\delta\gamma)\,\phi_M(\delta\gamma)\,\mu(d\delta),$$

where ϕ_M is the characteristic function of M. The right member is easily seen to be a continuous function of γ, for all $\gamma \in \Gamma_0$; hence $K(\gamma)$ is equal a.e. to a continuous function $\chi_0(\gamma)$, which is clearly a character on Γ_0. From Lemma 4 it follows also that, for positive $x \in L(G)$, if $E(\xi) \subseteq \Gamma_0$ then $\xi(\gamma) = \chi_0(\gamma)\,|\xi(\gamma)|$ a.e.

The proof is completed by extending the function χ_0 to all of Γ. To do this we write Γ as the union of disjoint cosets $\gamma_\alpha\Gamma_0$, and consider the open-closed subgroup Γ_1 generated by any finite number of cosets. Then Γ_1 is again σ-finite, and we may repeat the above argument to obtain a continuous character χ_1 on Γ_1. Lemma 4 guarantees that for two such subgroups Γ_1 and Γ_1' the characters χ_1 and χ_1' will agree on $\Gamma_1 \cap \Gamma_1' \supseteq \Gamma_0$, so that χ_1 is indeed an extension of χ_0. Clearly, if $x \geq 0$ and $E(\xi) \subseteq \Gamma_1$ then $\xi(\gamma) = \chi_1(\gamma)\,|\xi(\gamma)|$.

Finally, χ on all of Γ is defined by $\chi(\gamma) = \chi_1(\gamma)$ for $\gamma \in \Gamma_1$. Since the union of all such subgroups Γ_1 is precisely Γ, and since as shown above the subgroup characters are mutually consistent, the function χ is well-defined. It is clearly a continuous character. The remaining property, that $x \geq 0$ implies $\xi(\gamma) = \chi(\gamma)|\xi(\gamma)|$, can be proved as follows. The set $E(\xi)$ intersects at most a countable number of cosets $\gamma_n\Gamma_0$ in sets of positive measure. Let ξ_n be the restriction to $\gamma_n\Gamma_0$ of ξ, and put $x_n = T^{-1}\xi_n$. Then $x = \Sigma_{n=1}^{\infty} x_n$, and by Lemma 1 the sets $E(x_n)$ are pairwise disjoint, so that the x_n are themselves positive elements. From this it follows that $\xi_n(\gamma) = \chi_n(\gamma)|\xi_n(\gamma)| = \chi(\gamma)|\xi_n(\gamma)|$ for $\gamma \in \gamma_n\Gamma_0$; hence the result holds.

The proof of Theorem 3 is now immediate. For the continuous character χ on Γ constructed in Lemma 5 the mapping P on $L(G)$ to $L(\Gamma)$ defined by $(Px)(\gamma) = \chi(\gamma)^{-1}(Tx)(\gamma)$ carries positive elements of $L(G)$ into positive elements of $L(\Gamma)$; P is clearly an algebraic isomorphism of $L(G)$ onto $L(\Gamma)$. We have only to show that Px positive implies x positive. Suppose then that $Px = \xi$ is positive, but that $x = x_1 - x_2 + i(x_3 - x_4)$, with $x_j \geq 0$ and $E(x_1) \cap E(x_2) = E(x_3) \cap E(x_4) = \Lambda$, and correspondingly $\xi = \xi_1 - \xi_2 + i(\xi_3 - \xi_4)$. P is evidently an isometry, and therefore by Lemma 1 the sets $E(\xi_1) \cap E(\xi_2)$ and $E(\xi_3) \cap E(\xi_4)$ are null-sets. Therefore $\xi_2 = \xi_3 = \xi_4 = 0$; so $x = x_1$, and x is positive.

5. Proof of Theorem 1. Because of Theorem 3 we may apply Theorems K and (2B) to the real sub-algebras of $L(G)$, $L(\Gamma)$, to conclude that there is an isomorphism τ of G onto Γ such that $PS_gP^{-1} = \Sigma_{\tau g}$. Since τ is a homeomorphism we may regard the function χ as a continuous character on G, by defining $\chi(g) = \chi(\tau g)$. By Theorem (2C), P is given on the real subalgebras by the formula $(Px)(\tau g) = cx(g)$, and, because of the linearity, this formula must hold throughout all of $L(G)$. Therefore $(Tx)(\tau g) = c\chi(g)\,x(g)$, which proves (1B). Theorem (1A) is an easy consequence of this formula.

We note finally that Theorem (2A) shows that Kawada's theorem follows from Theorem 1.

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