ON THE BARYCENTRIC HOMOMORPHISM IN A SINGULAR COMPLEX

PAUL V. REICHELDERFER

INTRODUCTION

0.1. Radó has introduced and studied the following approach to singular homology theory (see [2;3;4] for details). With a general topological space X associate a complex $R = R(X)$ in the following manner. For integers $p \geq 0$, let $v_0, \cdots, v_p$ be a sequence of $p + 1$ points in Hilbert space E_∞, which are not required to be distinct or linearly independent, and let $|v_0, \cdots, v_p|$ denote their convex hull. Suppose that T is a continuous mapping from $|v_0, \cdots, v_p|$ into X. Then the sequence $v_0, \cdots, v_p$ jointly with T determines a p-cell in R, which is denoted by $(v_0, \cdots, v_p, T)^R$. The free Abelian group C_p^R generated by the p-cells in R is termed the group of integral p-chains in R. For integers $p < 0$, C_p^R is defined to be the group consisting of the zero element alone. The boundary operator $\partial_p^R : C_p^R \to C_{p-1}^R$ is defined, in the usual manner, as the trivial homomorphism if $p \leq 0$, and by the relation

$$\partial_p^R (v_0, \cdots, v_p, T)^R = \sum_{i=0}^{p} (-1)^p (v_0, \cdots, \hat{v}_i, \cdots, v_p, T)^R$$

if $p > 0$. Since $\partial_{p-1}^R \partial_p^R = 0$, one introduces the subgroup Z_p^R of p-cycles in C_p^R and the subgroup B_p^R of p-boundaries in C_p^R in the customary way, and defines the quotient group of Z_p^R with respect to B_p^R to be the homology group H_p^R.

0.2. The approach to singular homology theory pursued by Radó differs from other approaches in that absolutely no identifications are made. Thus two p-cells $(v_0', \cdots, v_p', T')^R$ and $(v_0'', \cdots, v_p'', T'')^R$ are equal only if they are identical; that is, if $v_i' = v_i''$ for $i = 0, \cdots, p$ and $T' \equiv T''$ on $|v_0', \cdots, v_p'|$ $= |v_0'', \cdots, v_p''|$. In [3;4], Radó introduces a technique for making identifications in a general Mayer complex and applies his procedure to study identifications in R, particularly those which yield homology groups isomorphic to the H_p^R. It is a primary purpose of the present paper to pursue the matter further in

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order to establish stronger results than those obtained by Radó.

The identification scheme of Radó for the complex R is briefly described in §0.3 below; the reader should consult [3, §1] or [4, §5] for details.

0.3. Let $\{G_p\}$ be a collection of subgroups G_p of the group C_p^R of integral p-chains in R such that $\partial_p^R G_p \subset G_{p-1}$ for every integer p; such a system is termed an *identifier* for R. Let C_p^m be the quotient group of C_p^R with respect to G_p, and denote that element of C_p^m to which a chain c_p^R in C_p^R belongs by $\{c_p^R\}$. The restriction on the groups G_p clearly implies that the element $\{\partial_p^R c_p^R\}$ in C_{p-1}^m is independent of the choice of the representative c_p^R of the element $\{c_p^R\}$ in C_p^m; thus one may define homomorphisms $\partial_p^m: C_p^m \longrightarrow C_{p-1}^m$ by the formula $\partial_p^m \{c_p^R\} = \{\partial_p^R c_p^R\}$. The resulting system of groups C_p^m together with the operator ∂_p^m constitutes a Mayer complex m with homology groups H_p^m. Define a natural homomorphism $\pi_p: C_p^R \longrightarrow C_p^m$ by the formula $\pi_p c_p^R = \{c_p^R\}$. It is readily verified that π_p is a chain mapping; hence it induces homomorphisms $\pi_{*p}: H_p^R \longrightarrow H_p^m$. If for every integer p these homomorphisms are isomorphisms onto, then the identifier $\{G_p\}$ is termed unessential for R. Radó notes that a necessary and sufficient condition in order that an identifier G_p be unessential for R is that every cycle z_p^R in G_p should be the boundary of some chain c_{p+1}^R in G_{p+1}. (See [3, §§1.3, 1.4, 1.5] or [4, §5].)

0.4. One of the principal results in this paper may now be described. Let $\beta_p^R: C_p^R \longrightarrow C_p^R$ be the barycentric homomorphism in R (see [3, §3.1] or [4, §6]; also §1.3), and denote by $N(\beta_p^R)$ the nucleus of this homomorphism for every integer p.

THEOREM. *The system of nuclei $N(\beta_p^R)$ of the barycentric homomorphisms in in R constitutes an unessential identifier for R* (see §3.2).

This result is combined with those of Radó in [3] to obtain stronger theorems concerning identifiers than any previously obtained. Since further definitions are necessary before these results can be described, the reader is requested to consult §3 for their statements.

0.5. In the process of proving the theorem above, various results of independent interest have been attained. The reader is referred especially to §§1.6, 1.7, 1.10, 2.2 for theorems which show the structural description of the barycentric homomorphism and of the barycentric homotopy operator.

I. Further Relations in the Auxiliary Complex K

1.1. As in Radó [3;4], the auxiliary complex K is the "formal complex", in the sense of [1], for the set E_∞ of points in Hilbert space. For integers $p \geq 0$, p-cells in K are ordered sequences $(v_0, \cdots, v_p)$ of $p+1$ points in E_∞, which are not required to be distinct or linearly independent. These p-cells are taken as the base for a free Abelian group C_p, which is termed the group of finite integral p-chains in K. For $p < 0$, the group C_p is defined to be the group composed of the zero element alone. (See [3, §2.1] or [4, §6].)

1.2. In K the following known homomorphisms will be used. (See [3, §2.2] or [4, §6].)

(i) For integers j, p such that $0 \leq j \leq p$, $p > 0$, the homomorphism

$$j_p\colon\ C_p \longrightarrow C_{p-1}$$

is defined by the relation $j_p(v_0, \cdots, v_p) = (-1)^j (v_0, \cdots, \hat{v}_j, \cdots, v_p)$, where the symbol ^ is placed over the point v_j to indicate that v_j is to be deleted. For $j = p = 0$, j_p is defined to be the trivial homomorphism. A homomorphism differing from this one only by the absence of the factor $(-1)^j$ has been used by Radó in [2, §2.6]. The definition given above has been chosen because it permits simplifications in later definitions and formulas.

(ii) For integers $p > 0$, the *boundary operator*

$$\partial_p\colon\ C_p \longrightarrow C_{p-1}$$

is defined by the formula

$$\partial_p (v_0, \cdots, v_p) = \sum_{j=0}^{p} (-1)^j (v_0, \cdots, \hat{v}_j, \cdots, v_p).$$

For integers $p \leq 0$, ∂_p is defined to be the trivial homomorphism.

(iii) For integers $p \geq 0$ and an arbitrary point v in E_∞, the *cone homomorphism* $h_p^v\colon C_p \longrightarrow C_{p+1}$ is defined by the relation

$$h_p^v(v_0, \cdots, v_p) = (-1)^{p+1} (v_0, \cdots, v_p, v).$$

For integers $p < 0$, h_p^v is defined to be the trivial homomorphism.

(iv) For integers j, p such that $0 \leq j \leq p-1$, the *transposition homomorphism* $t_{pj}\colon C_p \longrightarrow C_p$ is defined by the relation

$$t_{pj}(v_0, \cdots, v_j, v_{j+1}, \cdots, v_p) = (v_0, \cdots, v_{j+1}, v_j, \cdots, v_p).$$

Observe that $t_{pj}(v_0, \cdots, v_p) = (v_0, \cdots, v_p)$ if and only if $v_j = v_{j+1}$.

(v) The *barycentric homomorphism* $\beta_p : C_p \longrightarrow C_p$ is defined as follows. For integers $p < 0$, β_p is the trivial homomorphism; for $p = 0$, $\beta_0 = 1$; and for $p > 0$, β_p is defined by the recursion formula

$$\beta_p(v_0, \cdots, v_p) = h^b_{p-1}\, \beta_{p-1}\, \partial_p(v_0, \cdots v_p),$$

where b is the barycenter of the points $v_0, \cdots, v_p$.

(vi) The barycentric homotopy operator ρ_p used by Radó [1; 3, §2.2 (iv); 4, §6] will not be used in this paper. In its stead, a modification ρ_{*p} is presently introduced, which has a simpler form, satisfies all the important identities which hold for the ρ_p, and has useful properties not possessed by ρ_p. The *modified barycentric homotopy operator*

$$\rho_{*p} : C_p \longrightarrow C_{p+1}$$

is defined as follows. For integers $p < 0$, ρ_{*p} is the trivial homomorphism; for $p = 0$, ρ_{*p} is defined by the relation

$$\rho_{*0}(v_0) = -h_0^{v_0}(v_0) = (v_0, v_0);$$

and for $p > 0$, ρ_{*p} is defined by the recursion formula

$$\rho_{*p}(v_0, \cdots, v_p) = -h^b_p[1 + \rho_{*p-1}\, \partial_p]\, (v_0, \cdots, v_p),$$

where b is the barycenter of the points $v_0, \cdots, v_p$.

1.3. Amongst the preceding homomorphisms the following identities hold (see [2, §2; 3, §2.3]):

$$\partial_p = \sum_{j=0}^{p} j_p \qquad (p \geq 0);$$

$$\partial_{p+1}\, h^v_p + h^v_{p-1}\, \partial_p = 1 \qquad (p > 0);$$

$$\partial_p\ \beta_p = \beta_{p-1}\, \partial_p \qquad (-\infty < p < +\infty);$$

$$\beta_p\, t_{pj} = -\beta_p \qquad (0 \leq j \leq p-1);$$

$$\partial_{p+1}\, \rho_{*p} + \rho_{*p-1}\, \partial_p = \beta_p - 1 \qquad (0 \leq p < +\infty).$$

Of these identities, only the last is new; it may be established by an inductive reasoning similar to that used to prove the corresponding identity for the conventional barycentric homotopy operator ρ_p.

1.4. For integers k, p such that $0 \leq k \leq p$, the homomorphism

$$k_{*p} : C_p \longrightarrow C_p$$

is defined by the relation

$$k_{*p}(v_0, \cdots, v_p) = (-1)^{p+k}(v_0, \cdots, \hat{v}_k, \cdots, v_p, v_k),$$

and the homomorphism

$$\gamma_p : C_p \longrightarrow C_p$$

is defined by the formula $\gamma_p = \sum_{k=0}^{p} k_{*p}$. Obviously one has the identities

$$k_{*p}(v_0, \cdots, v_p) = -k_{p+1} h_p^{v_k} (v_0, \cdots, v_p), \ p \geq 0,$$

$$k_{*p}(v_0, \cdots, v_p) = h_{p-1}^{v_k} k_p (v_0, \cdots, v_p), \ p > 0.$$

Now the reader will easily verify the relations

$$j_p \, k_{*p} = \begin{cases} (k-1)_{*p-1} \, j_p & , \ 0 \leq j < k \leq p, \\ k_{*p-1}(j+1)_p & , \ 0 \leq k \leq j < p, \\ k_p & , \ 0 \leq k \leq j = p; \end{cases}$$

$$k_{*p-1} \, j_p = \begin{cases} (j-1)_p \, k_{*p} & , \ 0 \leq k < j \leq p, \\ j_p (k+1)_{*p} & , \ 0 \leq j \leq k < p. \end{cases}$$

From these relations the following identity is readily established:

$$\gamma_{p-1} \, \partial_p = \partial_p \, (\gamma_p - 1).$$

Using the identity, the reader will easily prove the following result.

LEMMA. *If $P(x)$ be any polynomial having integral coefficients, then*

$$P(\gamma_{p-1}) \, \partial_p = \partial_p \, P(\gamma_p - 1).$$

Explicitly, if $P(x) = \sum_{i=0}^{m} a_i x^i$, where the a_i are integers, then

$$\sum_{i=0}^{m} a_i \, \gamma_{p-1}^{i} \, \partial_p = \sum_{i=0}^{m} a_i \, \partial_p \left[\gamma_p^i - i \, \gamma_p^{i-1} + \cdots + (-1)^i\right].$$

where γ_p^i means that the homomorphism γ_p is to be repeated i times.

1.5. For integers k,p such that $0 \leq k \leq p$, the homomorphism

$$b_{pk}\colon C_p \longrightarrow C_{p+1}$$

is defined by the relation

$$b_{pk}(v_0, \cdots, v_p) = (-1)^k \Big[v_0, \cdots, v_k, b(v_0, \cdots, v_k),$$
$$b(v_0, \cdots, v_k, v_{k+1}), \cdots, b(v_0, \cdots, v_k, \cdots, v_p)\Big],$$

where $b(v_0, \cdots, v_q)$ is the barycenter of the points $v_0, \cdots, v_q$. Verification of the following simple relations is left to the reader:

$$-h_p^{b(v_0,\cdots,v_p)}(v_0, \cdots, v_p) = b_{pp}(v_0, \cdots, v_p);$$

$$-h_p^{b(v_0,\cdots,v_p)} b_{p-1\,k}(v_0, \cdots, v_{p-1}) = b_{pk} h_{p-1}^{v_p}(v_0, \cdots, v_{p-1}) \qquad (0 \leq k \leq p-1);$$

$$-h_p^{b(v_0,\cdots,v_p)} b_{p-1\,k}\, j_p(v_0, \cdots, v_p) = b_{pk}\, j_{*p}(v_0, \cdots, v_p) \qquad (0 \leq k \leq p-1,\ 0 \leq j \leq p);$$

$$-h_p^{b(v_0,\cdots,v_p)} b_{p-1\,k}\, \partial_p(v_0, \cdots, v_p) = b_{pk}\, \gamma_p(v_0, \cdots, v_p) \qquad (0 \leq k \leq p-1);$$

$$\begin{cases} -h_p^{b(v_0,\cdots,v_p)} b_{p-1\,k}\, \partial_p\, \gamma_p^{i-1}\, j_{*p}(v_0, \cdots, v_p) = b_{pk}\, \gamma_p^{i}\, j_{*p}(v_0, \cdots, v_p) & (0 \leq k \leq p-1,\ 0 \leq j \leq p,\ 1 \leq i) \\ -h_p^{b(v_0,\cdots,v_p)} b_{p-1\,k}\, \partial_p\, \gamma_p^{i}(v_0, \cdots, v_p) = b_{pk}\, \gamma_p^{i+1}(v_0, \cdots, v_p) & (0 \leq k \leq p-1,\ 0 \leq i). \end{cases}$$

If $P(x)$ be any polynomial having integral coefficients, then, for $0 \leq k \leq p-1$, we have

$$-h_p^{b(v_0,\cdots,v_p)} b_{p-1\,k}\, \partial_p\, P(\gamma_p)(v_0, \cdots, v_p) = b_{pk}\, \gamma_p\, P(\gamma_p)(v_0, \cdots, v_p).$$

1.6. For the homomorphisms β_p and ρ_{*p}, the following structural descriptions are now obtained.

THEOREM. *The following relations hold:*

$$\rho_{*0} = b_{00},$$
$$\rho_{*p} = b_{pp} + \sum_{j=1}^{p} b_{pp-j}\, \gamma_p \cdots (\gamma_p - j + 1) \qquad (p > 0).$$

Proof. It is sufficient to verify these formulas for a given p-cell $(v_0, \cdots, v_p)$. For $p = 0$, the formula $\rho_{*0}(v_0) = b_{00}(v_0)$ is obvious from the definitions. So assume that

$$\rho_{*p-1} = b_{p-1\, p-1} + \sum_{j=1}^{p-1} b_{p-1\, p-1-j}\, \gamma_{p-1} \cdots (\gamma_{p-1} - j + 1) \quad (p \geq 1).$$

Using §1.2, §1.4, §1.5, and this assumption, and letting $b = b(v_0, \cdots, v_p)$, one obtains

$$\rho_{*p}(v_0, \cdots, v_0)$$
$$= -h_p^b(v_0, \cdots, v_p) - h_p^b\, \rho_{*p-1}\, \partial_p(v_0, \cdots, v_p)$$
$$= b_{pp}(v_0, \cdots, v_p) - h_p^b\, b_{p-1\, p-1}\, \partial_p(v_0, \cdots, v_p)$$
$$- \sum_{j=1}^{p-1} h_p^b\, b_{p-1\, p-1-j}\, \gamma_{p-1} \cdots (\gamma_{p-1} - j + 1)\, \partial_p(v_0, \cdots, v_p)$$
$$= b_{pp}(v_0, \cdots v_p) + b_{p\,p-1}\, \gamma_p(v_0, \cdots, v_p)$$
$$- \sum_{j=1}^{p-1} h_p^b b_{p-1\, p-1-j} \partial_p (\gamma_p - 1) \cdots (\gamma_p - j)\, (v_0, \cdots, v_p)$$
$$= b_{pp}(v_0, \cdots, v_p) + b_{pp-1}\, \gamma_p(v_0, \cdots, v_p)$$
$$+ \sum_{j=2}^{p} b_{p\,p-j}\, \gamma_p(\gamma_p - 1) \cdots (\gamma_p - j + 1)\, (v_0, \cdots, v_p)$$
$$= b_{pp} + \sum_{j=1}^{p} b_{p\,p-j}\, \gamma_p \cdots (\gamma_p - j + 1)\, (v_0, \cdots, v_p).$$

So the proof is complete by induction.

1.7. THEOREM. *The following relations hold*:

$$\beta_0 = 0_1\ b_{00},$$

$$\beta_p = 0_{p+1}\ b_{p0}\ \gamma_p(\gamma_p - 1)\cdots(\gamma_p - p + 1),\ p > 0.$$

The proof is similar to that for the theorm in the preceding section.

1.8. From these formulas for β_p and ρ_{*p} and the identities in §1.3, many further interesting relations may be obtained. For example, it is easy to establish the following results:

$$\beta_p = [\partial_{p+1} - (p+1)_{p+1}]\ \rho_{*p} \qquad (p \geq 0);$$

$$\partial_p = -p_p\ \rho_{*p-1}\ \partial_p \qquad (p \geq 0);$$

$$\beta_p = (p+1)_{p+1}\ (p+2)_{p+2}\ \rho_{*p+1}\ \rho_{*p} \qquad (p \geq 0).$$

These relations are not needed for the present purposes; they may be studied on a later occasion.

In order to clarify the structural descriptions for β_p and ρ_{*p} given in §§1.6, 1.7, it is convenient to introduce another homomorphism.

1.9. For integers $p \geq 0$, let $i_0, \cdots, i_p$ be any rearrangement of the sequence $0, \cdots, p$, and put $\epsilon_{i_0 \cdots i_p}$ equal to +1 or to −1 according as $i_0, \cdots, i_p$ is obtained from $0, \cdots, p$ by an even or by an odd number of transpositions. With each rearrangement one associates a homomorphism

$$\tau_p\colon\ C_p \longrightarrow C_p$$

defined by the formula

$$\tau_p(v_0, \cdots, v_p) = \epsilon_{i_0 \cdots i_p}\ (v_{i_0}, \cdots, v_{i_p}).$$

Sometimes, for clarity, the more explicit notation $\tau_p(i_0, \cdots, i_p)$ is used for this homomorphism. For integers j such that $0 \leq j \leq p$, denote by T_{pj} the class of all $\tau_p(i_0, \cdots, i_p)$ for which $i_0 < \cdots < i_j$ — that is, for which $i_0, \cdots, i_j$ are in natural order. Obviously T_{pp} consists of just one element, namely $\tau_p(0, \cdots, p) = 1$; and T_{p0} consists of the τ_p obtained by all possible rearrangements of $0, \cdots, p$. Moreover, $T_{p\,j-1} \supset T_{pj}$ for $1 \leq j \leq p$. Clearly the number of elements in the class T_{pj} is $(p+1)\,p \cdots (j+2)$ for $0 \leq j \leq p-1$. For each integer j in $0 \leq j \leq p$, define a homomorphism

$$P_{pj}\colon C_p \longrightarrow C_p$$

by the formula

$$P_{pj} = \sum \tau_p \qquad (\tau_p \in T_{pj}).$$

Observe that $P_{pp} = 1$. The reader will readily verify these identities:

$$k_{*p}\, P_{pj} = P_{pj},\ 0 \leq j < k \leq p;$$

$$\sum_{k=0}^{j} k_{*p}\, P_{pj} = P_{p\,j-1},\ 0 < j \leq p.$$

From these identities, the following result is established.

LEMMA. *The following relations hold*:

$$P_{pp} = 1,$$

$$P_{p\,p-j} = \gamma_p(\gamma_p - 1)\cdots(\gamma_p - j + 1),\ 1 \leq j \leq p.$$

Proof. That $P_{pp} = 1$ was noted above. From the second relation above it follows that

$$P_{p\,p-1} = \sum_{k=0}^{p} k_{*p}\, P_{pp} = \gamma_p\, P_{pp} = \gamma_p,$$

so the general formula is established for $j = 1$. Now suppose that

$$P_{p\,p-j+1} = \gamma_p(\gamma_p - 1)\cdots(\gamma_p - j + 2) \qquad (2 \leq j \leq p).$$

Using the preceding identities, one finds

$$\begin{aligned}\gamma_p\, P_{p\,p-j+1} &= \sum_{k=0}^{p} k_{*p}\, P_{p\,p-j+1}\\ &= \sum_{k=0}^{p-j+1} k_{*p}\, P_{p\,p-j+1} + \sum_{k=p-j+2}^{p} k_{*p}\, P_{p\,p-j+1}\\ &= P_{p\,p-j} + (j-1)\, P_{p\,p-j+1};\end{aligned}$$

$$P_{p\,p-j} = (\gamma_p - j + 1)\, P_{p\,p-j+1} = \gamma_p(\gamma_p - 1)\cdots(\gamma_p - j + 1).$$

Thus the lemma is established.

1.10. Combining the results of the preceding lemma with those in the theoems in §§1.6, 1.7, one obtains the following description for the homomorphisms β_p and ρ_{*p}.

THEOREM. *The following relations hold*:

$$\beta_p = 0_{p+1}\, b_{p0}\, P_{p0} = \sum_{\tau_p \in T_{p0}} 0_{p+1}\, b_{p0}\, \tau_p \qquad (p \geq 0);$$

$$\rho_{*p} = \sum_{k=0}^{p} b_{pk}\, P_{pk} = \sum_{k=0}^{p} \sum_{\tau_p \in T_{pk}} b_{pk}\, \tau_p \qquad (p \geq 0).$$

1.11. Let $v_0, \cdots, v_p$ $(p \geq 0)$ be any sequence of $p+1$ points in E_∞. In §§1.2, 1.4, 1.5, 1.9, homomorphisms j_p, t_{pj}, k_{*p}, b_{pk}, τ_p, have been introduced which, when applied in any appropriate combination h_p to the special chain $(v_0, \cdots, v_p)$, yield a special chain either of the form $+(y_0, \cdots, y_q)$ or of the form $-(y_0, \cdots, y_q)$. In the sequel, $[h_p(v_0, \cdots, v_p)]$ is defined to be the p-cell $(y_0, \cdots, y_q)$, and $|h_p(v_0, \cdots, v_p)|$ denotes its convex hull $|y_0, \cdots, y_q|$. For example,

$$[0_{p+1}\, b_{p0}\, \tau_p(i_0, \cdots, i_p)(v_0, \cdots, v_p)]$$
$$= (b(v_{i_0}), b(v_{i_0}, v_{i_1}), \cdots, b(v_{i_0}, v_{i_1}, \cdots, v_{i_p})).$$

If for two sequences of points $u_0, \cdots, u_p$ and $v_0, \cdots, v_p$ it is true that

$$(b(u_0), b(u_0, u_1), \cdots, b(u_0, u_1, \cdots, u_p))$$
$$= (b(v_0), b(v_0, v_1), \cdots, b(v_0, v_1, \cdots, v_p))$$

then clearly $u_j = v_j$ for $0 \leq j \leq p$. From the remarks in §1.9 and the preceding theorem, one thus obtains the following result.

LEMMA. *If the points* $v_0, \cdots, v_p$ $(p \geq 0)$ *are distinct, then the chain* $\beta_p(v_0, \cdots, v_p)$ *contains* $(p+1)!$ *terms; that is, for distinct elements* τ_p' *and* τ_p'' *in* T_{p0}, *we have*

$$[0_{p+1}\, b_{p0}\, \tau_p'(v_0, \cdots, v_p)] \neq [0_{p+1}\, b_{p0}\, \tau_p''(v_0, \cdots, v_p)].$$

1.12. LEMMA. *Let* $v_0, \cdots, v_p$ $(p \geq 0)$ *be any set of* $p+1$ *points in* E_∞, *not necessarily distinct or linearly independent. A necessary and sufficient condition that a point v belong to the convex hull of the points*

(i) $b(v_0),\ b(v_0, v_1), \cdots, b(v_0, v_1, \cdots, v_p)$

is that it possess a representation of the form

$$\text{(ii)}\quad v = \sum_{j=0}^{p} \mu_j v_j \qquad \left(\sum_{j=0}^{p} \mu_j = 1,\ \mu_0 \geq \mu_1 \geq \cdots \geq \mu_p \geq 0 \right).$$

Proof. If v belongs to the convex hull of the points (i), then it has a representation of the form

$$\text{(iii)}\quad v = \sum_{i=0}^{p} \lambda_i\, b(v_0, \cdots, v_i) \qquad \left(\sum_{i=0}^{p} \lambda_i = 1,\ 0 \leq \lambda_i,\ 0 \leq i \leq p \right).$$

Thus

$$v = \sum_{i=0}^{p} \lambda_i \sum_{j=0}^{i} \frac{v_j}{i+1} = \sum_{j=0}^{p} \sum_{i=j}^{p} \frac{\lambda_i}{i+1} v_j,$$

which gives a representation of form (ii) for v. Conversely, if v has a representation of form (ii), put $\lambda_i = (i+1)(\mu_i - \mu_{i+1})$ for $0 \leq i \leq p-1$, $\lambda_p = (p+1)\mu_p$. It follows at once that v has a representation of form (iii), and hence belongs to the convex hull of the set of points (i).

1.13. For integers $p \geq 0$, if $u_0, \cdots, u_p$ is any sequence of $p+1$ points in E_∞, then $|u_0, \cdots, u_p|$ will denote its convex hull. Let k be any integer such that $0 \leq k \leq p$, and consider the sequence of $p+2$ points

(i) $u_0, \cdots u_k,\ b(u_0, \cdots u_k), \cdots, b(u_0, \cdots, u_k, \cdots, u_p)$,

that is (see §1.5), the sequence of points occurring in $b_{pk}(u_0, \cdots, u_p)$. Let

(ii) $w_0, \cdots, w_{p+1}$

be any rearrangement of the sequence of points (i). Designate by $x_0 = w_{h_0} = u_{i_0}$ the first u_i $(0 \leq i \leq k)$ occurring in the sequence (ii). In general, let $x_l = w_{h_l} = u_{i_l}$ $(0 \leq l \leq k)$ be the $(l+1)$st u_i $(0 \leq i \leq k)$ occurring in the sequence (ii), and put $x_l = u_l$ for $k+1 \leq l \leq p$ in case $k < p$. Now clearly $x_0, \cdots, x_p$ is a rearrangement of the sequence $u_0, \cdots, u_p$ in which the last $p-k$ elements are unaltered; the sequence (i) is a rearrangement of the sequence

(iii) $x_0, \cdots, x_k, b(x_0, \cdots, x_k), \cdots, b(x_0, \cdots, x_k, \cdots, x_p)$

in which the last $p+1-k$ elements are unaltered; and the sequence (ii) is a rearrangement of the sequence (iii) in which the points $x_0, \cdots, x_k$ appear in the same order as in (iii); that is, $x_l = w_{h_l}$ for $0 \leq l \leq k$, where $0 \leq h_0 < h_1 < \cdots < h_k \leq p$. Now let q be any integer such that $0 \leq q \leq p+1$. It will be shown that

(iv) $b(w_0, \cdots, w_q) \in |b(x_0), b(x_0, x_1), \cdots, b(x_0, x_1, \cdots, x_p)|$ $\quad (0 \leq q \leq p+1)$.

Case $q = 0$. Then $b(w_0) = w_0$. If w_0 is one of the $u_i\ (0 \leq i \leq k)$, it follows by the choice above that $h_0 = 0$ and $w_0 = x_0 = b(x_0)$. If w_0 is not one of the $u_i\ (0 \leq i \leq k)$, there must be a $l \geq k$ such that $w_0 = b(u_0, \cdots, u_k, \cdots, u_l) = b(x_0, \cdots, x_k, \cdots, x_l)$. Thus relation (iv) is established when $q = 0$.

General case. By a rearrangement, the points $w_0, \cdots, w_q$ may be ordered into two sets

$$w_{h_0} = x_0, \cdots, w_{h_l} = x_l \qquad (0 \leq l \leq k,\ 0 \leq h_0 < \cdots < h_l \leq p),$$

$$\begin{cases} w_{h_{l+1}} = b(u_0, \cdots, u_k, \cdots, u_{i_{l+1}}) = b(x_0, \cdots, x_{i_{l+1}}) \\ w_{h_{l+2}} = b(u_0, \cdots, u_k, \cdots, u_{i_{l+2}}) = b(x_0, \cdots, x_{i_{l+2}}) \\ \cdots \\ w_{h_q} = b(u_0, \cdots, u_k, \cdots, u_{i_q}) = b(x_0, \cdots, x_{i_q}) \\ \qquad (k \leq i_{l+1} < i_{l+2} < \cdots < i_q \leq p). \end{cases}$$

The special cases which arise when one of these sets is missing are left to the reader. Now clearly

$$b(w_0, \cdots, w_q) = b(w_{h_0}, \cdots, w_{h_q})$$

$$= \sum_{j=0}^{l} \frac{1}{q+1}\left[1 + \sum_{h=l+1}^{q} \frac{1}{i_h+1}\right] x_j + \sum_{j=l+1}^{i_{l+1}} \frac{1}{q+1} \sum_{h=l+1}^{q} \frac{1}{i_h+1} x_j$$

$$+ \sum_{j=i_{l+1}+1}^{i_{l+2}} \frac{1}{q+1} \sum_{h=l+2}^{q} \frac{1}{i_h+1} x_j + \cdots + \sum_{j=i_{q-1}+1}^{i_q} \frac{1}{q+1}\,\frac{1}{i_q+1}\, x_j .$$

In view of this equation and of the lemma in §1.12, the relation (iv) now follows.

1.14. From the facts presented above, the following result is presently established.

LEMMA. *Let* $v_0, \cdots, v_p$ $(p \leq 0)$ *be any sequence of* $p+1$ *points in* E_∞. *Fix* $\tau_{p+1} \in T_{p+1\,0}\,(0 \leq k \leq p)$, $\tau_p \in T_{pk}$ (see §1.9). *Then there exists a* $\tau_p' \in T_{p0}$ *such that* (see §1.11).

$$|0_{p+2}\, b_{p+1\,0}\, \tau_{p+1}\, b_{pk}\, \tau_p(v_0, \cdots, v_p)| \subset |0_{p+1}\, b_{p0}\, \tau_p'(v_0, \cdots, v_p)|.$$

Proof. Evidently $[\tau_p(v_0, \cdots, v_p)] = (v_{i_0}, \cdots, v_{i_p})$, where $i_0, \cdots, i_p$ is a rearrangement of $0, \cdots, p$ such that $i_0 < \cdots < i_k$. Put $u_j = v_{i_j}$ for $0 \leq j \leq p$, so that $[\tau_p(v_0, \cdots, v_p)] = (u_0, \cdots, u_p)$. Then

$$[b_{pk}\, \tau_p(v_0, \cdots, v_p)]$$
$$= (u_0, \cdots, u_k, b(u_0, \cdots u_k), \cdots, b(u_0, \cdots, u_k, \cdots, u_p)),$$

and $[\tau_{p+1}\, b_{pk}\, \tau_p(v_0, \cdots, v_p)] = (w_0, \cdots, w_{p+1})$, where $w_0, \cdots, w_{p+1}$ is a rearrangement of

$$u_0, \cdots, u_k, b(u_0, \cdots, u_k), \cdots, b(u_0, \cdots, u_k, \cdots, u_p).$$

Finally,

$$[0_{p+2}\, b_{p+1\,0}\, \tau_{p+1}\, b_{pk}\, \tau_p(v_0, \cdots, v_p)]$$
$$= [b(w_0), b(w_0, w_1), \cdots, b(w_0, w_1, \cdots, w_{p+1})].$$

The reasoning of §1.13 shows that there is a rearrangement $x_0, \cdots, x_p$ of $u_0, \cdots, u_p$, and hence of $v_0, \cdots, v_p$, such that

$$|0_{p+2}\, b_{p+1\,0}\, \tau_{p+1}\, b_{pk}\, \tau_p(v_0, \cdots, v_p)|$$
$$\subset |b(x_0), b(x_0, x_1), \cdots, b(x_0, x_1, \cdots, x_p)|.$$

Let τ_p' be that element of T_{p0} such that $[\tau_p'(v_0, \cdots, v_p)] = (x_0, \cdots, x_p)$. Since

$$[0_{p+1}\, b_{p0}\, \tau_p'(v_0, \cdots, v_p)] = (b(x_0), b(x_0, x_1), \cdots, b(x_1, x_1, \cdots, x_p)),$$

the lemma is established.

1.15. If c_p is a p-chain in K, and A is a convex subset in E_∞, then the inclusion $c_p \subset A$ will mean that either $c_p = 0 \in C_p$ or else

$$c_p = \sum_{j=1}^{n} m_j (v_{0j}, \cdots, v_{pj}),$$

where the m_j are nonzero integers and $|v_{0j}, \cdots, v_{pj}| \subset A$ for $1 \leq j \leq n$. One readily verifies the following inclusions (see [3, §2.4]):

$$j_p(v_0, \cdots, v_p) \subset |v_0, \cdots, v_p| \qquad (0 \leq j \leq p),$$

$$\partial_p(v_0, \cdots, v_p) \subset |v_0, \cdots, v_p| \qquad (p \geq 0),$$

$$\beta_p(v_0, \cdots, v_p) \subset |v_0, \cdots, v_p| \qquad (p \geq 0),$$

$$\rho_{*p}(v_0, \cdots, v_p) \subset |v_0, \cdots, v_p| \qquad (p \geq 0),$$

$$t_{pj}(v_0, \cdots, v_p) \subset |v_0, \cdots, v_p| \qquad (0 \leq j \leq p-1),$$

$$k_{*p}(v_0, \cdots, v_p) \subset |v_0, \cdots, v_p| \qquad (0 \leq k \leq p),$$

$$\gamma_p(v_0, \cdots, v_p) \subset |v_0, \cdots, v_p| \qquad (p \geq 0),$$

$$b_{pk}(v_0, \cdots, v_p) \subset |v_0, \cdots, v_p| \qquad (0 \leq k \leq p),$$

$$\tau_p(v_0, \cdots, v_p) \subset |v_0, \cdots, v_p| \qquad (\tau_p \in T_{p0}),$$

$$P_{pj}(v_0, \cdots, v_p) \subset |v_0, \cdots, v_p| \qquad (0 \leq j \leq p).$$

II. Relations in the Complex R = R(X).

2.1. If A is a convex subset of E_∞, then for integers $p \geq 0$, C_p^A denotes that subgroup of C_p generated by those p-cells $(v_0, \cdots, v_p)$ for which $|v_0, \cdots, v_p| \subset A$; for $p < 0$, we have $C_p^A = 0 \in C_p$ (see §1.1). Suppose $T: A \longrightarrow X$ is a continuous mapping (see §0.1). For integers $p \geq 0$ define a homomorphism

$$T_p: C_p^A \longrightarrow C_p^R$$

by the relation $T_p(v_0, \cdots, v_p) = (v_0, \cdots, v_p, T)^R$ for $(v_0, \cdots, v_p) \in C_p^A$. For $p < 0$, let T_p be the trivial homomorphism. For chains c_p in C_p^A the notation $T_p c_p = (c_p, T)^R$ is used. In terms of this notation one finds the relation (see §0.1): $\partial_p^R (c_p, T)^R = (\partial_p c_p, T)^R$.

Now suppose that, for certain integers p,

$$h_p: C_p \longrightarrow C_q$$

is a homomorphism from the group C_p of p-chains into the group C_q of q-chains in K with the property that for all p-cells $(v_0, \cdots, v_p)$ in K one has

$$h_p(v_0, \cdots, v_p) \subset |v_0, \cdots, v_p|.$$

Then clearly one may define for these integers p a homomorphism

$$h_p^R\colon C_p^R \longrightarrow C_q^R$$

by the formula $h_p^R(v_0, \cdots, v_p, T)^R = (h_p(v_0, \cdots, v_p), T)^R$ in case $p \geq 0$, and one may make h_p^R the trivial homomorphism if $p < 0$. In view of the inclusions in §1.15, one observes that this definition creates the following homomorphisms in R (see [3, §3.1]):

$$j_p^R\colon C_p^R \longrightarrow C_{p-1}^R \quad (0 \leq j \leq p);$$

$$\beta_p^R\colon C_p^R \longrightarrow C_p^R \quad (-\infty < p < +\infty); \qquad \gamma_p^R\colon C_p^R \longrightarrow C_p^R \quad (p \geq 0);$$

$$\rho_{*p}^R\colon C_p^R \longrightarrow C_{p+1}^R \quad (-\infty < p < +\infty); \qquad b_{pk}^R\colon C_p^R \longrightarrow C_{p+1}^R \quad (0 \leq k \leq p);$$

$$t_{pj}^R\colon C_p^R \longrightarrow C_{p-1}^R \quad (0 \leq j \leq p-1); \qquad \tau_p^R\colon C_p^R \longrightarrow C_p^R \quad (\tau_p \in T_{p_0});$$

$$P_{pj}^R\colon C_p^R \longrightarrow C_p^R \quad (0 \leq j \leq p).$$

2.2. From the relations in §1.3, one derives the following (see [3, §3.1]):

$$\partial_p^R \beta_p^R = \beta_{p-1}^R \partial_p^R \qquad (-\infty < p < +\infty);$$

$$\beta_p^R t_{pj}^R = -\beta_p^R \qquad (0 \leq j \leq p-1);$$

$$\partial_{p+1}^R \rho_{*p}^R + \rho_{*p-1}^R \partial_p^R = \beta_p^R - 1 \qquad (0 \leq p < +\infty).$$

The theorems in §§1.6, 1,7 give rise to these formulas for β_p^R and ρ_{*p}^R:

$$\rho_{*0}^R = b_{00}^R,$$

$$\rho_{*p}^R = b_{pp}^R + \sum_{j=1}^{p} b_{p\,p-j}^R \gamma_p^R, \cdots, (\gamma_p^R - j + 1) \quad (p > 0);$$

$$\beta_0^R = 0_p^R b_{00}^R;$$

$$\beta_p^R = 0_{p+1}^R b_{p0}^R \gamma_p^R (\gamma_p^R - 1), \cdots, (\gamma_p^R - p + 1) \quad (p > 0).$$

From the theorem in §1.10, one obtains the following description for β_p^R and ρ_{*p}^R.

THEOREM. *The following relations hold:*

$$\beta_p^R = \partial_{p+1}^R b_{p0}^R P_{p0}^R = \sum_{\tau_p \in T_{p0}} \partial_{p+1}^R b_{p0}^R \tau_p^R \quad (p \geq 0);$$

$$\rho_{*p}^R = \sum_{k=0}^{p} b_{pk}^R P_{pk}^R = \sum_{k=0}^{p} \sum_{\tau_p \in T_{pk}} b_{pk}^R \tau_p^R \quad (p \geq 0).$$

2.3. The writer is indebted to T. Radó for suggestions which led to the results presently presented in §§2.3-2.7, 2.9, 2.10, 2,12. The new facts contributed by this paper are contained in §§2.8, 2.11, 2.13. For integers $p \geq 1$, any chain of the form $(1 + t_{pj}^R)(v_0, \cdots, v_p, T)^R$ $(0 \leq j \leq p-1)$ is termed an *elementary t-chain* in R (see [3, §3.2] or [4, §7]), and the subgroup of C_p^R generated by these elementary t-chains is denoted by T_p^R. For $p < 1$, T_p^R is defined to be the subgroup of C_p^R composed of the zero element alone.

LEMMA. *If* $c_p^R \in T_p^R$, *then*

(i) $\partial_p^R c_p^R \in T_{p-1}^R$,

(ii) $\beta_p^R c_p^R = 0$,

(iii) $\rho_{*p}^R c_p^R \in T_{p+1}^R$.

This lemma differs from that in Radó [3, §3.2], only by the fact that the barycentric homotopy operator ρ_p^R has been replaced by the modified operator ρ_{*p}^R (see §1.2). It may be established by the same reasoning as that employed by Radó.

2.4. For integers $p \geq 1$, any chain of the form

$$(v_0, \cdots, v_j, v_{j+1}, \cdots, v_p, T)^R$$

with $v_j = v_{j+1}$ for some j such that $0 \leq j \leq p-1$ is called an elementary d-chain in R (see [3, §3.3] or [4, §7]), and the subgroup of C_p^R generated by these elementary d-chains is denoted by D_p^R. For $p < 1$, D_p^R is defined to be that subgroup of C_p^R composed of the zero element alone.

LEMMA. *If* $c_p^R \in D_p^R$, *then*

(i) $\partial_p^R c_p^R \in D_{p-1}^R$,

(ii) $\beta_p^R c_p^R = 0$,

(iii) $\rho^R_{*p}\ c^R_p \in D^R_{p+1}$.

This is the lemma in [3, §3.3], except that the modified barycentric homotopy operator ρ^R_{*p} is used in place of ρ^R_p; it is proved in the same way.

2.5. LEMMA. *Let $(v_0, \cdots, v_p, T)^R$ be any p-cell in R ($p \geq 1$). Suppose that the sequence $w_0, \cdots, w_p$ is obtainable from the sequence $v_0, \cdots, v_p$ by n transpositions. Then there is an element t^R_p in T^R_p such that*

$$(v_0, \cdots, v_p, T)^R = (-1)^n\ (w_0, \cdots, w_p, T)^R + t^R_p.$$

Proof. By assumption there exist $n+1$ sequences $v_{0j}, \cdots, v_{pj}$ for $0 \leq j \leq n$ where $v_{i0} = v_i$ and $v_{in} = w_i$ for $0 \leq i \leq p$ such that

$$(v_{0j}, \cdots, v_{pj}, T)^R = t^R_{pi_j}\ (v_{0\,j-1}, \cdots, v_{p\,j-1}, T)^R$$

for some integer i_j satisfying $0 \leq i_j \leq p-1$, $1 \leq j \leq n$. Clearly

$$(v_0, \cdots, v_p, T)^R = (-1)^n\ (w_0, \cdots, w_p, T)^R$$
$$+ \sum_{j=1}^{n} (-1)^{j-1}\ (1 + t^R_{pi_j})\ (v_{0\,j-1}, \cdots, v_{p\,j-1}, T)^R,$$

and the lemma is established.

2.6. LEMMA. *Let $(v_0, \cdots, v_p, T)^R$ be any p-cell in R ($p \geq 1$), for which $v_i = v_k$ for some i, k such that $0 \leq i \leq k \leq p$. Then there are elements t^R_p in T^R_p and d^R_p in D^R_p such that*

$$(v_0, \cdots, v_p, T)^R = t^R_p + d^R_p.$$

Moreover, $2(v_0, \cdots, v_p, T)^R$ is in T^R_p.

Proof. Since the sequence $v_0, \cdots, v_{i-1}, v_k, v_i, \cdots, v_{k-1}, v_{k+1}, \cdots, v_p$ is obtained from $v_0, \cdots, v_i, \cdots, v_k, \cdots, v_p$ by $k-i$ transpositions, and $v_i = v_k$ by assumption, if follows that

$$(-1)^{k-i}\ (v_0, \cdots, v_{i-1}, v_k, v_i, \cdots, v_{k-1}, v_{k+1}, \cdots, v_p)^R$$

is an element d^R_p of D^R_p. Moreover, from the lemma in §2.5 it follows that there is an element t^R_p in T^R_p such that $(v_0, \cdots, v_p, T)^R = d^R_p + t^R_p$, and the first part of the lemma is proven. Now the sequence $v_0, \cdots, v_k, \cdots, v_i, \cdots, v_p$ is obtained from $v_0, \cdots, v_i, \cdots, v_k, \cdots, v_p$ by $2(k-i)-1$ transpositions.

Again, from the lemma in §2.5 it follows that there is an element t_p^R in T_p^R such that

$$(v_0, \cdots, v_i, \cdots, v_k, \cdots, v_p, T)^R = -(v_0, \cdots, v_k, \cdots, v_i, \cdots, v_p, T)^R + t_p^R .$$

Since $v_i = v_k$, one obtains $2(v_0, \cdots, v_p, T)^R = t_p^R$; and the second part of the lemma is demonstrated.

2.7. For integers $p \geq 0$, a chain c_p^R is termed an elementary n-chain in R if it has the form

$$c_p^R = \sum_{r=1}^{n} m_r(v_0, \cdots, v_p, T_r)^R ,$$

where

(i) for $1 \leq r \leq n$, the m_r are nonzero integers;

(ii) for $1 \leq r_1 < r_2 \leq n$, the transformations T_{r_1} and T_{r_2} are not identical on $|v_0, \cdots, v_p|$;

(iii) the points $v_0, \cdots, v_p$ are distinct. The p-cell $(v_0, \cdots, v_p)$ in K (see §1.11) is called the base for c_p^R, and the notation $c_p^R = c_p^R(v_0, \cdots, v_p)$ is used when it is desirable to display the base.

2.8. LEMMA. *Suppose that c_p^R is an elementary n-chain in R for which $\beta_p^R c_p^R = 0$. Then $\beta_{p+1}^R \rho_{*p}^R c_p^R = 0$.*

Proof. With the notation of §2.7, one finds (see §§2.1, 2.2).

$$\text{(i)}\quad \beta_p^R c_p^R = \sum_{\tau_p \in T_{p0}} \sum_{r=1}^{n} m_r(0_{p+1}\, b_{p0}\, \tau_p(v_0, \cdots, v_p), T_r)^R = 0;$$

$$\text{(ii)}\quad \beta_{p+1}^R \rho_{*p}^R c_p^R = \sum_{\tau_{p+1} \in T_{p+1\,0}} \sum_{k=0}^{p} \sum_{\tau_p \in T_{pk}} \sum_{r=1}^{n} m_r(0_{p+2}\, b_{p+1\,0}\, \tau_{p+1}\, b_{pk}\, \tau_p(v_0, \cdots, v_p), T_r)^R .$$

In view of §2.7 (iii), and §1.11, it follows from (i) that for each $\tau_p' \in T_{p0}$, one has

$$\text{(iii)}\quad \sum_{r=1}^{n} m_r(0_{p+1}\, b_{p0}\, \tau_p'(v_0, \cdots, v_p), T_r)^R = 0 \qquad (\tau_p' \in T_{p0}),$$

Fix

$$\tau_{p+1} \in T_{p+10}, \ \tau_p \in T_{pk} \qquad (0 \leq k \leq p).$$

From the lemma in §1.14 follows the existence of a $\tau_p' \in T_{p0}$ such that

$$\text{(iv)} \quad |0_{p+2}\, b_{p+10}\, \tau_{p+1}\, b_{pk}\, \tau_p(v_0, \cdots, v_p)| \subset |0_{p+1}\, b_{p0}\, \tau_p'(v_0, \cdots, v_p)|.$$

From (iii) and (iv) one concludes that for each

$$\tau_{p+1} \in T_{p+10}, \ \tau_p \in T_{pk} \qquad (0 \leq k \leq p),$$

we have

$$\text{(v)} \quad \sum_{r=1}^{n} m_r\,(0_{p+2}\, b_{p+10}\, \tau_{p+1}\, b_{pk}\, \tau_p(v_0, \cdots, v_p),\, T)^R = 0.$$

In view of (ii) and (v) the lemma is now established.

2.9. For integers $p \geq 0$, the class N_p^R is defined to be that subset of C_p^R composed of the chain $0 \in C_p^R$ and of all c_p^R having a representation of the form

$$c_p^R = \sum_{s=1}^{n} c_{ps}^R\,(v_{0s}, \cdots, v_{ps})$$

where

(i) for $1 \leq s \leq n$ the $c_{ps}^R(v_{0s}, \cdots, v_{ps})$ are elementary n-chains (see 2.7);

(ii) for $1 \leq s_1 < s_2 \leq n$, the point sets $v_{0s_1}, \cdots, v_{ps_1}$ and $v_{0s_2}, \cdots, v_{ps_2}$ are distinct. For $p < 0$, the class N_p^R consists of the chain $0 \in C_p^R$ alone. Each of the elementary n-chains $c_{ps}^R(v_{0s}, \cdots, v_{ps})$ $(1 \leq s \leq n)$, is termed a n-composant of c_p^R. Observe that the sets N_p^R are *not* generally subgroups of C_p^R.

2.10. LEMMA. *Let*

$$c_p^R = \sum_{s=1}^{n} c_{ps}^R\,(v_{0s}, \cdots, v_{ps})$$

be any nonzero element in N_p^R. *A necessary and sufficient condition in order that*

$\beta_p^R\, c_p^R = 0$ *is that* $\beta_p^R\, c_{ps}^R = 0$ *for every n-composant* c_{ps}^R $(1 \leq s \leq n)$.

Proof. Trivially the condition suffices. It is presently shown to be necessary. With explicit notations (see §§2.7, 2.9),

$$\beta_p^R\, c_p^R = \sum_{s=1}^{n} \beta_p^R\, c_{ps}^R = \sum_{s=1}^{n} \sum_{r=1}^{n_s} m_{rs}\,(\beta_p(v_{0s}, \cdots, v_{ps})\, T_{rs})^R$$

$$= \sum_{s=1}^{n} \sum_{r=1}^{n_s} \sum_{\tau_p \in T_{p0}} m_{rs}\,(0_{p+1}\, b_{p0}\, \tau_p(v_{0s}, \cdots, v_{ps}),\, T_{rs})^R = 0\,.$$

In view of §2.9 (ii) and of the remarks in §1.11, it is clear (see §0.2) that, for $1 \leq s \leq n$ we have

$$\beta_p^R\, c_{ps}^R = \sum_{r=1}^{n_s} \sum_{\tau_p \in T_{p0}} m_{rs}(0_{p+1}\, b_{p0}\, \tau_p(v_{0s}, \cdots, v_{ps}),\, T_{rs})^R = 0$$

and hence the assertion in the lemma is verified.

2.11. LEMMA. *Let* c_p^R *be any element in* N_p^R *for which* $\beta_p^R\, c_p^R = 0$. *Then*

$$\beta_{p+1}^R\, \rho_{*p}^R\, c_p^R = 0\,.$$

This result is an immediate consequence of the lemmas in §§2.8, 2.10.

2.12. LEMMA. *Every chain* c_p^R *has a representation of the form* (see §§2.3, 2.4, 2.9)

$$c_p^R = t_p^R + d_p^R + n_p^R \qquad (t_p^R \in T_p^R,\ d_p^R \in D_p^R,\ n_p^R \in N_p^R)\,.$$

Generally this representation is not unique.

Proof. The nonuniqueness of the representation will be evident from the proof of its existence which follows. For chains $c_p^R = 0 \in C_p^R$, the result is trivial, so assume that $c_p^R \neq 0$. Then c_p^R has a unique representation of the form

$$\text{(i)}\quad c_p^R = \sum_{j=1}^{n} m_j(v_{0j}, \cdots, v_{pj},\, T_j)^R\,,$$

where the m_j are nonzero integers and the p-cells $(v_{0j_1}, \cdots, v_{pj_1}, T_{j_1})^R$ and $(v_{0j_2}, \cdots, v_{pj_2}, T_{j_2})^R$ are distinct for $1 \leq j_1 \leq j_2 \leq n$. The proof is made by an induction on n. If $n = 1$, then $c_p^R = m_1(v_{01}, \cdots, v_{p1}, T_1)^R$. If, for some integers i, k such that $0 \leq i < k \leq p$, one finds $v_{i1} = v_{k1}$, then the fact that c_p^R has a representation of the prescribed form follows from the lemma in §2.6. On the other hand, if all the $v_{01}, \cdots, v_{p1}$ are distinct, then c_p^R is an elementary n-chain (see §2.7). Thus the lemma is established in case $n = 1$. Suppose that the lemma is true for all chains c_p^R having a representation of the form (i) with at most $n = N - 1$ terms ($N > 1$). For chains c_p^R whose representations (i) have N terms it is convenient to consider several cases.

Case 1. Assume there is some term in the representation (i) of c_p^R — without loss of generality one may assume it to be the first — for which there are integers i, k such that $0 \leq i < k \leq p$ and $v_{i1} = v_{k1}$. By the lemma in §2.6 there are elements t_{p1}^R in T_p^R and d_{p1}^R in D_p^R such that

$$m_1(v_{01}, \cdots, v_{p1}, T_1)^R = t_{p1}^R + d_{p1}^R.$$

By assumption there are elements t_{p2}^R in T_p^R, d_{p2}^R in D_p^R, and n_p^R in N_p^R such that

$$\sum_{j=2}^{N} m_j(v_{0j}, \cdots, v_{pj}, T_j)^R = t_{p2}^R + d_{p2}^R + n_p^R.$$

Thus

$$c_p^R = (t_{p1}^R + t_{p2}^R) + (d_{p1}^R + d_{p2}^R) + n_p^R,$$

and since T_p^R and D_p^R are subgroups of C_p^R, the existence of a representation of the prescribed form for c_p^R follows in Case 1.

Case 2. Assume that for each j ($1 \leq j \leq N$) the $v_{0j}, \cdots, v_{pj}$ are distinct. By rearranging terms one may obtain from (i) a representation of the form

$$\text{(ii)}\quad c_p^R = \sum_{s=1}^{m} \sum_{r=1}^{n_s} m_{rs}(v_{0s}, \cdots, v_{ps}, T_{rs})^R, \quad \sum_{s=1}^{m} n_s = N,$$

satisfying these conditions: none of the m_{rs} is zero; for the same s ($1 \leq s \leq m$), $1 \leq r_1 < r_2 \leq n_s$, the mappings $T_{r_1 s}$ and $T_{r_2 s}$ are not identical on $|v_{0s}, \cdots, v_{ps}|$; for $1 \leq s_1 < s_2 \leq m$, the p-cells $(v_{0s_1}, \cdots, v_{ps_1})$ and $(v_{0s_2}, \cdots, v_{ps_2})$ are distinct in K (see §1.1). Now for each s ($1 \leq s \leq m$) clearly each of the chains

$$c_{ps}^R = \sum_{j=1}^{n_s} m_{rs}(v_{0s}, \cdots, v_{ps}, T_{rs})^R$$

is an elementary n-chain in R (see §2.7). The proof is carried forth by an inductive reasoning on m. If $m = 1$ then c_p^R is an elementary n-chain in R, and the representation (ii) already has the prescribed form. So assume that c_p^R, whose representation (i) has at most N terms, has a representation of the prescribed form whenever its representation (ii) has at most $m = M - 1$ terms ($M > 1$). Suppose now that C_p^R is a chain whose representation (i) has N terms while its representation (ii) has M terms

$$\sum_{s=1}^{M} n_s = N$$

Subcase 2.1. Assume that for $1 \leq s_1 < s_2 \leq M$ the point sets $v_{0s_1}, \cdots, v_{ps_1}$ and $v_{0s_2}, \cdots, v_{ps_2}$ are distinct. From §2.9 it is clear that c_p^R is itself an element in N_p^R and representation (ii) has the prescribed form.

Subcase 2.2. Assume that there are distinct integers s — with no loss of generality one may assume these to be $s = 1$ and $s = 2$ — such that the sets $v_{01}, \cdots, v_{p1}$ and $v_{02}, \cdots, v_{p2}$ are the same. It follows that the sequence $v_{02}, \cdots, v_{p2}$ is obtainable from $v_{01}, \cdots, v_{p1}$ by a positive number l of transpositions. Hence by the lemma in §2.5 there exists for each r in $1 \leq r \leq n_1$ an element t_{pr}^R in T_p^R such that

$$(v_{01}, \cdots, v_{p1}, T_{r1})^R = (-1)^l (v_{02}, \cdots, v_{p2}, T_{r1})^R + t_{pr}^R \qquad (1 \leq r \leq n_1).$$

Since T_p^R is a subgroup of C_p^R, the chain

$$\sum_{r=1}^{n_1} m_{r1} t_{pr}^R$$

is an element t_{p*}^R in T_p^R. Consequently,

$$c_p^R = t_{p*}^R + \left[\sum_{r=1}^{n_1} (-1)^l m_{r1} (v_{02}, \cdots, v_{p2}, T_{r1})^R \right.$$
$$\left. + \sum_{s=2}^{M} \sum_{r=1}^{n_s} m_{rs} (v_{0s}, \cdots, v_{ps}, T_{rs})^R \right].$$

Clearly the terms in square brackets may be rearranged into the form (ii) with an integer $m \leq M - 1$, and their representation in form (i) has an integer $n \leq N$. By the inductive assumption there are elements $t_{p\#}^R$ in T_p^R, d_p^R, in D_p^R and n_p^R in

N_p^R such that $c_p^R = (t_{p*}^R + t_{p\#}^R) + d_p^R + n_p^R$, and the existence of a representation of the prescribed form for c_p^R now follows in Case 2. Indeed, it is obvious in this case that $d_p^R = 0 \in C_p^R$. So the lemma is completely established.

2.13. LEMMA. *If c_p^R is any chain in C_p^R for which $\beta_p^R c_p^R = 0$, then*

$$\beta_{p+1}^R \rho_{*p}^R c_p^R = 0.$$

The proof follows at once from the lemmas in §§2.3, 2.4, 2.11, 2.12.

RESULTS

3.1. In [3, §4.1] (see also [4, §8]) Radó has established a lemma from which one derives the following statement by replacing the barycentric homotopy operator ρ_p^R by the modified barycentric homotopy operator ρ_{*p}^R (see §§1.2, 2.1).

LEMMA. *Let $\{G_p\}$ be an identifier for R* (see §0.3) *such that the following conditions hold:*

(i) *$c_p^R \in G_p$ implies that $\beta_p^R c_p^R = 0$;*

(ii) *$c_p^R \in G_p$ implies that $\rho_{*p}^R c_p^R \in G_{p+1}$.*

Then $\{G_p\}$ is unessential.

3.2. For each integer p let $N(\beta_p^R)$ be the nucleus of the homomorphism $\beta_p^R\colon C_p^R \longrightarrow C_p^R$ (see §2.1). Since β_p^R is a chain mapping (see §2.2) it is clear that the nuclei $N(\beta_p^R)$ constitute an identifier for R (see §0.3). Now in view of the lemma in §2.13, conditions (i) and (ii) of the lemma above are clearly fulfilled for the identifier $\{N(\beta_p^R)\}$, and furthermore, this choice of an identifier yields the maximum amount of information that may be obtained from that lemma. Thus the $\{N(\beta_p^R)\}$ constitute an unessential identifier for R, and one of the main results is now established (see §0.4). It is summarized in the following statement.

THEOREM. *The system of nuclei $N(\beta_p^R)$ of the barycentric homomorphisms $\beta_p^R\colon C_p^R \longrightarrow C_p^R$ constitutes an unessential identifier for R.*

3.3. In order to compare this result with those in Radó [3; 4], first observe that it follows from the lemmas in §§2.3, 2.4 that

$$N(\beta_p^R) \supset T_p^R + D_p^R \qquad (-\infty < p < +\infty).$$

Moreover, since C_p^R is a free group, it is clear that the division hull of $N(\beta_p^R)$

must be identical with the group $N(\beta_p^R)$. Thus the group $N(\beta_p^R)$ also contains the the division hull of the group $T_p^R + D_p^R$ for all integers p. An example is now given to show that the group $N(\beta_p^R)$ generally contains more.

3.4. Denote by d_0, d_1, d the points (1, 0, 0, ⋯), (0, 1, 0, 0, ⋯), (1/2, 1/2, 0, 0, ⋯) respectively, let X be Euclidean x-space, and define transformations by the following relations:

$$T_1\colon x = v_0 - 1/2 \qquad (v \in |d_0, d_1|);$$

$$T_2\colon x = \begin{cases} 0 & (v \in |d_0, d\;|); \\ v_0 - 1/2 & (v \in |d\;, d_1|); \end{cases}$$

$$T_3\colon x = \begin{cases} v_0 - 1/2 & (v \in |d_0, d\;|); \\ 0 & (v \in |d\;, d_1|); \end{cases}$$

$$T_4\colon x = 0 \qquad (v \in |d_0, d_1|).$$

Clearly

$$c_1^R = (d_0, d_1, T_1)^R - (d_0, d_1, T_2)^R - (d_0, d_1, T_3)^R + (d_0, d_1, T_4)^R$$

belongs to C_1^R and $\beta_1^R c_1^R = 0$. Moreover, c_1^R is an elementary n-chain (see §2.7). An elementary reasoning shows that it cannot belong to the division hull for the group $T_1^R + D_1^R$.

3.5. In order to describe the largest unessential identifier for R obtained by Radó, a further definition is needed. For integers $p \geq 0$, let $(v_0, \cdots, v_p, T)^R$ be any p-cell in R (see §0.1). Let $w_0, \cdots, w_p$ be any set sequence of $p + 1$ linearly independent points in E_∞. Then there is a linear mapping

$$\alpha\colon |w_0, \cdots, w_p| \longrightarrow |v_0, \cdots, v_p$$

such that $\alpha(w_i) = v_i$ for $0 \leq i \leq p$. The p-chain

$$c_p^R = (v_0, \cdots, v_p, T)^R - (w_0, \cdots, w_p, T\alpha)^R$$

is termed an *elementary a-chain* in R (see [3, §3.4]), and the subgroup of C_p^R generated by the elementary a-chains is denoted by A_p^R. For $p < 0$, A_p^R consists of the zero element alone. In [3, §3.4] Radó has a simple characterization for the group A_p^R which he uses to define the group in [4, §7].

3.6. For each integer p, put $\Gamma_p^R = A_p^R + D_p^R + T_p^R$ (see §§2.3, 2.4, 3.5), and let $\hat{\Gamma}_p^R$ denote the division hull of Γ_p^R. Then Radó shows that $\{\hat{\Gamma}_p^R\}$ is an unessential identifier in R (see [3, §4.7] or [4, §9]), and this is his best result. If one sets $\Delta_p^R = A_p^R + N(\beta_p^R)$ (see §3.2) and lets $\hat{\Delta}_p^R$ denote the division hull of Δ_p^R, then clearly $\Delta_p^R \supset \Gamma_p^R$, and hence $\hat{\Delta}_p^R \supset \hat{\Gamma}_p^R$. If one modifies the reasoning of Radó in [3, §4] by replacing the barycentric homotopy operator ρ_p^R by the modified barycentric homotopy operator ρ_{*p}^R (see §2.1), one finds that $\hat{\Delta}_p^R$ is an unessential identifier for R. Thus one obtains the following result.

THEOREM. *If* $\hat{\Delta}_p^R$ *is the division hull of the group* $A_p^R + N(\beta_p^R)$ *then the system* $\{\hat{\Delta}_p^R\}$ *is an unessential identifier for* R.

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OHIO STATE UNIVERSITY