HOMOLOGY BASES WITH APPLICATIONS TO LOCAL CONNECTEDNESS

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I. INTRODUCTION

1. The present investigation originated in an attempt to extend certain results of Wilder [11] concerning the imbedding of compact metric spaces in locally connected spaces. The means that suggested themselves for the treatment of the problem necessitated an analysis of the homology group of a compact space with respect to certain sequences of direct-sum decompositions. In the earlier part of this paper we introduce the concept of sequential decomposability of a vector space with respect to a countable set of subspaces. An equivalence between sequential decompositions is set up with the usual properties. Decomposition theorems with uniqueness properties for elements of the vector space are noted. In terms of a nonnegative real-valued function on elements of the vector space, a topology is introduced in which the vector space is a Hausdorff space if and only if it is metric; the completeness of this space is also discussed. When the vector space is a complete metric space, there exists a countable set of elements associated with the sequential decomposition which becomes a base for the space in the sense of infinite combinations.

Since the homology group of a compact space with coefficients over a field is a vector space, the preceding results are applicable; conditions are discussed under which the homology group is a metric space, and further conditions under which it is complete. The existence problem arising here is disposed of readily by showing that compact G_δ subsets of locally compact lc^r spaces (see 8.1 below) satisfy the requirements of the preceding abstract situation. This gives further an intrinsic approach to the existence of a fundamental system of 0-cycles discussed by Wilder in another setting [13, Ch. VI, §5].

In an investigation of this kind it is often desirable from many points of view to have a specialized class of objects with a well-defined property; hence we next occupy ourselves with determining the existence of a fundamental

system of nontrivial 0-cycles for the sets described above. The existence of such a fundamental system plays a mildly important role in the sequel.

With this done, the stage is set for a partial solution of the problem whence arose the whole discussion, and a sufficient condition is given under which a compact Hausdorff space M is imbeddable as a G_δ subset of a 0-lc continuum, which is obtained by adjoining a denumberable infinitude of disjoint arcs to M. To improve the value of the preceding result, it is pointed out that the hypothesis of the theorem is verified by a class $\mathfrak{R}$ of compact spaces which includes definitely nonmetric spaces. Using the same technique as was employed in obtaining the preceding result, we show further that the class $\mathfrak{R}$ contains as a proper subset the class of all compacta. Another condition is given that a separable compact G_δ subset of a space S be a subset of a locally compact subset $S' \subset S$. A localization of the preceding condition gives the last application of the abstract development.

II. Some Properties of Vector Spaces

2. **Sequential decomposition.** In the sequel all vector spaces will be taken over a given field, $\mathfrak{F}$. Let $\mathbf{H} = \{H_k\}$ be a monotonic decreasing sequence of subspaces of a vector space V such that each H_k is finite co-dimensional [4, Ch. 2, Def. 2], then evidently [8, Ch. II, (23.1)] there always exists a countable set of linearly independent elements $\mathbf{g} = \{g_i\}$ of V such that (i) $\mathbf{g}_k = \{g_i \mid 1 \le i \le i_k\}$ is a set of generators for a subspace G_k, (ii) V is the direct sum of G_k and H_k, and (iii) $\{g_i \mid i > i_k\} \subset H_k$. Similarly, if $\mathbf{g} = \{g_i\}$ is a countable linearly independent set of elements of a vector space V then there exists a sequence $\mathbf{H}$ of subspaces having the properties described above such that (i), (ii), and (iii) again hold. This pair $(\mathbf{H}, \mathbf{g})$ will be called a *sequential decomposition* of of V. Evidently $\mathbf{g}$ is not unique.

In terms of a fixed sequential decomposition we may obviously [8, Ch. II, (23.1)] assert that to each element $v \in V$ there corresponds a unique infinite linear form which we may denote by

$$\sum_{i=1}^{\infty} \alpha^i g_i, \quad \text{where} \quad v = \sum_{i=1}^{i_k} \alpha^i g_i \pmod{H_k}$$

for each positive integer k, $\alpha^i \in \mathfrak{F}$. The $\sum_{i=1}^{i_k} \alpha^i g_i$ will be called the k-*decomposition* (rel $\mathbf{H}, \mathbf{g}$) of $v \in V$. We note that at this stage of the development it may, and at times certainly will, happen that the same linear form will correspond to distinct elements of V. Evidently if to two elements v_1, $v_2 \in V$, there corresponds the same infinite linear form, then $v_1 = v_2 \pmod{H_k}$ for each k.

3. **Topology for the vector space.** We now define the positive real-valued function ρ on the product space $V \times V$, relative to a fixed sequential decomposition $(\mathbf{H}, \mathbf{g})$, as follows: if v_1, $v_2 \in V$, then (1) $\rho(v_1, v_2) = 1/k$ for some positive integer $k \iff k$ is the largest index such that $v_1 = v_2 \pmod{H_k}$; (2) if there is no upper bound to the k's such that $v_1 = v_2 \pmod{H_k}$, then $\rho(v_1, v_2) = 0$. It is quite simple to prove that:

$$(1) \qquad\qquad \rho(v_1, v_2) = \rho(v_2, v_1)$$

and

$$(2) \qquad\qquad \rho(v_1, v_3) \leq \rho(v_1, v_2) + \rho(v_2, v_3).$$

We note, in view of an earlier comment, that $\rho(v_1, v_2) = 0$ does not necessarily imply $v_1 = v_2$. Hence it is clear that the function ρ cannot always serve as a metric for the vector space V. Under suitable conditions this will be possible, however, as will be noted below. At this stage ρ may be called a *pseudo-metric*.

For each $v \in V$ and each positive integer k, the family of sets of the form

$$U(v) = \{ v' | \rho(v', v) < i/k \}$$

will, by finite intersection and arbitrary union, induce a topology for V in which the family above is a family of open sets. With this topology, V is not necessarily a Hausdorff space [see, for example 8, p. 25]; however, it is readily seen that in order that V be Hausdorff it is necessary and sufficient that it be metrizable by the function ρ. Then we may write

$$v = \sum_{i=1}^{\infty} a^i g_i = \lim_{k \to \infty} \sum_{i=1}^{i_k} a^i g_i, \qquad\qquad a^i \in \mathfrak{F}.$$

We shall call $(\mathbf{H}, \mathbf{g})$ and $(\mathbf{H}', \mathbf{g}')$ *equivalent* sequential decompositions if, when $\mathbf{H} = \{H_k\}$ and $\mathbf{H}' = \{H_j'\}$, for each k there exists a j such that $H_k \supset H_j'$, and conversely. It is relatively easy to show that if $(\mathbf{H}, \mathbf{g})$ and $(\mathbf{H}', \mathbf{g}')$ are equivalent sequential decompositions and $\mathfrak{O}$ and $\mathfrak{O}'$ and ρ and ρ' are respectively topologies and pseudo-metrics relative to $(\mathbf{H}, \mathbf{g})$ and $(\mathbf{H}', \mathbf{g}')$, then, when V is Hausdorff in both cases, the topologies and the metrics are equivalent in the usual sense. The proof in this instance proceeds along lines usual in such cases.

4. **A fundamental system for** V. When V is a Hausdorff space in the topology described above, then we may assert the following readily proved properties:

(1) *For any nonzero element* $\alpha \in \mathfrak{F}$, *and any* $v \in V$, *we have* $\rho(\alpha v, 0) = \rho(v, 0)$.

(2) *If* (**H, g**) *is the sequential decomposition, then* $\lim\limits_{i \to \infty} g_i = 0$.

(3) *If* (**H, g**) *is the sequential decomposition, and*

$$v_1 = \sum_{i=1}^{\infty} \alpha^i g_i \quad and \quad v_2 = \sum_{i=1}^{\infty} \mathfrak{b}^i g_i,$$

then $v_1 \neq v_2 \iff \alpha^i \neq \mathfrak{b}^i$ *for some* i.

(4) *If furthermore* V *is the complete enclosure* [10, p. 28] *of the vector space* G *spanned by* **g**, *then* G *is dense in* V *and, in addition to the preceding, to each infinite linear form* $\sum \alpha^i g_i$, $\alpha^i \in \mathfrak{F}$ *and* $g_i \in$ **g**, *there corresponds a unique element of* V.

When this last property is satisfied, the set **g** becomes a sort of base for the space V, and it will be called a *fundamental system* for V. With appropriate particularization, this is identical with the *fundamental system of r-cycles* treated by Wilder [13, Ch. VI, §5]. Later results of this paper will give a new intrinsic approach to the problem of the existence of this system of cycles.

III. Homology Bases

5. Preliminaries. The homology theory used in this paper is the usual Čech homology theory, originally formulated in [6], augmented in the sense of [8, p. 248]. We recall, for later clarity, that if $\mathfrak{U}$ is a covering of a space, then the nerve of the covering $\mathfrak{U}$, written $N(\mathfrak{U})$, is the abstract simplicial complex defined as follows: Every element of the covering is a vertex of the nerve, and for each natural number k every set of $k + 1$ elements with a nonvacuous intersection forms a k-simplex of the nerve. We shall consistently use lower case Greek or Roman letters to denote individual cycles: thus z^r is an r-cycle; capitals will indicate homology classes of cycles, thus Z^r; $z^r(\mathfrak{U})$ will denote the representative on the nerve of the covering $\mathfrak{U}$ of the cycle z^r; likewise if H_k^r is a subset of $H^r(S, \mathfrak{F})$, then $H_k^r(\mathfrak{U})$ is the corresponding subset of the homology group $H^r(\mathfrak{U})$ on the nerve of the covering $\mathfrak{U}$; and so on. Finally, π_{uv} will be the usual projection mapping. Note that here, as well as later, subscripts referring to a covering, say $\mathfrak{U}$, will be printed as the corresponding Roman letter; this is done for obvious typographical reasons.

If $\mathfrak{H}^r = \{H^r(\mathfrak{U})\}$ is the inverse system of homology groups, with coefficients from a field $\mathfrak{F}$, on nerve complexes for the family of all finite coverings of a compact space S, then the limit group is the homology group $H^r(S, \mathfrak{F})$ of the

space S. The limit group, and also the homology groups of the nerve complexes, are vector spaces; hence all that has been noted in the preceding section is applicable here.

In terms of the notation found in [13, p. 147], h_{uv} will be the homomorphism of $H^r(\mho)$ in $H^r(\mathcal{U})$ induced by the projection of $N(\mho)$ in $N(\mathcal{U})$. We shall further use h_u to denote the homomorphism which associates with an element of the homology group of the space its representative in $H^r(\mathcal{U})$. Evidently, if $\mathcal{W} > \mho > \mathcal{U}$ then $h_{uw} = h_{uv} h_{vw}$ and $h_u = h_{uv} h_v$.

5.1. DEFINITION. A sequential decomposition $(\mathbf{H}^r, \Gamma^r)$ is *compatible* with a sequence of coverings $\{\mathcal{E}_k\}$ if and only if H_k is the kernel of h_{e_k}.

The existence of a sequential decomposition compatible with a given sequence of coverings is obvious. Concerning this notion we make the following remarks which will be used later.

5.2. REMARK. If $(\mathbf{H}^r, \Gamma^r)$ and $\{\mathcal{E}_k\}$ are compatible, then the sequence of representative cycles, γ^r has the following properties: (i) the subset of $\gamma^r(\mathcal{E}_k)$ defined by $\{\gamma_i^r(\mathcal{E}_k) \mid 1 \leq i \leq i_k\}$ is lirh (that is, linearly independent relative to homology) on $N(\mathcal{E}_k)$ for each k; and (ii) $\gamma_i^r(\mathcal{E}_k) \sim 0$ on $N(\mathcal{E}_k)$ for $i > i_k$.

5.3. REMARK. If $(\mathbf{H}^r, \Gamma^r)$ and $\{\mathcal{E}_k\}$ are compatible, and $Z_1^r, Z_2^r \in H^r(S, \mathfrak{F})$, then $\rho(Z_1^r, Z_2^r) = 1/k \iff k$ is the largest index such that $z_1^r - z_2^r$ belongs to the kernel of the homomorphism h_{e_k}.

6. **Metrizability.** In the preceding part on vector spaces it was noted that a necessary and sufficient condition for the vector space to be Hausdorff under the topology introduced was the metrizability by the function ρ. We shall formulate here a sufficient condition that $H^r(S, \mathfrak{F})$ be metrizable in this way. As a preliminary we introduce the following lemma.

6.1. LEMMA. (i) *Let there exist for each covering* $\mathcal{U}$ *of* S *a number* $k(\mathcal{U})$ *and a refinement* $\mathcal{U}' > \mathcal{U}$ *such that* $H_{k(u)}^r(\mathcal{U}')$ *belongs to the kernel of* $h_{uu'}$; *and* (ii) *let* $\Gamma^r \in H^r(S, \mathfrak{F})$ *be such that for each* k *there exists a covering* $V(k)$ *with the property that*

$$\mho' > \mho(k) \implies h_v \cdot \Gamma^r \subset h_v \cdot H_k^r = H_k^r(\mho').$$

Then $\Gamma^r = 0$.

Proof. Let $\mathcal{U}$ be an arbitrary covering of S; we must show that $h_u \Gamma^r = 0$. Let $k(\mathcal{U})$ be the number given by hypothesis (i), and $\mathcal{U}'$ the refinement such that $h_u \cdot H_{k(u)}^r$ belongs to the kernel of $h_{uu'}$. By hypothesis (ii), for $k = k(\mathcal{U})$,

we have $\mho(k)$ such that

$$\mho' > \mho(k) \implies h_v, \Gamma^r \subset h_v, H_k^r .$$

Let $\mathcal{W}$ be a common refinement of $\mathcal{U}'$ and $\mho(k)$. Then we have

$$h_{uw} h_w H_k^r \ = \ h_{uu'} h_{u'w} h_w H_k^r \ = \ h_{uu'} h_{u'} H_k^r \ = \ 0 .$$

But $h_w \Gamma^r \subset h_w H_k^r$. Hence we evidently have the desired result.

We may now state and prove a theorem giving a sufficient condition for the metrizability of $H^r(S, \mathfrak{F})$.

6.2. THEOREM. *If for each covering* $\mathcal{U}$ *of* S *the hypothesis* (i) *of the preceding lemma is satisfied, then* $H^r(S, \mathfrak{F})$ *is metrizable.*

Proof. Evidently all we need prove is that, for $Z_1^r, Z_2^r \in H^r(S, \mathfrak{F})$, we have $\rho(Z_1^r, Z_2^r) = 0 \implies Z_1^r = Z_2^r$. Now $\rho(Z_1^r, Z_2^r) = 0$ implies that $Z_1^r - Z_2^r \in H_k^r$ for each k. But then [13, p. 147] for each covering $\mathcal{U}$ of S we have $h_u(Z_1^r - Z_2^r) \subset h_u H_k^r$, and thus hypothesis (ii) of the preceding lemma is more than satisfied, so that $Z_1^r - Z_2^r = 0$ as required.

7. **Completeness.** Before establishing the completeness of $H^r(S, \mathfrak{F})$, we introduce the following definition for convenience of reference.

7.1. DEFINITION. A compact space S will be said to have property P_r (rel $\mathbf{H}^r, \Gamma^r$) if and only if for each covering $\mathcal{U}$ of S there exists an index $k = k(\mathcal{U})$ such that H_k^r is contained in the kernel of h_u. A space S will be said to have the property P_r (rel $\mathbf{H}^r, \Gamma^r$) with respect to a compact subset $M \subset S$, if the property described is satisfied with respect to the homology theory of M itself.

The following lemmas are quite obvious.

7.2. LEMMA. *If the compact space* S *has property* P_r (rel $\mathbf{H}^r, \Gamma^r$), *then hypothesis* (i) *of Lemma* 6.1 *is satisfied.*

7.3. LEMMA. *If* $\mho > \mathcal{U}$ *and* $k'(\mho)$ *and* $k'(\mathcal{U})$ *are respectively the least indices greater than* 1 *for which* H_k^r *is contained in the kernel of* h_v *and* h_u, *then* $k'(\mho) \geq k'(\mathcal{U})$.

7.4. THEOREM. *If the compact space* S *has property* P_r (rel $\mathbf{H}^r, \Gamma^r$), *then* $H^r(S, \mathfrak{F})$ *is a complete metric space.*

Proof. By Lemma 7.2 and Theorem 6.2, metrizability follows obviously. Let

$\{Z_i^r\}$ be a Cauchy sequence of elements of $H^r(S, \mathfrak{F})$, and let $\mathbf{U}$ be a complete family of coverings of S. For each $\mathcal{U} \in \mathbf{U}$ let $k(\mathcal{U})$ be the smallest index greater than 1 for which the defining condition of property P_r holds. Let $j(\mathcal{U})$ be the smallest natural number such that for all s we have $Z_{j(u)}^r - Z_{j(u)+s}^r \in H_{k(u)}^r$. We now let $z^r(\mathcal{U}) = z_{j(u)}^r(\mathcal{U})$. We shall first show that the z^r thus defined by its coordinates on a complete family of coverings is a cycle.

Suppose $\mathcal{U}, \mathcal{V} \in \mathbf{U}$ such that $\mathcal{V} > \mathcal{U}$; we must show that $\pi_{uv} z^r(\mathcal{V}) \sim z^r(\mathcal{U})$. As a consequence of Lemma 7.3, we need only consider the single case $k(\mathcal{V}) \geq k(\mathcal{U})$. Under this hypothesis, we have $j(\mathcal{V}) \geq j(\mathcal{U})$, and so we have $z_{j(u)}^r(\mathcal{U}) - z_{j(v)}^r(\mathcal{U}) \sim \phi^r(\mathcal{U})$, $\Phi^r \in H_k^r$; hence $z_{j(u)}^r(\mathcal{U}) \sim z_{j(v)}^r(\mathcal{U})$. But $\pi_{uv} z_{j(v)}^r(\mathcal{V}) \sim z_{j(v)}^r(\mathcal{U})$. Therefore, as was required, $\pi_{uv} z_{j(v)}^r(\mathcal{V}) \sim z_{j(u)}^r(\mathcal{U})$. Hence $z^r(\mathcal{U})$ is the coordinate on $N(\mathcal{U})$ of a cycle z^r as defined above.

We now complete the proof by showing that $\lim_{i \to \infty} Z_i^r = Z^r$. Let n_k be the smallest natural number such that $\rho(Z_i^r, Z_j^r) \leq 1/k$ (k fixed) for all $i, j \geq n_k$. Let $\mathbf{U}'$ be a subset of $\mathbf{U}$ such that for $\mathcal{U} \in \mathbf{U}'$ the $k(\mathcal{U})$ as defined above in this proof is greater than or equal to k. Hence $j(\mathcal{U}) \geq n_k$, and thus $Z_{j(u)}^r(\mathcal{U}) - Z_i^r(\mathcal{U}) \in h_u H_k^r$ for all $i \geq h_k$. Now, in virtue of Lemma 7.3, it is evident that $\mathbf{U}'$ is a complete family of coverings, so that the proof of the theorem is complete.

In this connection we might note the following rather obvious consequences of the preceding result.

7.5. *When the compact space S has the property P_r (rel $\mathbf{H}^r, \Gamma^r$), then*

(1) *for each r-cycle z^r of S there exists a unique homology relation of the form*

$$z^r \sim \sum_{i=1}^{\infty} a^i \gamma_i^r, \qquad\qquad a^i \in \mathfrak{F};$$

(2) *$H^r(S, \mathfrak{F})$ is the complete enclosure of the vector space $H_\Gamma^r(S, \mathfrak{F})$ generated by Γ^r;*

(3) *$H_\Gamma^r(S, \mathfrak{F})$ is dense in $H^r(S, \mathfrak{F})$;*

(4) *$H^r(S, \mathfrak{F})$ is equivalent to the product space $P = P_1^r \times P_2^r \times \cdots \times P_i^r \times \cdots$, $P_i^r = \{\Gamma^r \mid \Gamma^r = a\Gamma_i^r, \ a \in \mathfrak{F}\}$.*

IV. EXISTENCE THEOREMS

8. **Locally connected spaces.** We proceed now to establish the existence of a situation in which the preceding formal development is completely verified.

The results of this section will, further, constitute a new intrinsic approach to the problem of the existence of fundamental systems of r-cycles treated in [13, Ch. VI, §5].

8.1. DEFINITION. A space S is r-lc at a point $x \in S$, if and only if each open set which contains x also contains a pair of open sets, P, Q, such that (i) $x \in Q \subset P$, and (ii) the number of r-cycles on Q that are lirh on P is finite. A space S, is r-lc if and only if it is r-lc at each $x \cdot S$; and finally S is lc^r, if and only if it is p-lc for each p such that $0 \le p \le r$.

As an immediate consequence of the remarks made heretofore and a theorem proved elsewhere [13, p. 186, Th. 5.1], we have the following result.

8.2. THEOREM. (i) Let M be a compact G_δ subset of a locally compact lc^r space S, and (ii) let $\{ U_k \}$ be a sequence of open sets such that $U_{k+1} \subset U_k$ and $\cap U_k = M$. Then there exists a sequential decomposition (H^r, Γ^r) of $H^r(M, \mathfrak{F})$ such that: (1) $\Phi_k^r \in H_k^r \iff \Phi_k^r \sim 0$ on $\overline{U}_k$; (2) $\{ \gamma_i^r \mid 1 \le i \le i_k \}$ is a basis for cycles of M with respect to homologies on $\overline{U}_k$.

The sequential decomposition given by the preceding theorem will be called compatible with the sequence of open sets $\{ U_k \}$.

8.3. THEOREM. Let $M \subset S$ satisfy hypothesis (i), and let $\{ U_k \}$ and $\{ U_j' \}$ be two sequences of open sets satisfying hypothesis (ii) of Theorem 8.2. Let $(\mathbf{H}^r, \Gamma^r)$ and $(\mathbf{H}'^r, \Gamma'^r)$ be compatible sequential decompositions given by the conclusion of Theorem 8.2. Then the sequential decompositions are equivalent.

Proof. For each k, there exists a j such that $U_k \supset U_j'$. By hypothesis, $\Phi_j'^r \in H_j'^r \implies \Phi_j'^r \sim 0$ on $\overline{U}_j'^r$ and so also on $\overline{U}_k$; and H_k^r is the subspace consisting of all classes of r-cycles which have representatives homologous to zero on $\overline{U}_k$; hence $\Phi_j'^r \in H_k^r$, so that $H_j'^r \subset H_k^r$ as required. The converse procedure completes the proof that the sequential decompositions are equivalent.

Because it will be used frequently in the sequel, we formulate the following hypothesis here for convenience of reference.

8.4. HYPOTHESIS. Let M be a compact G_δ subset of a locally compact lc^r space S, and let $\{ U_k \}$ be a sequence of open sets such that $U_k \supset U_{k+1}$ and $\cap U_k = M$; further let $(\mathbf{H}^r, \Gamma^r)$ be a sequential decomposition compatible with $\{ U_k \}$.

We now treat the possibility of obtaining a sequence $\{ \mathfrak{C}_k \}$ of coverings of S compatible with a given sequential decomposition.

8.5. THEOREM. *Let hypothesis 8.4 be satisfied. Then there exists a sequence $\{\mathcal{E}_j\}$ of coverings of S compatible with a sequential decomposition $(\mathbf{H}'^r, \Gamma'^r)$ of $H^r(M, \mathcal{F})$ such that $(\mathbf{H}'^r, \Gamma'^r)$ and $(\mathbf{H}^r, \Gamma^r)$ are equivalent.*

Proof. Let $\mathcal{E}_1$ be a covering of S given by a well-known lemma [8, p. 263; or 13, p. 134, Lemma 8.7], due originally to Čech, such that $\mathcal{E}_1$ is a refinement of a covering $\mathcal{U}$ [13, p. 167, Theorem 19.2], such that $\gamma_1^r(\mathcal{U}), \cdots, \gamma_{i_1}^r(\mathcal{U})$ are lirh on $N(\mathcal{U})$ for all $\mathcal{U} > \mathcal{U}$. Let P be the open set containing M such that if the nucleus of a cell of $N(\mathcal{E}_1)$ is on $\bar{P}$, then it is on M. There exists a k such that a subset $P^* \subset P$ has the additional property that $U_{k-1} \supseteq P^* \supseteq U_k$. Since H_k^r consists of classes of r-cycles of M which are homologous to zero on U_k, it is evident that H_k^r is contained in the kernel of h_{e_1}. Let us designate the kernel of h_{e_1} by $H_1'^r$. Since $\mathcal{E}_1 > \mathcal{U}$, we have the following inclusions: $H_1^r \supset$ kernel of $h_u \supset H_1'^r$.

Evidently we may repeat this procedure and eventually obtain a sequence of coverings $\{\mathcal{E}_j\}$ of S satisfying the following: (1) only a finite number of elements of $\mathcal{E}_j$ meet M for each j; (2) for each j there exists a k_j such that $H_j'^r \supset H_{k_j}^r$; and (3) conversely, for each k there is a j_k such that $H_k^r \supset H_{j_k}'^r$. Consequently the sequential decompositions are equivalent as required.

9. **Existence of a fundamental system.** This section will be devoted to the proof of the existence of a fundamental system of r-cycles.

9.1. THEOREM. *Let hypothesis 8.4 be verified. Then the space S has the property P_r (rel $\mathbf{H}^r, \Gamma^r$) with respect to the subset M.*

Proof. There exists a complete family $\mathbf{U}$ of coverings of S satisfying the following conditions: (1) if $\mathcal{U} \in \mathbf{U}$, then at most a finite number of elements of $\mathcal{U}$ meet M; (2) there exists an open set $P \supset M$ such that if the nucleus of a cell of $N(\mathcal{U})$ meets P, then it meets M [13, p. 134, Lemma 8.7]. Hence for each $\mathcal{U} \in \mathbf{U}$, there exists a $k(\mathcal{U})$ such that the relation $z_1^r(\mathcal{U}) \sim z_2^r(\mathcal{U})$ on $\bar{U}_k$ implies that $z_1^r(\mathcal{U}) \sim z_2^r(\mathcal{U})$ on M. Since $\Phi_{k(v)}^r \in H_{k(v)}^r$ implies that $\phi_{k(v)}^r \sim 0$ on $\bar{U}_{k(v)}$, it follows that $\phi_{k(v)}^r \sim 0$ on M. This means that $H_{k(v)}^r$ belongs to the kernel of h_v, and hence S has the property P_r with respect to the subset M.

9.2. COROLLARY. *Under the same hypothesis, hypothesis (i) of Lemma 6.1 is satisfied relative to the subset M.*

It is now obvious that the result given in 7.4 is applicable here, and hence the set $\gamma^r = \{\gamma_i^r\}$ is a *fundamental system of r-cycles* as desired.

As a conclusion to this part we give a necessary and sufficient condition that a countable lirh set, $\gamma^r = \{\gamma_i^r\}$, of r-cycles of a compact G_δ subset of a

locally compact lc^r space S be a *fundamental* system of r-cycles for the subset M; that is, $H^r(M, \mathfrak{F})$ is the complete enclosure of the vector space generated by Γ^r.

9.3. THEOREM. *Let M be a compact G_δ subset of a locally compact lc^r space S. Suppose further that $\gamma^r = \{ \gamma_i^r \}$ is a countable set of r-cycles of M such that for any open set U containing M at most a finite number of elements of γ^r forms a lirh set on $\overline{U}$ and the remainder are homologous to zero on $\overline{U}$. Then a necessary and sufficient condition that γ^r be a fundamental system of r-cycles for M is that $H_\Gamma^r(M, \mathfrak{F})$, the vector space generated by Γ^r, be dense in $H^r(M, \mathfrak{F})$.*

Proof. Let $\{ U_k \}$ be a sequence of open sets such that $U_k \supset U_{k+1}$ and $\cap U_k = M$; the existence of this sequence follows from the properties of M and S. Then by Theorem 8.2 there exists a sequential decomposition $(\mathbf{H}^r, \Gamma^r)$ of $H^r(M, \mathfrak{F})$ compatible with $\{ U_k \}$.

The necessity follows as an immediate consequence of the remark which follows Corollary 9.2.

To prove sufficiency, proceed as follows. Let $Z^r \in H^r(M, \mathfrak{F})$; then for $1/k > 0$ there exists a $W_k^r \in H^r(M, \mathfrak{F})$ such that $\rho(Z^r, W_k^r) \leq 1/k$, which in the present instance implies that $z^r \sim w_k^r$ on $\overline{U}_k$. This is true for each integer k, and hence we obtain a sequence $\{ W_k^r \} \subset H^r(M, \mathfrak{F})$. By hypothesis, we have

$$w_k^r = \sum_{i=1}^{\infty}{}' \, a_k^i \, \gamma_i^r \, ,$$

where the $'$ on the $\sum$ indicates that all but a finite number of the coefficients a_k^i are zero. Now let $k_2 > k_1$; then

$$z^r \sim w_{k_1}^r \sim \sum_{i=1}^{i_{k_1}} a_{k_1}^i \gamma_i^r \quad \text{on} \quad \overline{U}_{k_1}$$

and

$$z^r \sim w_{k_2}^r \sim \sum_{i=1}^{i_{k_2}} a_{k_2}^i \gamma_i^r \quad \text{on} \quad \overline{U}_{k_2} \, .$$

Therefore $w_{k_1}^r \sim w_{k_2}^r$ on $\overline{U}_{k_1}$, and since for $i > i_k$ we have $\gamma_i^r \sim 0$ on $\overline{U}_k$, we have

$$\sum_{i=1}^{i_{k_1}} \left(a_{k_1}^i - a_{k_2}^i \right) \gamma_i^r \sim 0 \quad \text{on} \quad \overline{U}_{k_1} \, .$$

But the γ_i^r are lirh on $\bar{U}_{k_1}$ for $i \leq i_{k_1}$, and hence $a_{k_1}^i = a_{k_2}^i$ for $i \leq i_{k_1}$. Continuing this argument, recalling the completeness, and applying Lemma 6.1, we have $Z^r = \lim_{k \to \infty} W_k^r$, and the proof of the theorem is complete.

V. NONTRIVIAL 0-CYCLES

10. Special fundamental system. In this part, we ultimately establish the existence of a fundamental system of nontrivial 0-cycles [13, p. 142, Def. 11.4] for compact G_δ subsets of a locally compact 0-lc space. It has been shown [6, p. 168; or 13, p. 142, Th. 11.5] that if x and y are distinct points of a compact space, then there exists a nontrivial 0-cycle carried by them. We now prove the following lemma.

10.1. LEMMA. *If S is a compact space, then for any covering $\mathfrak{U}$ of S, and 0-cycle z^0, there exists a refinement $\mathfrak{V} > \mathfrak{U}$, and a set of nontrivial 0-cycles carried by pairs of points, $\gamma_1^0, \gamma_2^0, \cdots, \gamma_k^0$, such that*

$$z^0(\mathfrak{V}) \sim \sum_{i=1}^{k} a^i \gamma_i^0(\mathfrak{V}), \quad a^i \in \mathfrak{F}, \quad on \quad N(\mathfrak{V}).$$

Proof. Let $\mathfrak{V}$ be an irreducible refinement [8, p. 247] of U. Then on $N(\mathfrak{V})$, we have

$$z^0(\mathfrak{V}) = a^1(V_{i_1} - V_{i_2}) + a^2(V_{i_2} - V_{i_3}) + \cdots + a^{k-1}(V_{i_{k-1}} - V_{i_k})$$
$$+ a^k(V_{i_k} - V_{i_{k+1}}).$$

For each $V_i \in \mathfrak{V}$ there exists a $p_i \in V_i$ such that $p_i \not\subset V_h$ if $i \neq h$. For each m let γ_m^0 be a nontrivial 0-cycle carried by $p_{i_m} \cup p_{i_{m+1}}$. Obviously $\gamma_m^0(\mathfrak{V}) = V_{i_m} - V_{i_{m+1}}$, and hence $z^0(\mathfrak{V})$ is homologous to $\sum_{i=1}^{k} a^i \gamma_i^0(\mathfrak{V})$ on $N(\mathfrak{V})$.

11. Existence theorem. The proof of the existence theorem now follows.

11.1. THEOREM. *Let M be a compact G_δ subset of a locally compact 0-lc space S. Then there exists a fundamental system of nontrivial 0-cycles carried by pairs of points of M satisfying hypothesis 8.4 for $r = 0$.*

Proof. Let $\{\mathcal{E}_k\}$ be a sequence of coverings of S composed of connected open sets such that the part of $\mathcal{E}_k$ that meets M is finite and further $\mathcal{E}_{k+1} > \mathcal{E}_k$. If n_k is the number of components [see 8, p. 15] of the subcollection of $\mathcal{E}_k$ that meets M, then we have $n_k \leq n_{k+1}$. Since M is a G_δ subset of S, then the sequence $\{\mathcal{E}_k\}$ can be so selected that, with E_k^M denoting an element $E_k \in \mathcal{E}_k$

such that $E_k \cap M \neq 0$, we have $\cup E_k^M \supset \cup E_{k+1}^M$ and $\cap (\cup E_k^M) = M$.

Let $\{p_j\}$ $(j = 1, \cdots, n_1)$ be points of M such that for $j_1 \neq j_2$, p_{j_1} and p_{j_2} belong to distinct components of $\mathcal{E}_1 \cap M$. Then there exist $n_1 - 1$ nontrivial 0-cycles of M with coordinates on the coverings $\mathcal{E}_1 \cap M$ forming a lirh set on $N(\mathcal{E}_1 \cap M)$ and consequently on any $\mathcal{E} \cap M$, where $\mathcal{E} > \mathcal{E}_1$; hence the cycles themselves are lirh on M.

For $\mathcal{E}_2 \cap M$, when $n_2 > n_1$, we may add $n_2 - n_1$ nontrivial 0-cycles to this collection in such a way that they belong to the kernel of h_{e_1}, but have coordinates on $\mathcal{E}_2 \cap M$ which together with the previous set form a lirh set.

A continuation of this process yields a countable lirh set of nontrivial 0-cycles, $\gamma^0 = \{\gamma_i^0\}$, which together with a sequence $\mathbf{H}^0$ of subsets of $H^0(M, \mathfrak{F})$ gives rise to a sequential decomposition $(\mathbf{H}^0, \Gamma^0)$ compatible with both $\{\mathcal{E}_k\}$ and $\{U_k\}$, where $U_k = \cup E_k^M$. This latter follows since $\gamma_1^0, \cdots, \gamma_{i_k}^0$ is a basis for cycles on M with respect to homology on $\overline{U}_k$, for the covering $\mathcal{E}_k \cap M$ is composed of connected open sets, and coordinates of γ_i^0 $(i = 1, \cdots, i_k)$ form a maximum lirh set on $\mathcal{E}_k \cap M$. Thus by Theorem 9.3 it remains only to show that the vector subspace $H_\Gamma^0(M, \mathfrak{F})$, generated by Γ^0, is dense in $H^0(M, \mathfrak{F})$.

If this is not true, then there exists $Z^0 \in H^0(M, \mathfrak{F})$ such that $\rho(Z^0, H_\Gamma^0(M, \mathfrak{F})) > 0$. As in Theorem 9.3, for some $k > 0$ the cycles $z^0, \gamma_1^0, \cdots, \gamma_{i_k}^0$ are lirh on $\overline{U}_k$. Wherefore [13, p. 167, Th. 19.2] there exists a covering $\mathcal{U}$ of the space such that $z^0(\mathcal{U}), \gamma_1^0(\mathcal{U}), \cdots, \gamma_{i_k}^0(\mathcal{U})$ are lirh on $\overline{U}_k$ for all $\mathcal{U} > \mathcal{U}$. By Lemma 10.1 we have the existence of a refinement $\mathcal{U} > \mathcal{U}$ such that

$$z^0(\mathcal{U}) \sim \sum_{j=1}^{s} a^j w_j^0(\mathcal{U}), \qquad\qquad (a^j \in \mathfrak{F}, \quad s \text{ finite}),$$

where the $w_j^0(\mathcal{U})$ $(j = 1, \cdots, s)$ are coordinates on $N(\mathcal{U})$ of nontrivial 0-cycles w_j^0 of M carried by pairs of points. Since the w_j^0 are homologous to a linear combination of the γ_i^0 on $\overline{U}_k$ $(i = 1, \cdots, i_k)$, we may say the same about the coordinates on $N(\mathcal{U})$, namely

$$w_j^0(\mathcal{U}) \sim \sum_{i=1}^{i_k} b_j^i \gamma_i^0(\mathcal{U}).$$

Thus we have the homology on $\overline{U}_k$,

$$z^0(\mathcal{U}) \sim \sum_{j=1}^{s} a^j \sum_{i=1}^{i_k} b_j^i \gamma_i^0(\mathcal{U}),$$

in contradiction to what was noted above. Hence $H_\Gamma^0(M, \mathfrak{F})$, as required.

VI. Applications

12. Related results. As announced in the beginning, the origin of the preceding results lies in an attempt to investigate the problem of imbedding compact spaces in locally connected spaces. Problems of a related nature have already received the attention of various authors. References to these may be found in Wilder [11]; in the latter paper necessary and sufficient conditions are given under which a compact subset of a metric space may be made connected and locally connected by the addition of a set of nonintersecting open arcs with diameters which form a null sequence. Another result in this direction was that of MacKay [9], according to which a Peano continuum may be made 1-connected and locally 1-connected in the sense of homology by the addition of a denumerable set of nonintersecting open 2-cells with diameters forming a null sequence.

13. Preliminary remarks. By application of the formal results obtained in the earlier parts of this paper, we are able to obtain a partial solution of the imbedding problem which gave rise to this investigation. Before proceeding to the first theorem we set down the following notation.

13.1. NOTATION. If γ^0 is a set of nontrivial 0-cycles carried by pairs of points of a space S, and V is an open set contained in S, then by $\gamma^0 \cap \overline{V}$ is meant the collection of 0-cycles of γ^0 whose entire carriers are subsets of $\overline{V}$.

14. A sufficient condition. A sufficient condition for the imbeddability of a compact Hausdorff space in a 0-lc continuum is given by the following theorem.

14.1. THEOREM. *Let M be a compact Hausdorff space and let $\gamma^0 = \{\gamma_i^0\}$ be a countable sequence of nontrivial 0-cycles carried by pairs of points of M, respectively $p_{i1} \cup p_{i2}$, such that:* (i) $H^0(M, \mathfrak{F})$ *is a Hausdorff space in a topology induced by a sequential decomposition* $(\mathbf{H}^0, \Gamma'^0)$, *where γ'^0 is a subset of γ^0;* (ii) *if $\mathfrak{E}$ is any finite covering of M, then* diam $(p_{i1} \cup p_{i2}) < \mathfrak{E}$ [13, p. 106, Def. 3.5] *except for at most a finite number of i;* (iii) *if $\mathfrak{U}$ is any open set of M and $p \in U$, then there exists an open set V, $p \in V \subset U$, such that $H^0(\overline{V}, \mathfrak{F})$ satisfies hypothesis* (i) *in terms of a subset of $\gamma^0 \cap \overline{V}$ (in place of γ^0). Then M is imbeddable as a G_δ subset of a 0-lc continuum formed by the union of M and a countable set of arcs.*

Proof. Let $\{a_i\}$ be a countable collection of disjoint open arcs adjoined to M such that the end-points of $\overline{a}_i$ are p_{i1} and p_{i2}. Let M' be the set composed of $M \cup A$, where $A = \cup a_i$. We shall topologize M' as follows: if U is to be an

open set of M', then $U = U^* \cup V^*$, where U^* is an open subset of M by its original topology, and V^* is an open set of A, where A is topologized by the convention that every open set of an a_i is open in A. The set V^* may be empty if and only if $U^* \subset M - \overline{P}$, where P is the set composed of all carriers $p_{i1} \cup p_{i2}$; U^* may be empty under any hypothesis. If $U^* \cap P \neq 0$, then V^* *must* contain all but at most a finite number of entire arcs a_i such that $p_{i1} \cup p_{i2} \subset U^*$, and complements of closed subsets of the remaining finite number, and it *may* also contain other sets open in A. For each a_i such that only one end-point, say p_{i1}, is in U^*, V^* must contain an open subset of $\overline{a}_i$ containing p_{i1}.

(a) M' *is a compact Hausdorff space containing M as a compact G_δ subset.* In order to show that M' is Hausdorff, it seems evident that we need only verify the separation axiom, and that only for points of $\overline{P}$; the other possibilities have obvious verifications. Therefore let q_1 and q_2 be two distinct points of $\overline{P}$. Since M is compact Hausdorff and hence normal, there exist sets U_1 and U_2 such that $\overline{U}_1 \cap \overline{U}_2 = 0$ and $q_1 \in U_1$ and $q_2 \in U_2$. These sets will correspond to the U^*'s of the open sets of M'. The only difficulty in selecting suitable V^*'s comes when U_1 and U_2 are such that, for a carrier $(p_{i1} \cup p_{i2})$ of some cycle, we have $p_{i1} \in U_1$ and $p_{i2} \in U_2$. Since M is normal, there exist open sets U_1' and U_2' such that $U_1' \cap U_2' = 0$, $\overline{U}_1 \subset U_1'$, $\overline{U}_2 \subset U_2'$. Let $U_3' = M - (\overline{U}_1 \cup \overline{U}_2)$; then $\mathfrak{U}' = (U_1', U_2', U_3')$ is a covering of M, and hence by hypothesis we have diam $(p_{i1} \cup p_{i2}) < U'$ except for at most a finite number of carriers, $(p_{i1} \cup p_{i2})$, such that $p_{i1} \in U_1$ and $p_{i2} \in U_2$. For such a carrier $\overline{a}_i^*$ can be a closed interval of a_i separating p_{i1} and p_{i2} on $\overline{a}_i$; and then $\overline{a}_i - \overline{a}_i^* = \alpha_{i1} \cup \alpha_{i2}$, two disjoint open sets such that $p_{i1} \in \alpha_{i1}$ and $p_{i2} \in \alpha_{i2}$. It should now be evident, without further elaboration, that the disjoint open sets V_1^* and V_2^*, the sets corresponding to the V^* of the defining topology, may be chosen so that disjoint open sets of M' can be constructed which contain q_1 and q_2 respectively; and so M' is a Hausdorff space.

The identity mapping is here also a continuous mapping, and hence M as a subset of M' is compact. Let $\mathcal{E}$ be any covering of M'; then a finite subcollection of $\mathcal{E}$ covers M. By the topology of the space M', at most a finite number of arcs a_i are not completely contained in this subcollection of $\mathcal{E}$. The closures of these arcs are also compact subsets of M', and hence evidently there exists a finite subcollection of $\mathcal{E}$ which covers M', so that M' is a compact Hausdorff space.

Evidently, if γ'^0 is the countable subset of hypothesis (i), the $\mathbf{H}^0$ of the sequential decomposition can be chosen so that if $H_k^0 + G_k^0 = H^r(M, \mathfrak{F})$, direct, then $\Gamma_i'^0$ ($i = 1, 2, \cdots, k$) is a base for G_k^0. This selection merely simplifies the construction of the next paragraph.

We proceed now to the construction of a collection of open sets $\{Q_k\}$ with which $(\mathbf{H}^0, \Gamma'^0)$ will be compatible; incidentally, it also shows that M is a G_δ subset of M'. Let f_m be a fixed homeomorphism of the open interval $0 < x < 1$ onto the open arc a_m. Then if $[a_m : 2^t]$ denotes the complement in a_m of the set $f_m(\cup x \mid 1/2^{t+2} < x < 1 - 1/2^{t+1})$, we shall denote by A_k the set defined by

$$\bigcup_{m > k}^{\infty} \left(a_m \cup [a_k : 2^k] \cup \cdots \cup [a_1 : 2^k] \right).$$

This will be the complement in A of a subset closed in M' and contained in A, so that we may in this way construct a countable collection $\{Q_k = M \cup A_k\}$ of open sets closing down on M with the property that $Q_k \supset Q_{k+1}$. By the topology of the space M' and the construction of $\{Q_k\}$, we may readily see that, with respect to this collection of sets closing down on M, the sequence γ'^0 of 0-cycles on M is such that $\gamma_i'^0 \sim 0$ on $\overline{Q}_k$ for $i > k$. Furthermore, from the properties of the sequential decomposition $(\mathbf{H}^0, \Gamma'^0)$, it is evident that either $\{\gamma_i'^0\}$ $(1 \le i \le k)$, or a subsequence of it, is a base for 0-cycles of M relative to homology on $\overline{Q}_k$. Hence the desired compatibility is established, and in subsequent parts of the proof it will be possible to apply Theorem 9.1.

(b) *M' is connected.* Now let z^0 be any 0-cycle of M'. By Lemma 10.1, for any covering U there exists a refinement $\mathbb{U} > \mathfrak{U}$ such that

$$z^0(\mathbb{U}) \sim \sum_{j=1}^{s} a^j g_j^0(\mathbb{U}),$$

where the $g_j^0(\mathbb{U})$ are coordinates on the covering $\mathbb{U}$ of nontrivial 0-cycles carried by pairs of points. We must show that $z^0(\mathbb{U}) \sim 0$. This will amount to showing that each $g_j^0(\mathbb{U}) \sim 0$. Consider now the nontrivial 0-cycle g_j^0 carried by the pair of points $q_{j1} \cup q_{j2}$; there are three possibilities that must be taken into account.

Case 1: $q_{j1}, q_{j2} \in M$. Under this hypothesis, by Theorem 9.1 and 7.5 (1) there exists a unique homology, $g_j^0 \sim \sum_{i=1}^{\infty} a^i \gamma_i'^0$. Since $\gamma_i'^0 \sim 0$ on M' for all i, then, as a consequence of Corollary 9.2, the complete hypothesis of Lemma 6.1 is satisfied, so that $g_j^0 \sim 0$ on M' as required for the connectedness of M'.

Case 2: $q_{j1}, q_{j2} \in A$. Suppose $q_{j1} \in a_m$ and $q_{j2} \in a_n$. Then evidently $g_j^0 \sim z_{mn}^0$, where z_{mn}^0 is a nontrivial 0-cycle carried by a pair of points, $p_{m1} \cup p_{n1}$ (there are points of the carrier sets of 0-cycles in γ^0). Hence z_{mn}^0 is a 0-cycle of M, and we may proceed as above with the same result.

Case 3: $q_{j1} \in M$ and $q_{j2} \in A$. If $q_{j2} \in a_i$, then $g_j^0 \sim z_{1i}^0 - z_{21}^0$, where z_{1i}^0 is a nontrivial 0-cycle carried by the pair $q_{j1} \cup p_{i1}$, and z_{21}^0 is a nontrivial 0-cycle carried by the pair $p_{i1} \cup q_{j2}$. Evidently $z_{21}^0 \sim 0$ on M'; hence $g_j^0 \sim z_{1i}^0$ on M', where z_{1i}^0 is on M. Again we may proceed as above with the same result.

Thus we have shown that in all possibilities we have $g_j^0(\mathfrak{U}) \sim 0$ on M', so that the space M' is connected.

(c) *The space M' is* 0-lc. To examine M' for 0-local connectedness, it is evident that we need only investigate neighborhoods of M. Let $p \in M \subset M'$; and let U be any open subset of M' containing p. Then by the topology of M', we have $U = U^* \cup V^*$, where $p \in U^*$; further, by the hypothesis, there exists a $U_1^* \subseteq U^*$ such that $H^0(\overline{U}_1^*, \mathfrak{F})$ is a Hausdorff space under a topology induced by a sequential decomposition based on a subset of $\gamma^0 \cap \overline{U}_1^*$. Let V_1^* be a set contained in V^* such that $\overline{V}_1^* \subset U$, and containing all but a finite number of entire arcs a_i corresponding to carriers of the subset of $\gamma^0 \cap \overline{U}_1^*$ just above, and complements of a_i of closed intervals of the remaining finite number. Then $U_1 = U_1^* \cup V_1^* \subset U$. Let k be the finite number of arcs a_i such that the entire arc fails to belong to U. Then for any covering $\mathfrak{U}$ (by the same process as was used above in proving the connectedness of M'), at most $2k$ 0-cycles of U_1 are lirh in U; hence by 8.1 the space M' is 0-lc.

15. **The class $\mathfrak{R}$.** In order to improve the value of the preceding theorem, it is desirable that the class $\mathfrak{R}$ of compact spaces which satisfy the hypothesis includes, in addition to all metric spaces, a definite nonmetric space. This is partially accomplished by showing that the nonmetric compact Hausdorff space of Alexandroff-Urysohn [used in another problem, 1, p. 76] fulfills the requirements of the theorem; details of this procedure are tedious, though not difficult, and will not be given here. Using practically the same technique, we can show that the class of all compacta is a subset of $\mathfrak{R}$. Of course this last result yields immediately the conclusion that every compactum is imbeddable as a compact G_δ subset of a 0-lc continuum formed by the addition of a countable set of arcs; this is also a corollary of the Urysohn theorem on the imbedding of such spaces in the Hilbert parallelotope.

16. **A necessary condition.** In giving a necessary condition, we shall restrict ourselves to separable[1] compact G_δ subsets. Incidentally, we might remark that the Alexandroff-Urysohn nonmetric space, referred to above, is separable; this indicates that the restriction imposed does not limit the spaces

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[1] Because of the occasional variety that is observed in the use of this term, we note that in this paper a *separable* space will be one having a countable dense subset.

to which the theorem is applicable too sharply. As an immediate consequence of an obvious lemma and a theorem due to Wilder [13, p. 143, Th. 11.10], we may state the following theorem.

16.1. THEOREM. *If S is a compact 0-lc space, U is an open set of S, and p, $q \in U$, such that a nontrivial cycle carried by $p \cup q$ bounds in U, then $p \cup q$ lies in a connected 0-lc subset of U.*

The necessary condition is given in the following theorem.

16.2. THEOREM. *Let M be a separable compact G_δ subset of a locally compact space S. If M is a subset of a 0-lc open subset of S, then there exists a countable $\gamma^0 = \{\gamma_i^0\}$ of nontrivial 0-cycles of M carried by pairs of points such that: (i) if $p \in M$ and U is any open set containing p, then there exists a $V \subseteq U$, $p \in V$, such that a subset γ_v^0 of $\gamma^0 \cap (\overline{V} \cap M)$ gives rise to a sequential decomposition $(\mathbf{H}^0, \Gamma_v^0)$ which induces a Hausdorff topology in $H^0(\overline{V} \cap M, F)$; (ii) all but a finite number of carriers of the 0-cycles of γ_v^0, for any open set $P \supset \overline{V} \cap M$, are contained in 0-lc connected open sets of P.*

Proof. Let M^* be a countable dense subset of M, and let γ^0 be the collection of all nontrivial 0-cycles carried by pairs of points of M^*. Let p be an arbitrary point of M, and U any open set of S containing p. Then $\overline{U}^* = \overline{U} \cap M$ is a compact subset of M. We may then select a set V, open in M, containing p and such that $\overline{V} \subset U \cap M$.

Consider now a countable family of coverings $\{\mathcal{E}_i\}$, relative to the 0-lc open subset of S containing M, each consisting of connected open sets and such that the portion on $\overline{V}$ of any $\mathcal{E}_i$ is the star-closure refinement[2] of all its predecessors, and further such that if c_i is the number of components of the covering $\mathcal{E}_i \cap \overline{V} = \mathcal{E}_i^*$, then $c_i > c_j$ for $i > j$. We shall now select a sequence of points of $\overline{V} \cap M^*$ as follows: let the components of $\mathcal{E}_i^*$ be arbitrarily ordered, $C_1, C_2, \cdots, C_{c_1}$; then the components of $\mathcal{E}_2^*$ are indexed $C_{c_1+1}, \cdots, C_{c_1+c_2}$, in such a way that those which are contained in C_1 are indexed first; then the indexing proceeds to those of the remainder contained in C_2, and so on. A continuation of this process gives a countable sequence of elements, $\mathbf{C} = \{C_j\}$. Now in each element C_j choose a point, $p_j \in \overline{V} \cap M^*$, in such a way that (1) at most one $p_j \in C_k$ $(j \leq k)$, and (2) when a previously selected $p_j \in C_k$ $(j < k)$, then the selection of a new p_k is omitted. With pairs of $p_j's$ as carriers, we may associate nontrivial 0-cycles γ_j^0, carried by the pair $p_j \cup p_k$, where the pairing is made as follows: if $\mathcal{E}_i^*$ is the first covering of the sequence of coverings $\{\mathcal{E}_i^*\}$, such that p_j and

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[2] If $\mathrm{St}(\mathcal{U}, \mathcal{U})$ is a closure refinement [13, p. 133, Def. 8.5] of a covering $\mathcal{U}$, then $\mathcal{U}$ is called a *star-closure refinement* of $\mathcal{U}$.

p_k belong to distinct components of $\mathcal{E}_i^*$, then p_j is paired to p_k if and only if they belong to the same component of $\mathcal{E}_{i-1}^*$; this will give us a countable ordered subcollection γ_v^0 of $\gamma^0 \cap \bar{V}$, the elements of which have the following properties: (1) the set γ_v^0 is lirh on $\bar{V}$, (2) for any covering $\mathcal{E}_i^*$, the coordinates of the first $c_i - 1$ of the 0-cycles are lirh on the covering $\mathcal{E}_i^*$ and form a basis for homology on the covering, and the remainder are homologous to zero. Consequently we may show in a way similar to that used in the proof of Theorem 11.1 that this collection gives rise to a sequential decomposition which induces a Hausdorff topology in $H^0(\bar{V}, \mathcal{F})$.

The remainder of the proof follows easily. By a theorem due to Begle [2, p. 63] all but a finite number of 0-cycles of the collection γ_v^0 bound in P, so that by Theorem 16.1 all but a finite number of the carriers of the 0-cycles of this collection are contained in 0-lc connected sets of P.

Since the 0-lc property is a local characteristic of a space, we could give an obvious localization of the preceding result with the natural satisfaction of aesthetic completeness.

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