A THEOREM ON THE REPRESENTATION THEORY OF JORDAN ALGEBRAS

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1. Introduction. Let J be a Jordan algebra over a field Φ of characteristic neither 2 nor 3. Let $a \longrightarrow S_a$ be a (general) representation of J. If α is an algebraic element of J, then S_α is an algebraic element. The object of this paper is to determine the polynomial identity* satisfied by S_α. The polynomial obtained depends only on the minimal polynomial of α and the characteristic of Φ. It is the minimal polynomial of S_α if the associative algebra U generated by the S_a is the universal associative algebra of J and if J is generated by α.

2. Preliminaries. A (nonassociative) commutative algebra J over a field Φ is called a *Jordan algebra* if

$$(a^2 b)a = a^2(ba) \tag{1}$$

holds for all $a, b \in J$. In this paper it will be assumed that the characteristic of Φ is neither 2 nor 3.

It is well known that the Jordan algebra J is *power associative*;** that is, the subalgebra generated by any single element a is associative. An immediate consequence is that if $f(x)$ is a polynomial with no constant term then $f(a)$ is uniquely defined.

Let R_a be the multiplicative mapping in J, $a \longrightarrow xa = ax$, determined by the element a. From (1) it can be shown that we have

$$[R_a R_{bc}] + [R_b R_{ac}] + [R_c R_{ab}] = 0$$

and

$$R_a R_b R_c + R_c R_b R_a + R_{(ac)b} = R_a R_{bc} + R_b R_{ac} + R_c R_{ab}$$

for all $a, b, c \in J$, where $[AB]$ denotes $AB - BA$. Since the characteristic of Φ is not 3, either of these relations and the commutative law imply (1). Let

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* This problem was proposed by N. Jacobson.

** See, for example, Albert [1].

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$a \longrightarrow S_a$ be a linear mapping of J into an associative algebra U such that for all $a, b, c \in J$ we have

(2) $$[S_a S_{bc}] + [S_b S_{ac}] + [S_c S_{ab}] = 0$$

and

(3) $$S_a S_b S_c + S_c S_b S_a + S_{(ac)b} = S_a S_{bc} + S_b S_{ac} + S_c S_{ab} .$$

Such a mapping is called a *representation.*

It has been shown* that there exists a representation $a \longrightarrow S_a$ of J into an associative algebra U such that (a) U is generated by the elements S_a and (b) if $a \longrightarrow T_a$ is an arbitrary representation of J then $S_a \longrightarrow T_a$ defines a homomorphism of U. In this case the algebra U is called the *universal associative algebra* of J.

We shall now suppose that $a \longrightarrow S_a$ is an arbitrary representation of J, and α a fixed element of J. Let $s(r) = S_{\alpha^r}$, $A = s(1)$, $B = s(2)$. If we put $a = b = c = \alpha$ in (2), we get $AB = BA$. If we put $a = b = \alpha$, $c = \alpha^{r-2}$, $r \geq 3$, then (3) becomes

(4) $$s(r) = 2As(r-1) + s(r-2)B - A^2 s(r-2) - s(r-2)A^2 .$$

We now see that A and B generate a commutative subalgebra U_α containing $s(r)$ for all r. By the commutativity of U_α, (4) becomes

(5) $$s(r) = 2As(r-1) + (B - 2A^2)\, s(r-2) .$$

We now adjoin to the commutative associative algebra U_α an element C commuting with the elements of U_α such that $C^2 = B - A^2$. We have the following result.

LEMMA 1. *For all positive integers r, we have*

$$s(r) = (1/2)(A + C)^r + (1/2)(A - C)^r .$$

Proof. If $r = 1$, then

$$(1/2)(A + C)^r + (1/2)(A - C)^r = A = s(1) .$$

* For a general discussion of the theory of representations of a Jordan algebra and a proof of the existence of the universal associative algebra, see Jacobson [2].

If $r = 2$, then

$$(1/2)(A + C)^r + (1/2)(A - C)^r = A^2 + C^2 = s(2)\,.$$

Now suppose that $r \geq 3$ and that Lemma 1 holds for $r - 1$ and $r - 2$. By direct substitution it follows that $A + C$ and $A - C$ are roots of

$$x^2 = 2Ax + B - 2A^2\,,$$

and therefore of

$$x^r = 2Ax^{r-1} + (B - 2A^2)\, x^{r-2}\,.$$

Hence,

$$(A + C)^r = 2A(A + C)^{r-1} + (B - 2A^2)(A + C)^{r-2}$$

and

$$(A - C)^r = 2A(A - C)^{r-1} + (B - 2A^2)(A - C)^{r-2}\,.$$

Adding and dividing by 2, we have the desired result:

$$(1/2)(A + C)^r + (1/2)(A - C)^r = 2As(r - 1) + (B - 2A^2)\, s(r - 2) = s(r)\,.$$

An immediate consequence of Lemma 1 is that if $g(x)$ is an arbitrary polynomial with no constant term, then

$$S_{g(\alpha)} = (1/2)\, g(A + C) + (1/2)\, g(A - C)\,. \tag{6}$$

Now suppose further that α is an algebraic element of J and that $f(x)$ is a polynomial with no constant term, such that $f(\alpha) = 0$. Then by (6) we have

$$\begin{aligned} 0 &= 2S_{f(\alpha)} = f(A + C) + f(A - C)\,, \\ 0 &= 2S_{\alpha f(\alpha)} = (A + C)\, f(A + C) + (A - C)\, f(A - C)\,. \end{aligned} \tag{7}$$

The next step is to eliminate C from the system (7). To do this we need some additional tools.

3. Theory of elimination. Let Ω be the splitting field of $f(x)$ over the field Φ. Let $P = \Phi[x]$, $Q = P[y]$, $P' = \Omega[x]$, $Q' = P'[y]$ be polynomial rings in one and two variables over Φ and Ω, respectively. Then P and P' are principal ideal rings. If q_1 and q_2 are elements of Q, let (q_1, q_2) be the ideal of Q generated by q_1 and q_2, and let $\{q_1, q_2\}$ be a generator of the P-ideal $(q_1, q_2) \cap P$. Similarly, if q_1 and q_2 are elements of Q', let $((q_1, q_2))$ be the ideal of Q' generated by q_1 and q_2. Furthermore, let $\{\{q_1, q_2\}\}$ denote a generator of the P'-ideal $((q_1, q_2)) \cap P'$. We note that $\{q_1, q_2\}$ and $\{\{q_1, q_2\}\}$ are determined up to unit factors. The unit factors are nonzero elements of Φ and Ω respectively.

We shall establish the following lemma.

LEMMA 2. *If q_1 and q_2 are elements of Q, then $\{q_1, q_2\} = \{\{q_1, q_2\}\}$ up to a unit factor.*

Proof. Let $\omega_1, \omega_2, \cdots, \omega_m$ be a basis of Ω over Φ. Then $P' = \Sigma\omega_i P$ and $Q' = \Sigma\omega_i Q$. Therefore

$$((q_1, q_2)) = Q'q_1 + Q'q_2 = \Sigma\omega_i Qq_1 + \Sigma\omega_i Qq_2 = \Sigma\omega_i (q_1, q_2)$$

and

$$((q_1, q_2)) \cap P' = \Sigma\omega_i ((q_1, q_2) \cap P) = ((q_1, q_2) \cap P) P' = \{q_1, q_2\} P' .$$

It follows that $\{q_1, q_2\} = \{\{q_1, q_2\}\}$.

Let r and s be distinct elements of P', and let m and n be positive integers. We shall determine $\{\{(y-r)^m, (y-s)^n\}\}$.

LEMMA 3. *Let $S(m, n)$ be that positive integer satisfying*

$$S(m,n) \leq m + n - 1 ,$$

$$\binom{S(m,n) - 1}{n - 1} \neq 0 ,$$

and

$$\binom{N}{n-1} = 0 \qquad \text{if } S(m,n) \leq N \leq m + n - 2 ,$$

where $\binom{N}{M}$ is the binomial coefficient considered as an integer in Φ. Then we have

$$\{\{(y - r)^m , \quad (y - s)^n\}\} = (s - r)^{S(m,n)} .$$

Proof. We note that $S(m, n)$ depends only on m, n, and the characteristic p of Φ. If $p = 0$, or if $p \geq m + n - 1$, then $S(m, n) = m + n - 1$. In any case,

$$m + n - 1 \geq S(m,n) \geq n . \tag{8}$$

Replacing y by $y + r$, we may assume that $r = 0$, $s \neq 0$. Formally, modulo y^m, we have

$$(s - y)^{-n} = s^{-n}(1 - y/s)^{-n} \equiv s^{-n} \sum_{\mu=0}^{m-1} \binom{-n}{\mu} (-y/s)^{\mu}$$

$$= \sum_{\mu=0}^{m-1} s^{-n-\mu} \binom{n+\mu-1}{\mu} y^{\mu} = \sum_{\mu=0}^{m-1} s^{-n-\mu} \binom{n+\mu-1}{n-1} y^{\mu}$$

$$= \sum_{\nu=n}^{S(m,n)} s^{-\nu} \binom{\nu-1}{n-1} y^{\nu-n} .$$

Therefore there exists a $q \in Q'$ such that

$$qy^m + (y-s)^n(-1)^n \sum_{\nu=n}^{S(m,n)} s^{S(m,n)-\nu} \binom{\nu-1}{n-1} y^{\nu-n} = s^{S(m,n)} . \tag{9}$$

It follows that

$$\{\{y^m, (y - s)^n\}\} \mid s^{S(m,n)} .$$

Put

$$\{\{y^m, (y - s)^n\}\} = G , \qquad s^{S(m,n)}/G = H .$$

Then G and H are elements of P'. Furthermore, there exist q_1 and q_2 in Q' such that the y-degree of q_2 is less than m and such that $q_1 y^m + q_2(y-s)^n = G$. Hence

$$q_1 H y^m + q_2 H (y - s)^n = GH = s^{S(m,n)} . \tag{10}$$

Subtracting (9) from (10) and comparing terms not divisible by y^m, we obtain

$$q_2 H = (-1)^n \sum_{\nu=n}^{S(m,n)} s^{S(m,n)-\nu} \binom{\nu-1}{n-1} y^{\nu-n} . \tag{11}$$

Comparing coefficients of $y^{S(m,n)-n}$ in (11), we get

$$H \,\Big|\, \binom{S(m,n)-1}{n-1} ,$$

which is a nonzero element of Φ. Therefore H is a unit element, and this establishes Lemma 3.

In the following we shall use l.c.m. $(a_1, a_2, \cdots, a_n)$ for the least common multiple of $a_1, a_2, \cdots, a_n$.

LEMMA 4. *If* $((q_1, q_2)) \supseteq P'$, *then*

$$\{\{q_1q_2, q_3\}\} = \text{l.c.m.}(\{\{q_1, q_3\}\}, \{\{q_2, q_3\}\}).$$

Proof. Put $p_1 = \{\{q_1, q_3\}\}$, $p_2 = \{\{q_2, q_3\}\}$, and $p_3 = \text{l.c.m.}(p_1, p_2)$. We note that $((q_1, q_3)) \cap P' \supseteq ((q_1q_2, q_3)) \cap P'$, and therefore $p_1 | \{\{q_1q_2, q_3\}\}$. Similarly, $p_2 | \{\{q_1q_2, q_3\}\}$, and hence $p_3 | \{\{q_1q_2, q_3\}\}$. Now there exist D, E, F, G, H, I in Q' such that

$$Dq_1 + Eq_3 = p_1, \quad Fq_2 + Gq_3 = p_2, \quad Hq_1 + Iq_2 = 1.$$

Therefore

$$Dq_1q_2 + Eq_2q_3 = p_1q_2 \qquad \text{and} \qquad Fq_1q_2 + Gq_1q_3 = p_2q_1.$$

Hence there exist K, L, M, N in Q' such that

$$Kq_1q_2 + Lq_3 = p_3q_2 \qquad \text{and} \qquad Mq_1q_2 + Nq_3 = p_3q_1.$$

Hence

$$(HM + IK)q_1q_2 + (HN + IL)q_3 = p_3.$$

Therefore $\{\{q_1q_2, q_3\}\} | p_3$, and the proof of Lemma 4 is complete.

We shall now determine $\{D, E\}$, where

$$D = f(x+y) + f(x-y),$$

$$E = (x+y)\, f(x+y) + (x-y)\, f(x-y).$$

By Lemma 2, we have $\{D, E\} = \{\{D, E\}\}$. Since

$$E - (x-y)D = 2yf(x+y),$$

we have

$$\{\{D, E\}\} = \{\{D, yf(x+y)\}\}.$$

Put

$$\{\{f(x+y),\ f(x-y)\}\} = \Delta .$$

Let n be the degree of $f(x)$. Choose $F(y)$ and $G(y)$ in Q', with y-degree less than n, such that

$$F(y)\ f(x+y) + G(y)\ f(x-y) = \Delta .$$

Then $F(y)$ and $G(y)$ are completely determined. Now

$$F(-y)\ f(x-y) + G(-y)\ f(x+y) = \Delta .$$

Therefore we have $F(-y) = G(y)$, from which it follows that $F(0) = G(0)$, or $y \mid [F(y) - G(y)]$. Now

$$(F(y) - G(y))\ f(x+y) + G(y)\ D = \Delta .$$

Therefore $\{\{D, yf(x+y)\}\} \mid \Delta$. It is clear that $\Delta \mid \{\{D, yf(x+y)\}\}$. Thus we have

$$\{D, E\} = \{\{D, yf(x+y)\}\} = \Delta .$$

We must now determine

$$\Delta = \{\{f(x+y),\ f(x-y)\}\} .$$

Let $f(x) = \Pi (x - \alpha_i)^{n_i}$, where the α_i are distinct elements of Ω. Then

$$f(x+y) = \prod (x + y - \alpha_i)^{n_i}, \qquad f(x-y) = \prod (x - y - \alpha_j)^{n_j} .$$

If q_1 and q_2 are two relatively prime factors of $f(x+y)$, or of $f(x-y)$, then $((q_1, q_2)) \supseteq P'$. Therefore we can apply Lemmas 3 and 4 to obtain

$$\{D,E\} = \{\{f(x+y),\ f(x-y)\}\} = \underset{i,j}{\mathrm{l.c.m.}}\,(2x - \alpha_i - \alpha_j)^{S(n_i, n_j)} . \tag{12}$$

4. The equation for S_α. We shall establish the following result.

THEOREM. *Let α be an algebraic element of J satisfying the equation $f(\alpha) = 0$, where $f(x)$ is a polynomial with no constant term. Let*

$$f(x) = \prod (x - \alpha_i)^{n_i} ,$$

where the α_i *are distinct elements of the splitting field* Ω *of* $f(x)$. *Put*

$$\psi(x) = \underset{i,\,j}{\text{l.c.m.}}\,(x - (1/2)\alpha_i - (1/2)\alpha_j)^{S(n_i,\, n_j)} .$$

Then $\psi(S_\alpha) = 0$. *Furthermore, if the algebra* U *generated by the* S_a, $a \in J$, *is the universal associative algebra of* J, *if* $f(x)$ *is the minimal polynomial of* α, *and if* J *is generated by* α, *then* $\psi(x)$ *is the minimal polynomial satisfied by* S_α.

Proof. As before, we let $P = \Phi[x]$, $Q = P[y]$ be polynomial rings over Φ in one and two variables respectively, and put

$$D = f(x + y) + f(x - y)$$

and

$$E = (x + y)\ f(x + y) + (x - y)\ f(x - y) .$$

From (7) and (12) it follows that $\psi(S_\alpha) = 0$. We must now show that $\psi(x)$ is the minimal polynomial of S_α under the three given conditions. If we let $(f(x))$ be the principal ideal of P generated by $f(x)$, then J is isomorphic to the quotient ring $P/(f(x))$ under the natural mapping $g(\alpha) \longrightarrow g(x) + (f(x))$. Let V be the quotient ring $Q/(D, E)$. We now consider the linear mapping

$$g(x) \longrightarrow T_{g(x)} = (1/2)g(x + y) + (1/2)g(x - y) + (D,E) \tag{13}$$

of P into V. By the commutativity of V we have, for all g, h, $j \in P$,

$$[T_g T_{hj}] + [T_h T_{gj}] + [T_j T_{gh}] = 0 , \tag{14}$$

since each of the three terms vanishes. Furthermore, by direct substitution we have

$$2\,T_g T_h T_j + T_{ghj} = T_g T_{hj} + T_h T_{gj} + T_j T_{gh} . \tag{15}$$

We now determine the kernel K of the mapping (13). By definition, $g(x) \in K$ if and only if $g(x + y) + g(x - y) \in (D, E)$. Now

$$yf(x + y) = (1/2)E - (1/2)(x - y)\ D \in (D,E)$$

and

$$yf(x - y) = (1/2)(x + y)D - (1/2)\ E \in (D, E) .$$

Let $q(x)$ be an arbitrary element of P. Then, for suitable $h(x, y) \in Q$, we have

$$q(x+y)\ f(x+y) + q(x-y)\ f(x-y) = q(x)D + h(x,y)\ yf(x+y) - h(x,-y)\ yf(x-y) \in (D,E)\,.$$

Therefore $q(x)f(x) \in K$ for all $q(x)$, and thus $K \supseteq (f(x))$. Suppose $g(x) \in K$, $g(x) \notin (f(x))$. We may suppose that the degree of $g(x)$ is less than n, the degree of $f(x)$. Then $g(x+y) + g(x-y) = h_1D + h_2E$ for suitable h_1 and h_2 in Q. Since the degree of D is n and that of E is $n+1$, it follows that $h_1 = h_2 = 0$. Therefore $g(x+y) + g(x-y)$ is identically 0. This implies that $g(x)$ is identically zero, a contradiction; hence we have $K = (f(x))$. It follows that

$$g(\alpha) \longrightarrow T_{g(x)} = (1/2)g(x+y) + (1/2)g(x-y) + (D,E)$$

defines a single-valued linear mapping of J into V. Furthermore, (14) and (15) imply that this mapping is a representation, and from (12) it follows that T_x, the image of α, has $\psi(x) = \{D,E\}$ as its minimal polynomial. Now since U is the universal associative algebra of J, the mapping $S_{g(\alpha)} \longrightarrow T_{g(x)}$ defines a homomorphism* of U into V. It follows that $\psi(x)$ is the minimal polynomial of S_α. This completes the proof.

We conclude by mentioning two simple consequences of the main theorem. If $f(x) = x^n$, then $\psi(x) = x^{S(n,n)}$. Now (8) yields $S(n,n) \leq 2n-1$, and we have the following result.

COROLLARY 1. *If* $\alpha^n = 0$, *then* $S_\alpha^{2n-1} = 0$.
Similarly, we obtain the following result.

COROLLARY 2. *Let* $f(\alpha) = 0$, *where*

$$f(x) = \prod_{\mu=1}^{n} (x - \beta_\mu)\,.$$

Then $\Lambda(S_\alpha) = 0$, *where*

$$\Lambda(x) = \prod_{\mu \geq \nu} (x - (1/2)\beta_\mu - (1/2)\beta_\nu)\,.$$

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*In fact it can easily be shown that this mapping is an isomorphism of U onto V.

Proof. Suppose

$$f(x) = \prod (x - \alpha_i)^{n_i} ,$$

where the α_i are distinct. Now by (8),

$$S(n_i, n_j) \le n_i + n_j - 1 \le n_i n_j ,$$

and

$$\Lambda(x) = \prod_i (x - \alpha_i)^{n_i(n_i+1)/2} \prod_{j>i} (x - (1/2)\alpha_i - (1/2)\alpha_j)^{n_i n_j} .$$

Therefore $\psi(x) \mid \Lambda(x)$, and the second corollary follows.

References

1. A. A. Albert, *A structure theory for Jordan algebras*, Ann. of Math. 48 (1947), 546-567.

2. N. Jacobson, *General representation theory of Jordan algebras*, Trans. Amer. Math. Soc., scheduled to appear in vol. 70 (1951).

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