SET FUNCTIONS ASSOCIATED WITH LEBESGUE AREA

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1. **Introduction.** It will be convenient, for the purposes of this paper, to define Lebesgue area as in [6]. This definition is applicable to surfaces in metric spaces and is equivalent to the usual definition for surfaces in Euclidean space.

There is described in [7] a method for generating from a continuous function defined on a square into a metric space, a monotone function with range in m, the space of bounded sequences [1], having the same Lebesgue area. A corresponding remark holds for Fréchet surfaces instead of continuous functions.

Suppose that $\mathcal{S}$ is a surface and that $\mathcal{S}'$ is obtained from $\mathcal{S}$ by the procedure referred to. The purpose of this paper is to show the existence of set functions whose values on elementary configurations of m agree with the elementary areas of the configurations, and whose values on the point set occupied by $\mathcal{S}'$ are equal to the Lebesgue area of $\mathcal{S}$. We shall give definitions for two such set functions; for one of these functions it is necessary to assume that the Lebesgue area of $\mathcal{S}$ is finite in order to be sure that the equality holds.

The set functions can be interpreted for subsets of E_n. If a surface in E_n admits a monotone representation, then the value of each of the set functions on the point set carrying the surface is, with the proviso mentioned above, equal to the Lebesgue area of the surface.

2. **Preliminary remarks.** The definition of Lebesgue area which we shall use is given in this section. We shall see that there is no loss in generality in supposing that all of the surfaces with which we shall be concerned are in m.

We list here some definitions and notations that will be used in the sequel. If D is a domain in the plane and if a subset $\mathbb{D}$ of some topological space is homeomorphic to D, then $\mathbb{D}$ is a *2-domain.* If E is a set, then $\overline{E}$, E^0, and E^* will denote its closure, interior, and boundary, respectively. We reserve the letters f and g to represent linear functionals on m of norm one. For fixed f and g, π^{fg} is the transformation from m to E_2 defined by $\pi^{fg}(a) = (f(a), g(a))$ for each $a \in m$. If a, b, and c are the vertices of a triangle in m, then the area of the triangle is, by definition,

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$$(1/2) \sup_{f,g} \begin{vmatrix} f(a) & g(a) & 1 \\ f(b) & g(b) & 1 \\ f(c) & g(c) & 1 \end{vmatrix}$$

where, as indicated above, f and g are linear functionals of norm one.

If $\mathcal{S}$ is a surface, then $L(\mathcal{S})$ and $P(\mathcal{S})$ are the Lebesgue and Peano areas of $\mathcal{S}$. Similarly, $L(x)$ and $P(x)$ are the corresponding areas of x, where x is a suitable continuous function. If $\mathbb{D}$ is a 2-domain in m, ϕ is a transformation from $\overline{\mathbb{D}}$ to E_2, and $z \in E_2$, then $M(\phi, z)$ is the value of Federer's multiplicity function M determined by ϕ at z [4].

We denote the unit square by Q and the surface of the unit sphere by Σ. Let ψ be a transformation from Q onto Σ which is topological on Q^0 and constant on Q^*. If x is continuous on Q and constant on Q^*, then $\overline{x}$ is defined on Σ by $\overline{x}(\psi(q)) = x(q)$. In general, the composition of two functions F and G is denoted by $F * G$. Hence we can write $x = \overline{x} * \psi$. If D is contained in the domain of a function F, then $F \mid D$ is the function F restricted to D.

We recall here the definitions of Peano and Lebesgue areas [6].

Let x be continuous on Q into m.

If D is a domain contained in Q^0, and

$$G(x, D) = \sup_{f,g} \iint_{E_2} M(\pi^{fg} * x \mid \overline{D}, z)\, dz,$$

then

$$P(x) = \sup_{\sigma} \sum_{D \in \sigma} G(x, D),$$

where σ is a finite disjoint family of domains contained in Q^0.

We define the Peano area of a function defined on Σ in an analogous manner. For convenience we suppose that σ does not consist of Σ alone.

In [6], Lebesgue area was defined by means of quasilinear functions. It can be shown, as in [5], that this definition is equivalent to that given below.

A polyhedron is a surface which admits a representation x on Q such that there exists a curvilinear triangulation of Q consisting of curvilinear triangles $t_1, \cdots, t_n$, and for each $i\,(i = 1, \cdots, n)$,

(i) $x \mid t_i$ is topological,

(ii) $x(t_i)$ is a nondegenerate triangle (in m), the vertices of $x(t_i)$ being the images of the vertices of t_i.

If $\mathfrak{P}$ is a polyhedron in m, then it can be shown, just as the corresponding statement is proved for polyhedra in E_n, that $P(\mathfrak{P})$ is equal to the elementary area of $\mathfrak{P}$.

If $\mathfrak{S}$ is a surface (in m), then

$$L(\mathfrak{S}) = \liminf_{D(\mathfrak{P},\mathfrak{S}) \to 0} \left\{ \text{elementary area of } \mathfrak{P} \right\},$$

where $D(\mathfrak{P}, \mathfrak{S})$ is the Fréchet distance between $\mathfrak{P}$ and $\mathfrak{S}$.

The Lebesgue area of a surface represented on Σ is defined in an analogous manner.

If $\mathfrak{S}''$ is a surface in a metric space $\mathfrak{M}$, then there exists a surface $\mathfrak{S}'$ in m which is isometric to $\mathfrak{S}''$ (if x'' is a representation of $\mathfrak{S}''$, then there exists a representation x' of $\mathfrak{S}'$ such that $||x'(p) - x'(q)|| = d(x''(p), x''(q))$, the distances being in m and $\mathfrak{M}$ respectively). We define $L(\mathfrak{S}'')$ to be $L(\mathfrak{S}')$. The definition is valid since Lebesgue area in m satisfies Kolmogoroff's principle. As noted earlier, it is this definition of Lebesgue area which we shall use.

3. **The set function** μ. We make the following definitions.

DEFINITION 3.1. Let $S' \subset m$, and let σ' be a finite family of disjoint 2-domains contained in S'. Then

$$\mu(S') = \sup_{\sigma'} \sum_{\mathfrak{D} \in \sigma'} \mu'(\mathfrak{D})$$

where

$$\mu'(\mathfrak{D}) = \sup_{f, g} \iint_{E_2} M(\pi fg \mid \overline{\mathfrak{D}}, z)\, dz.$$

(If σ' is empty, then $\sum_{\mathfrak{D} \in \sigma'} \mu'(\mathfrak{D}) = 0$.)

DEFINITION 3.2. A compact subset of m consisting of a denumerable number of points, line-segments, triangles, and so on, is an *elementary configuration* in m. If E is an elementary configuration, then $|E|$ is the elementary area of E.

We observe that if S' is the monotone image of a 2-cell, then μ is additive with respect to its cyclic elements. If E is an elementary configuration, then $\mu(E) = |E|$.

Now let x be continuous on Q into m. Define $\tilde{x}$ on $Q \times Q$ by

$$\tilde{x}(p, q) = \inf \operatorname{diam} x(\zeta([0,1]))$$

for all continuous functions ζ defined on [0,1] such that $\zeta(0) = p, \zeta(1) = q$, and range $\zeta = \zeta([0,1])$ is contained in Q. If $\{p_i\}$ is an everywhere dense sequence of points in Q, define x' on Q into m by $x'(q) = \{\tilde{x}(p_i, q)\}$. Then x' is monotone and $L(x') = L(x)$ [7].

It is necessary, for our purposes, to extend this result to the case where x is defined on Σ. If $p \in \Sigma$, $q \in \Sigma$, then ζ is admissible if range $\zeta \subset \Sigma$. Then $\tilde{x}$ and x' are defined as before. In order to show that $L(x) = L(x')$, let y be defined on Q by $y = x * \psi$. The reader can verify that $x = \bar{y}$ and $(\bar{y})' = \bar{y}'$. We assume the results of Lemma 3.2 to obtain $L(\bar{y}) = L(y)$ and $L(\bar{y}') = L(y')$. Thus we have $L(x) = L(\bar{y}) = L(y) = L(y') = L(\bar{y}') = L(x')$.

LEMMA 3.1. *If x is light on Q (or Σ), and D is a domain in Q^0 (a 2-domain in Σ), then $M(\pi^{fg} * (x \mid D), z) = M(\pi^{fg} \mid x'(D), z)$.*

The proof is almost evident since x', being monotone as well as light, is a homeomorphism.

LEMMA 3.2. *If x is continuous on Q and constant on Q^*, then $P(\bar{x}) = P(x)$ and $L(\bar{x}) = L(x)$.*

The argument of Cesari in [3] is valid here.

We are now in a position to compare $\mu(x'(Q)$ with $P(x')$.

LEMMA 3.3. *If x is light on Q or light on Q^0 and constant on Q^*, then $\mu(x'(Q)) = P(x')$.*

The result follows from Lemma 3.1 or Lemmas 3.1 and 3.2.

LEMMA 3.4. *If x is continuous on Q into m, then $P(x') = \mu(x'(Q))$.*

Proof. The equality is a result of the cyclic additivity of P and μ, the preceding lemma, and the following three statements [5]:

(i) If $x'(Q)$ is a 2-cell, then there exists a light function y' which is Fréchet equivalent to x'.

(ii) If $x'(Q)$ is a 2-sphere, then there exists a function y' which is Fréchet equivalent to x' and which is light in Q^0 and constant on Q^*.

(iii) If $D(x', y') = 0$, then $x'(Q) = y'(Q)$.

LEMMA 3.5. *If $P(x) = L(x)$ and $P(x') \geqq P(x)$, then $L(x) = \mu(x'(Q))$.*

Proof. We have $P(x') \leqq L(x') = L(x) = P(x) \leqq P(x')$, and so $L(x) = P(x')$. The result now follows from the preceding lemma.

Suppose that we use, instead of x', the function x'' defined by $x''(q) = \{z^k(q)\}$ for $q \in Q$, where

$$z^k(q) = x'^{(k+1)/2}(q) \qquad \text{if } k \text{ is odd},$$

$$z^k(q) = x^{k/2}(q) \qquad \text{if } k \text{ is even}.$$

It is evident that $\|x''(p) - x''(q)\| = \|x'(p) - x'(q)\|$ for $p, q \in Q$, and so $L(x'') = L(x') = L(x)$. It is not hard to see that $P(x') \geqq P(x)$. We can now summarize our results.

THEOREM 3.1. *If $L(x) < +\infty$, then $L(x) = \mu(x'(Q))$. If $P(x) = L(x)$, then $L(x) = \mu(x''(Q))$. If x has range in E_3, then $L(x) = \mu(x''(Q))$.*

The last statement of the theorem results when we observe that we may consider E_3 as a subspace of m, and then use the fact that $P(x) = L(x)$ [2].

4. **The set function λ.** In defining λ we leaned heavily on Peano area. We shall define a (possibly) new set function λ by relying upon Lebesgue area.

Let $\mathcal{E}$ be the family of elementary configurations of m, and let $\mathfrak{F}$ be the family of subsets of m each homeomorphic to an element of $\mathcal{E}$.

In each class of homeomorphic subsets of m we introduce a metric d defined by

$$d(A, B) = \inf_h \sup_{p \in A} \|p - h(p)\|,$$

where h is a homeomorphism of A onto B.

LEMMA 4.1. *For each $F \in \mathfrak{F}$ there exists a sequence $\{E_n\}$, $E_n \in \mathcal{E}$, such that $d(E_n, F) \to 0$.*

Proof. There exists a set $E \in \mathcal{E}$ and a homeomorphism h such that $F = h(E)$. Since E is compact, h is uniformly continuous and so, for $\epsilon > 0$, there exists a $\delta > 0$ such that $\|h(p) - h(q)\| < \epsilon$ if $\|p - q\| < \delta$. Let T_n be a triangulation of E of mesh less than δ. Define h_n on E by putting $h_n(p) = h(p)$ if p is a vertex of T_n and extending h_n to be linear on each k-simplex of T_n. Then if $q \in E$, and p is a vertex of a k-simplex of T_n containing q, we have

$$\|h(q) - h_n(q)\| \leqq \|h(q) - h(p)\| + \|h(p) - h_n(p)\| + \|h_n(p) - h_n(q)\| < \epsilon + 0 + \epsilon = 2\epsilon.$$

We now let $E_n = h_n(E)$.

DEFINITION 4.1. Define λ' on $\mathfrak{F}$ by

$$\lambda'(F) = \liminf_{\substack{d(E,F)\to 0\\ E\in\mathcal{E}}} |E|,$$

and λ on subsets of m by

$$\lambda(S) = \sup_{\sigma} \sum_{F\in\sigma} \lambda'(F),$$

where σ is a finite family of disjoint subsets of S and $\sigma \in \mathfrak{F}$.

We note the following result.

LEMMA 4.2. *If $d(A_n, A) \to 0$, then $\lambda(A) \leqq \liminf_{n\to\infty} \lambda(A_n)$.*

We require the following information from [6]. If π is a plane (in m), then there exists a projection π^* of m onto π such that $|\pi^*(\Delta)| \leqq |\Delta|$ for each triangle Δ. Furthermore, if p and q are points of m then $||\pi^*(p) - \pi^*(q)|| \leqq 2||p - q||$.

LEMMA 4.3. *If $E \in \mathcal{E}$, and σ is a finite family of disjoint subsets F of E, each $F \in \mathfrak{F}$, then $\sum_{F\in\sigma} \lambda(F) \leqq |E|$.*

The proof will be sketched for the special case where E consists of a single triangle T in a plane π. Let $F \in \sigma$. By virtue of the remarks preceding this lemma, we may assume, without loss of generality, that there exists a sequence $\{E_n\}$, $E_n \in \mathcal{E}$, with $E_n \subset \pi$, $d(E_n, F) \to 0$, and $|E_n| \to \lambda'(F)$. If $\tilde{E} \in \mathcal{E}$, and $\tilde{E} \subset F^0$, then $\tilde{E} \subset E_n$ for sufficiently large n. Hence $\lambda'(F) \geqq$ Lebesgue measure of F^0 [6].

Let $\sigma' \subset \mathcal{E}$ be a family of disjoint subsets of F. Then it is easy to see that $\sum_{F'\in\sigma'} \lambda'(F') \leqq$ Lebesgue measure of F^0, and so $\lambda(F) =$ Lebesgue measure of F^0. The lemma results for the special case considered.

The general case follows when the lemma has been proved under the assumption that E consists of a finite number of triangles.

LEMMA 4.4. *If $F \in \mathfrak{F}$, then $\lambda(F) = \lambda'(F)$.*

Proof. It is sufficient to show that $\lambda(F) \leqq \lambda'(F)$. Let σ be as in Definition 4.1. If $d(E_n, F) \to 0$ such that $|E_n| \to \lambda'(F)$, then there exist homeomorphisms h_n of F onto E_n with $d(F_n^k, F^k) \to 0$, where $F^k \in \sigma$ and $F_n^k = h_n(F^k)$. We use the preceding lemma and the lower semi-continuity of λ to obtain $\sum_{F^k\in\sigma} \lambda(F_n^k) \leqq |E_n|$ and $\lambda(F^k) \leqq \liminf_{n\to\infty} \lambda(F_n^k)$. Consequently

$$\sum_{F^k \in \sigma} \lambda(F^k) \leqq \sum_{F^k \in \sigma} \left\{ \liminf_{n \to \infty} \lambda(F_n^k) \right\}$$

$$\leqq \liminf_{n \to \infty} \sum_{F^k \in \sigma} \lambda(F_n^k) \leqq \liminf_{n \to \infty} |E_n| = \lambda'(F),$$

and the lemma follows.

LEMMA 4.5. *If x is light on Q, then $L(x) = \lambda(x'(Q))$.*

Proof. If $\mathcal{S}'$ is the (Fréchet) surface determined by x', then there exists a sequence of polyhedra $\mathcal{P}_n$ such that $D(\mathcal{P}_n, \mathcal{S}') \to 0$ and {elementary area of $\mathcal{P}_n$} $\to L(\mathcal{S}') = L(x')$. We may suppose that $\mathcal{P}_n$ has a topological representation π_n on Q. Since $\pi_n(Q) \in \mathcal{E}$ for each n, and $d(\pi_n(Q), x'(Q)) = D(\mathcal{P}_n, \mathcal{S}') \to 0$, we see that

$$\lambda(x'(Q)) = \lambda'(x'(Q)) \leqq \liminf_{n \to \infty} |\pi_n(Q)| = L(x') = L(x).$$

Now if $E_n \to x'(Q)$ so that $|E_n| \to \lambda'(x'(Q))$, then we define x_n' on Q by $x_n' = h_n * x'$, where h_n is a homeomorphism of $x'(Q)$ onto E_n such that $\sup_{p \in x'(Q)} ||p - h_n(p)|| < d(x'(Q), E_n) + 1/n$. It is clear that x_n' represents a polyhedron $\mathcal{P}_n'$ whose elementary area is $|E_n|$. Also, $D(x_n', x') < d(E_n, x'(Q)) + 1/n$. Therefore $D(x_n', x') \to 0$ and

$$L(x') \leqq \liminf_{n \to \infty} L(x') = \liminf_{n \to \infty} |E_n| = \lambda'(x'(Q)) \leqq \lambda(x'(Q)).$$

LEMMA 4.6. *If x is light in Q^0 and constant on Q^*, then $L(x) = \lambda(x'(Q))$.*

The proof that $L(\bar{x}) = \lambda(\bar{x}'(\Sigma))$ is similar to that of Lemma 4.5. The lemma results from the observation that $L(x) = L(\bar{x}) = \lambda(\bar{x}'(\Sigma)) = \lambda(x'(Q))$.

THEOREM 4.1. *We have $L(x) = \lambda(x'(Q))$.*

The proof of this theorem is similar to that of Lemma 3.4.

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