SOME HYPERGEOMETRIC IDENTITIES

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1. Introduction. T. W. Chaundy [3] has given some hypergeometric identities of which the most general is

$$(1) \qquad F(a, b; c; x) = h \sum_{n=0}^{\infty} \frac{(h - \alpha n + 1)_{n-1} (e)_n}{n! \; (c)_n}$$

$$\times \; {}_4F_3 \left[\begin{matrix} a, \; b, \; 1 + h(1-\alpha)^{-1}, \; -n \\ e, \; h(1-\alpha)^{-1}, \; h - \alpha n + 1 \end{matrix} \right] (-x)^n \; F(e+n, \; h + (1-\alpha)n; \; c+n; \; x).$$

In this paper we give a generalisation of (1), namely,

$$(2) \qquad {}_{p+s}F_{q+t} \left[\begin{matrix} a_p, \; b_s; \\ c_q, \; d_t; \end{matrix} \; x \right] = h \sum_{n=0}^{\infty} \frac{(h - \alpha n + 1)_{n-1}}{n!} \frac{(b_s)_n \, (e_q)_n}{(d_t)_n \, (c_q)_n}$$

$$\times \; {}_{p+2}F_{q+2} \left[\begin{matrix} a_p, \; 1 + h(1-\alpha)^{-1}, \; -n \\ e_q, \; h(1-\alpha)^{-1}, \; h - \alpha n + 1 \end{matrix} \right] (-x)^n$$

$$\times \; {}_{s+q+1}F_{t+q} \left[\begin{matrix} b_s + n, \; e_q + n, \; h + (1-\alpha)n; \\ d_t + n, \; c_q + n; \end{matrix} \; x \right],$$

where $(h - \alpha n + 1)_{-1}$ means $(h - \alpha n)^{-1}$ and a_λ, $(a_\lambda)_n$, $a_\lambda + n$ denote $a_1 \cdots, a_\lambda$; $(a_1)_n \cdots (a_\lambda)_n$; and $a_1 + n, \cdots, a_\lambda + n$, respectively; and from (2), we deduce some other identities.

2. Proof of (2). The following is a simple extension of Dr. Chaundy's proof. Comparing the coefficients in (2) of $(a_p)_N / N!$, we have to prove that

$$\frac{(b_s)_N \, x^N}{(c_q)_N \, (d_t)_N} = \{ h + (1-\alpha)N \} \sum_{n=N}^{\infty} \frac{(h - \alpha n + 1)_{n-1} \, (b_s)_n \, (e_q)_n \, (-n)_N}{n! \, (d_t)_n \, (c_q)_N \, (e_q)_N \, (h - \alpha n + 1)_N} (-x)^n$$

$$\times \; {}_{s+q+1}F_{t+q} \, .$$

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Writing $n = N + r$, we find that this reduces to

$$1 = \{h + (1 - \alpha) N\} \sum_{r=0}^{\infty} \frac{[h + (1-\alpha) N + 1 - \alpha r]_{r-1} (b_s + N)_r (e_q + N)_r}{(d_t + N)_r (c_q + N)_r \; r!} (-x)^r$$

$$\times \; {}_{s+q+1}F_{t+q} \left[\begin{matrix} b_s + N + r, \; e_q + N + r, \; h + (1-\alpha)(N+r); \\ d_t + N + r, \; c_q + N + r; \end{matrix} \; x \right]$$

The term independent of x on the right is unity. It remains to be proved that the coefficient of any positive power of x vanishes on the right, that is, when $M > 0$,

$$\frac{(b_s + N)_M (e_q + N)_M}{(d_t + N)_M (c_q + N)_M} \sum_{r=0}^{M} (-1)^r \frac{[h + (1-\alpha) N + 1 - \alpha r]_{M-1}}{r! \, (M-r)!} = 0.$$

But this is the coefficient of x^{M-1} in

$$\frac{(b_s + N)_M (e_q + N)_M}{M (d_t + N)_M (c_q + N)_M} (1-x)^{-h-(1-\alpha)N-1} [1 - (1-x)^{\alpha}]^M,$$

in which the lowest term is x^M.

This completes the formal proof of (2). The rearrangement of the infinite series requires absolute convergence, which is secured when x is "sufficiently small", at least for the case $p = q + 1$, $s = t$, in which we are particularly interested.

3. A special case. If in (2) we write $s = t$, $b_k = d_k$ for $k = 1, 2, \cdots, s$, and $e_k = c_k$ for $k = 1, \cdots, q$, then we obtain

$$(3) \quad (1-x)^h \; {}_pF_q \left[\begin{matrix} a_p; \\ c_q; \end{matrix} \; x \right]$$

$$= h \sum_{n=0}^{\infty} \frac{(h - \alpha n + 1)_{n-1}}{n!} \; {}_{p+2}F_{q+2} \left[\begin{matrix} a_p, \; 1 + h(1-\alpha)^{-1}, \; -n \\ c_q, \; h(1-\alpha)^{-1}, \; h - \alpha n + 1 \end{matrix} \right] \left(\frac{-x}{(1-x)^{1-\alpha}} \right)^n.$$

4. Other cases. If

$$(4) \qquad {}_{p+2}F_{q+2} \left[\begin{matrix} a_p, \; 1 + h(1-\alpha)^{-1}, \; -n \\ e_q, \; h(1-\alpha)^{-1}, \; h - \alpha n + 1 \end{matrix} \right] = \frac{(\sigma_\mu)_n}{(\rho_\nu)_n},$$

then (2) and (3) reduce to simpler expressions.

4.1. In the case $p = q + 1$, (2) becomes

$$(5)\quad {}_{q+s+1}F_{q+t}\left[\begin{matrix} a_{q+1},\ b_s; \\ c_q,\ d_t; \end{matrix}\ x\right] = h \sum_{n=0}^{\infty} \frac{(h - \alpha n + 1)_{n-1}\,(b_s)_n\,(e_q)_n\,(\sigma_\mu)_n}{n!\,(d_t)_n\,(c_q)_n\,(\rho_\nu)_n}(-x)^n$$

$$\times\ {}_{q+s+1}F_{q+t}\left[\begin{matrix} b_s + n,\ e_q + n,\ h + (1-\alpha)\,n; \\ d_t + n,\ c_q + n; \end{matrix}\ x\right];$$

and (3) becomes

$$(6)\qquad (1-x)^h\ {}_{q+1}F_q\left[\begin{matrix} a_{q+1}; \\ c_q; \end{matrix}\ x\right] = h \sum_{n=0}^{\infty} \frac{(h - \alpha n + 1)_{n-1}}{n!}\,\frac{(\sigma_\mu)_n}{(\rho_\nu)_n}\left(\frac{-x}{(1-x)^{1-\alpha}}\right)^n,$$

which, for appropriate values of α, gives a relation between hypergeometric functions of argument x and $-x(1-x)^{\alpha-1}$.

4.2. In the case $q = 1$, $\alpha = 1/2$, $a_1 = a$, $a_2 = 2h$, $c = 2a$, (4) is summed by Watson's Theorem [1, p. 16], and vanishes for odd powers of n. Then (6) becomes (see [2, formula (4.22), with $\alpha + \beta = a$, $\alpha = h$])

$$(7)\qquad (1-x)^h\ {}_2F_1\left[\begin{matrix} a,\ 2h; \\ 2a; \end{matrix}\ x\right] = {}_2F_1\left[\begin{matrix} h,\ a-h; \\ a+1/2; \end{matrix}\ \frac{-x^2}{4(1-x)}\right]$$

and the corresponding formula (5) is

$$(8)\qquad {}_{s+2}F_{s+1}\left[\begin{matrix} a,\ 2h,\ b_s; \\ 2a;\ d_s; \end{matrix}\ x\right] = \sum_{m=0}^{\infty} \frac{(b_s)_{2m}\,(h)_m\,(a-h)_m}{(d_s)_{2m}\,m!\,(a+1/2)_m}\left(\frac{-x^2}{4}\right)^m$$

$$\times\ {}_{s+2}F_{s+1}\left[\begin{matrix} b_s + 2m,\ 2a + 2m,\ h + m; \\ d_s + 2m,\ 2a + m; \end{matrix}\ x\right].$$

If $\alpha = -1$, $q = 2$, $a_1 = \beta$, $a_2 = \gamma$, $a_3 = \delta$, $e_1 = 1 + \beta - \gamma$, $e_2 = 1 + \beta - \delta$, $h = \beta$, (4) can be summed by Dougall's formula [1, p. 25],

$$(9)\qquad {}_5F_4\left[\begin{matrix} \beta,\ 1+\beta/2,\ \gamma,\ \delta,\ -n \\ \beta/2,\ 1+\beta-\gamma,\ 1+\beta-\delta,\ 1+\beta+n \end{matrix}\right] = \frac{(1+\beta)_n\,(1+\beta-\gamma-\delta)_n}{(1+\beta-\gamma)_n\,(1+\beta-\delta)_n};$$

equation (5) becomes

$$(10)\quad {}_{s+3}F_{s+2}\left[\begin{matrix}\beta, \gamma, \delta, \ b_s\,; \\ c_1,\ c_2,\ d_s\,;\end{matrix}\ x\right]$$

$$= \beta \sum_{n=0}^{\infty} \frac{(\beta+n+1)_{n-1}\ (b_s)_n\ (1+\beta)_n\ (1+\beta-\gamma-\delta)_n}{n!\ \qquad (d_s)_n\ (c_1)_n\ (c_2)_n}\ (-x)^n$$

$$\times\ {}_{s+3}F_{s+2}\left[\begin{matrix}b_s+n,\ 1+\beta-\gamma+n,\ 1+\beta-\delta+n,\ \beta+2n; \\ d_s+n,\ c_1+n,\ c_2+n\,;\end{matrix}\ x\right];$$

and (6) becomes Whipple's formula [2, p. 250, where references are given]:

$$(11)\quad (1-x)^{\beta}\ {}_3F_2\left[\begin{matrix}\beta, \quad \gamma, \quad \delta; \\ 1+\beta-\gamma,\ 1+\beta-\delta;\end{matrix}\ x\right]$$

$$= {}_3F_2\left[\begin{matrix}\beta/2,\ (1+\beta)/2,\ 1+\beta-\gamma-\delta; \\ 1+\beta-\gamma, \qquad 1+\beta-\delta;\end{matrix}\ \frac{-4x}{(1-x)^2}\right].$$

4.3. If $\alpha = -1$, $q = 4$, $a_1 = \beta$, $a_2 = \gamma$, $a_3 = \delta$, $a_4 = \epsilon$, $a_5 = \theta$,

$e_1 = 1+\beta-\gamma$, $e_2 = 1+\beta-\delta$, $e_3 = 1+\beta-\epsilon$, $e_4 = 1+\beta-\theta$, $h = \beta$,

then using Whipple's transformation [1, p. 25],

$$(12)\quad {}_7F_6\left[\begin{matrix}\beta,\ 1+\beta/2,\ \gamma\ , \quad \delta\ , \quad \epsilon\ , \quad \theta\ , \quad -n \\ \beta/2,\ 1+\beta-\gamma,\ 1+\beta-\delta,\ 1+\beta-\epsilon,\ 1+\beta-\theta,\ 1+\beta+n\end{matrix}\right]$$

$$= \frac{(1+\beta)_n\ (1+\beta-\epsilon-\theta)_n}{(1+\beta-\epsilon)_n\ (1+\beta-\theta)_n}\ {}_4F_3\left[\begin{matrix}1+\beta-\gamma-\delta,\ \epsilon,\ \theta,\ -n \\ 1+\beta-\gamma,\ 1+\beta-\delta,\ \epsilon+\theta-\beta-n\end{matrix}\right],$$

in place of (4), we obtain

$$(13)\quad {}_{s+5}F_{s+4}\left[\begin{matrix}\beta,\ \gamma,\ \delta,\ \epsilon,\ \theta,\ b_s\ ; \\ c_1,\ c_2,\ c_3,\ c_4,\ d_s\,;\end{matrix}\ x\right]$$

$$= \beta \sum_{n=0}^{\infty} \frac{(\beta+n+1)_{n-1}\ (b_s)_n\ (1-\beta-\gamma)_n\ (1-\beta-\delta)_n\ (1+\beta)_n\ (1+\beta-\epsilon-\theta)_n}{n!\ \qquad (d_s)_n \qquad\qquad (c_1)_n\ (c_2)_n\ (c_3)_n\ (c_4)_n}$$

$$\times\ {}_4F_3\left[\begin{matrix}1+\beta-\gamma-\delta,\ \epsilon,\ \theta,\ -n \\ 1+\beta-\gamma,\ 1+\beta-\delta,\ \epsilon+\theta-\beta-n\end{matrix}\right](-x)^n\ \times$$

$$\times {}_{s+5}F_{s+4}\left[\begin{matrix} b_s+n, \ 1+\beta-\gamma+n, \ 1+\beta-\delta+n, \\ d_s+n, \ c_1+n, \ c_2+n, \ c_3+n, \end{matrix}\right.$$

$$\left.\begin{matrix} 1+\beta-\epsilon+n, \ 1+\beta-\theta+n, \ \beta+2n; \\ c_4+n; \end{matrix} \ x\right].$$

If $b_k = d_k$ for $k = 1, \cdots, s$, $c_1 = 1+\beta-\gamma$, $c_2 = 1+\beta-\delta$, $c_3 = 1+\beta-\epsilon$, $c_4 = 1+\beta-\theta$, this reduces to

$$(14)\quad (1-x)^{\beta} \, {}_5F_4\left[\begin{matrix} \beta, \ \gamma, \ \delta, \ \epsilon, \ \theta; \\ 1+\beta-\gamma, \ 1+\beta-\delta, \ 1+\beta-\epsilon, \ 1+\beta-\theta; \end{matrix} \ x\right]$$

$$= \sum_{n=0}^{\infty} \frac{(\beta+n+1)_{n-1} \, (1+\beta)_n \, (1+\beta-\epsilon-\theta)_n}{n! \, (1+\beta-\epsilon)_n \, (1+\beta-\theta)_n}$$

$$\times \, {}_4F_3\left[\begin{matrix} 1+\beta-\gamma-\delta, \ \epsilon, \ \theta, \ -n \\ 1+\beta-\gamma, \ 1+\beta-\delta, \ \epsilon+\theta-\beta-n \end{matrix}\right]\left(\frac{-x}{(1-x)^2}\right)^n.$$

If

$$\beta = \frac{1}{2}a - b, \ \gamma = 1-b, \ \delta = -\frac{1}{2}a, \ \epsilon = 1+\frac{1}{2}a, \ \theta = b,$$

by Bailey's result [1, p.30, formula (1.3)],

$$(15)\qquad {}_4F_3\left[\begin{matrix} a, \ 1+a/2, \ b, \ -n \\ a/2, \ 1+a-b, \ 1+2b-n \end{matrix}\right] = \frac{(a-2b)_n \, (-b)_n}{(1+a-b)_n \, (-2b)_n},$$

this becomes

$$(16)\qquad (1-x)^{-b+a/2} \, {}_5F_4\left[\begin{matrix} -b+a/2, \ 1-b, \ -a/2, \ 1+a/2, \ b; \\ a/2, \ 1+a-b, \ -b, \ 1-2b+a/2; \end{matrix} \ x\right]$$

$$= {}_3F_2\left[\begin{matrix} (a-2b)/4, (a-2b+2)/4, \ a-2b; \\ 1-2b+a/2, \ 1+a-b; \end{matrix} \ \frac{-4x}{(1-x)^2}\right].$$

4.4. If we take $\alpha = 0$, $q = 0$ and use Vandermonde's theorem in place of (4), we obtain

$$(17)\qquad {}_{s+1}F_s\left[\begin{matrix} a, \ b_s; \\ d_s; \end{matrix} \ x\right]$$

$$= \sum_{n=0}^{\infty} \frac{(b_s)_n \ (h-a)_n}{n! \ (d_s)_n} \ (-x)^n \ {}_{s+1}F_s \left[\begin{matrix} b_s + n, \ h+n; \\ d_s + n; \end{matrix} \ x \right]$$

and if $b_k = d_k$ for $k = 1, \cdots s-1$, $b_s = b$, $d_s = h$ this reduces to Euler's identity,

$$(18) \qquad (1-x)^b \ {}_2F_1 \left[\begin{matrix} a, \ b; \\ h; \end{matrix} \ x \right] = {}_2F_1 \left[\begin{matrix} h-a, \ b; \\ h; \end{matrix} \ \frac{x}{x-1} \right].$$

4.5. Multiplying (7) by $(1-x)^{-h}$ and equating coefficients of x, we obtain

$$(19) \qquad {}_3F_2 \left[\begin{matrix} a-h, \ -n/2, \ (1-n)/2 \\ a+1/2, \ 1-h-n \end{matrix} \right] = \frac{(a)_n \ (2h)_n}{(2a)_n \ (h)_n},$$

which is a particular case of Saalschutz' theorem.
Similarly from (16) we get

$$(20) \qquad {}_3F_2 \left[\begin{matrix} a-2b, \ a/2-b+n, \ -n \\ 1+a/2-2b, \ 1+a-b \end{matrix} \right] = \frac{(1-b)_n \ (-a/2)_n \ (1+a/2)_n \ (b)_n}{(a/2)_n \ (1+a-b)_n \ (1+a/2-2b)_n}.$$

This is a special case of

$$(21) \qquad {}_3F_2 \left[\begin{matrix} a, \ b, \ -n \\ e, \ 2+a+b-e-n \end{matrix} \right] = \frac{(e-b-1)_n \ (e-a-1)_n \ (\omega+1)_n}{(e)_n \ (e-a-b-1)_n \ (\omega)_n},$$

where

$$\omega = \frac{(e-a-1)\ (e-b-1)}{e-a-b-1},$$

which is, in Whipple's notation, a particular case of the relation between the quantities $F_p(0; 4, 5)$ and $F_p(2; 4, 5)$. [1, p. 85; 4]. This gives a generalisation of (16),

$$(22) \quad (1-x)^{2a} \ {}_5F_4 \left[\begin{matrix} 2a, \ e-c-1, \ 2a-e+1, \ 1+\phi, \ 1+\theta; \\ 2a+c+2-e, \quad e, \quad \theta, \quad \phi; \end{matrix} \ x \right]$$

$$= {}_3F_2 \left[\begin{matrix} a, \ a+1/2, \quad c \ ; \\ e, \ 2+c+2a-e; \end{matrix} \ \frac{-4x}{(1-x)^2} \right],$$

where θ, ϕ are the roots of $m^2 - 2am + (e-c-1)\ (2a+1-e) = 0$. Comparing with (14), we have

$$(23)\quad {}_4F_3\left[\begin{matrix} e-\theta-1,\ 1+\phi,\ e-c-1,\ -n \\ 2a-\theta,\ e,\ \phi+e-c-2a-n \end{matrix}\right] = \frac{(c)_n \qquad (2a-\phi)_n}{(e)_n\ (1+2a-\phi-e+c)_n}\ .$$

This is a generalisation of (15); we obtain (15), (16) from (22), (23) by taking $a=(a-2b)/4$, $c=a-2b$, $e=1+a-b$, $\theta=-b$, $\phi=a/2$.

I should like to take this opportunity of thanking Dr. Chaundy for many kindnesses and especially for allowing me to see his most recent paper before it was published.

REFERENCES

1. W. N. Bailey, *Generalised hypergeometric series*, University Press, Cambridge, England, 1935.

2. ———, *Products of generalised hypergeometric series*, Proc. London Math. Soc. (2) 28 (1928), 242-254.

3. T. W. Chaundy, *Some hypergeometric identities*, J. London Math. Soc. 26 (1951), 42-44.

4. F. J. W. Whipple, *A group of generalised hypergeometric series: relations between 120 allied series of the type* $F\left[\begin{matrix}a,\ b,\ c\\ e,\ f\end{matrix}\right]$, Proc. London Math. Soc. (2) 23 (1925), 104-114.

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