# THE NUMBER OF FARTHEST POINTS

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**1. Introduction.** Consider a set $S$ in a metric space $E$. *For each point* $x \in E$, let $y(x)$ *denote a point of* $S$ *which has maximum distance from* $x$, and let $Y(x)$ be the set of all $y(x)$ with that property. It is our purpose here to study sets $S$ for which certain restrictions are placed on the number of points in $Y(x)$. In §2 we analyze those sets $S$ in the Minkowski plane for which $Y(x)$ has exactly one element for each $x \in S$. In §3 we characterize those sets in the Euclidean plane $E_2$ for which $Y(x)$ has at least two elements for each $x \in S$.

In order to achieve these ends we first establish some introductory results which hold in rather general spaces.

DEFINITION 1. Let $S$ be a set in a metric space. If $S$ is contained in a sphere of radius $r$, then its *r-convex hull* is the intersection of all closed spheres of radius $r$ which contain $S$.

A set $S$ is *r-convex* if it coincides with its $r$-convex hull [2, p. 128].

LEMMA 1. *Let* $S$ *be a set of diameter* $d$ *in a linear metric space. Then for each* $x \in S$ *the set* $Y(x)$ *lies in the boundary of the* $d$-*convex hull of* $S$.

*Proof.* If $Y(x) \neq 0$, choose any point $y(x)$. Then $S$ is contained in a sphere with center at $x$ and with radius $d(x, y)$, where $d(x, y)$ denotes the distance from $x$ to $y$. Since for $x \in S$ we have $d(x, y) \leq d$, there exists a point $z$ on the ray $\overrightarrow{yx}$ such that the sphere with center $z$ and with radius $d = d(z, y)$ contains $S$. The point $y$ is thus clearly on the boundary of the $d$-convex hull.

NOTE. By virtue of Lemma 1, all results for compact $S$ below will hold under the less restrictive assumption that $S$ contain the intersection of its closure with the boundary of its $d$-convex hull.

COROLLARY 1. *Let* $S$ *be a set in a linear metric space. Then for each* $x$ *the set* $Y(x)$ *is contained in the boundary of the convex hull of* $S$.

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This is an immediate consequence of the fact that $S$ is contained in the sphere with center $x$ and radius $d(x, y(x))$, provided $Y(x) \neq 0$.

LEMMA 2. *Suppose $S$ is a set in a linear metric space, and let $T$ be a set such that $Y(x) \neq 0$ for each $x \in T$. Then $d(x, y(x))$ is a continuous function of $x$ on $T$.*

*Proof.* Since $|d(x, z) - d(u, z)| \leqq d(x, u)$, and since

$$\left|\max_{z \in S} d(x, z) - \max_{z \in S} d(u, z)\right| = |d(x, y(x)) - d(u, y(u))|,$$

we have $|d(x, y(x)) - d(u, y(u))| < \epsilon$ if $d(x, u) < \epsilon$.

LEMMA 3. *Let $S$ be a compact set in a linear metric space. If $x_i \longrightarrow x$, then all limit points of the sequence $\{y(x_i)\}$ lie in $Y(x)$.*

*Proof.* Let $y_i = y(x_i)$ be a sequence of points. Let $y$ be a limit point of the sequence $\{y_i\}$. Then the continuity of $d(x, y(x))$ implies that $d(x, y) \geq d(x, q)$ for all $q \in S$. Hence we have $y \in Y(x)$.

LEMMA 4. *Let $S$ be a compact set in a linear metric space, and suppose $y(x)$ is single-valued on a set $T$. Then $y(x)$ is a continuous mapping of $T$ into $S$.*

*Proof.* Since $y(x)$ is single-valued, Lemma 3 implies that if $x_i \longrightarrow x$, then $y(x_i) \longrightarrow y(x)$.

**2. Sets in $M_2$ on which $y(x)$ is single-valued.** Let $M_2$ be a two-dimensional Minkowski space [2, p. 23]. We restrict our attention here to connected sets $S$ in $M_2$. (See § 4 for remarks about disconnected sets.)

THEOREM 1. *Let $S$ be a continuum (compact connected set) in $M_2$. If $y(x)$ is single-valued on $S$, then the set sum*

$$\sum\nolimits_{x \in S} Y(x)$$

*is the entire boundary $B$ of the convex hull of $S$; and this convex hull is $d$-convex, where $d$ is the diameter of $S$.*

*Proof.* According to Corollary 1, we have

$$\sum\nolimits_{x \in S} Y(x) \subseteq B.$$

By Lemma 4, the mapping $y(x)$ yields a continuous mapping of $S$ into $B$. Now the only connected sets in a simple closed curve are: (1) a point, (2) a simple arc, (3) the whole closed curve. For cases (1) and (2), let

$$A \equiv \sum_{x \in S} Y(x);$$

then the mapping $y(x)$ of $A$ into itself must have a fixed point $x_0 = y(x_0)$, so that $\{x_0\} = Y(x_0) = S$, in which case the theorem is trivial. Thus $A = B$ in all three cases. Moreover, since by Lemma 1 the set $A = B$ lies in the boundary of of the $d$-convex hull of $S$, the boundary of the $d$-convex hull must coincide with $B$.

Since there is no continuous mapping without fixed points of a closed two-cell into itself, Lemma 2 and Theorem 1 imply that, for single-valued $y(x)$, the connected bounded set $S$ must contain the entire boundary of its convex hull, but not all of the interior of that hull (unless $S$ consists of a single point). It may suffice, in some cases, to delete one single point from the interior of a convex set; for instance, in the case of a circular disc in $E_2$, the deletion of the center makes $y(x)$ single-valued throughout.

In the remaining theorems and lemmas we restrict our attention to sets in $E_2$.

DEFINITION 2. By a *normal* to a convex curve $C$ at a point $x \in C$ we mean a line perpendicular to a line of support to $C$ at $x$.

NOTATION. We designate a line of support at $x$ by $L(x)$, and the corresponding normal by $N(x)$. Further, for a point $y \in S$, we let $x(y)$ be a point in $S$ such that $y = y(x)$, and let $X(y)$ be the set of all $x(y)$.

THEOREM 2. *Suppose $S$ is the boundary of a compact convex set in $E_2$, and suppose $y(x)$ is single-valued on $S$. Then:*

(1) *The set $X(y)$ consists of all points of intersection of the normals to $S$ at $y$ with $S - y$. If $S$ has a tangent at $y$, then $x(y)$ is single-valued and continuous at $y$.*

(2) *The mapping $x(y)$ is monotonic; that is, the order of $x(y_1)$, $x(y_2)$, $x(y_3)$ on $S$ has the same sense as that of $y_1, y_2, y_3$.*

*Proof.* (1) If $x = x(y)$, then the circle with center $x$ and radius $d(x, y)$ contains $S$. Hence the tangent to this circle at the point $y$ is also a line of support to $S$, and the radius lies in a normal to $S$ at $y$.

Now, let $y_i \longrightarrow y$, $y_i \in S$, and choose $x_i = x(y_i)$. Then, due to the continuity of the mapping $y(x)$, each limit point of $\{x_i\}$ is in $X(y)$. Thus if $S$ has a tangent at a point $y$, then the mapping $x(y)$ is one-to-one and continuous at $y$.

To complete the proof of (1), suppose $S$ has a corner at $y$. Then the farthest points of intersection from $y$ of the normals at $y$ with $S$ fill out a closed subarc of $S$, which we denote by $S_1$; the end-points of $S_1$ we denote by $u_l$ and $u_r$. There exists a sequence $y_i \in S$ with $y_i \longrightarrow y$ such that the normals to $S$ at the $y_i$ are unique and approach the left normal at $y$. Hence, by the above, $x(y_i)$ converges to $u_l$, and hence $u_l \in X(y)$. Similarly, $u_r \in X(y)$. The three lines determined by $u_l$, $u_r$, and $y$ divide the plane into seven closed sets, and the arc $S_1$ is contained in that unbounded one which has $u_l u_r$ as part of its boundary. We denote that set by $A$. Since each of the two circles with centers $u_l$ and $u_r$ which pass through $y$ contains $S$, it follows by the law of cosines that $y(u) = y$ for all $u \in A$. Hence $S_1 \subseteq X(y)$. According to Theorem 1, the curve $S$ contains no straight line-segment, and thus any normal to $S$ intersects $S$ in exactly two points. Hence the common part $(S - S_1) \cdot X(y)$ is the null set, so that $S_1 = X(y)$.

(2) The above facts, together with the fact that each $u \in S$ is contained in some $X(y)$, imply that the transformation $x(y)$ maps connected sets into connected sets, even though the mapping need not be single-valued and therefore not necessarily continuous. The single-valuedness of $y(x)$ implies that if $y_1 \neq y_2$, then $X(y_1) \cdot X(y_2) = 0$. If the transformation $x(y)$ failed to be monotonic, it would have a fixed point $y = x(y)$; but this is impossible unless $S$ is a single point. Hence condition (2) must hold.

COROLLARY 2. *Suppose $C$ is the boundary of a compact convex set $S$. Let $\alpha\beta$ be a diameter of $C$, and let $N(\alpha, \beta)$ designate the common normal to $C$ through $\alpha$ and $\beta$. Then $y(x)$ is single-valued on $C$ if and only if for every pair of points $u$, $v \in C$ which lie on the same side of $N(\alpha, \beta)$, the normals $N(u)$ and $N(v)$ intersect at an interior point of $S$.*

*Proof.* First observe that, for any compact convex set $S$ with $\alpha\beta$ as a diameter, if $x \cdot \alpha\beta = 0$, then $x$ and $y(x)$ must lie on opposite sides of $N(\alpha, \beta)$.

To prove the necessity, observe that $\alpha$ and $\beta$ are involutory points in the sense that

$$y(y(\alpha)) = \alpha \text{ and } y(y(\beta)) = \beta.$$

Hence the necessity follows from the monotonicity of $y(x)$ as described in

Theorem 2.

To prove the sufficiency, first choose $x \in C-(C \cdot \alpha\beta)$. Suppose $y(x)$ is not single-valued, and choose $u, v \in Y(x)$. As mentioned above, $y(x)$ and $x$ lie on opposite sides of $N(\alpha, \beta)$. A circle with center $x$ and radius $d(x, u)$ is tangent to $C$ at both $u$ and $v$, and the normals $N(u)$ and $N(v)$ intersect at $x$, which is not interior to $S$. Hence $y(x)$ is single-valued for $x \in C-(C \cdot \alpha\beta)$. By continuity it follows also that

$$y(\alpha)=\beta, \; y(\beta)=\alpha.$$

This completes the proof.

In the following we shall extend the generalized notions of curvature described by Bonnesen and Fenchel [2, pp. 143-144]. Choose a point $x \in C$, where $C$ is a closed convex curve together with a line of support $L(x)$. The circle tangent to $L(x)$ at $x$ and passing through a point $p \in C-x$ must have its center $z(p)$ on the normal $N(x)$ to $L(x)$ at $x$. Establish an order on $N(x)$ in terms of the distance from $x$, and let

$$\left.\begin{aligned} E_s(x, \delta(x)) &\equiv \sup_p z(p) \\ E_l(x, \delta(x)) &\equiv \inf_p z(p) \end{aligned}\right\} \quad p \in \delta(x)-x,$$

where $\delta(x)$ is an arc of $C$ containing $x$. We define four types of centers of curvature as follows:

$$E_s(x) \equiv E_s(x, C), \quad E_l(x) = E_l(x, C).$$

$$E_o(x) \equiv \lim_{\delta(x)\to x} E_s(x, \delta(x)), \quad E_i(x) \equiv \lim_{\delta(x)\to x} E_l(x, \delta(x)).$$

Clearly $E_l(x) \le E_i(x) \le E_o(x) \le E_s(x)$ relative to $N(x)$.

DEFINITION 3. The sets

$$\sum E_s(x), \; \sum E_o(x), \; \sum E_i(x), \text{ and } \sum E_l(x)$$

($x$ ranges over $C$) are respectively called the *superior evolute*, the *outer evolute*, the *inner evolute*, and the *inferior evolute* of $S$, and are denoted by $E_s, E_o, E_i, E_l$.

THEOREM 3. *Suppose $C$ is the boundary of the compact convex set $S \subset E_2$. If $y(x)$ is single-valued on $C$, then the superior evolute, and hence all four* *evolutes, of $C$ must be contained in $S$.*

*Proof.* Since $y(x)$ is single-valued for each point $x_1 \in C$, the proof of Theorem 2 implies that for any normal $N(x_1)$, the set $N(x_1) \cdot (C - x_1)$ consists of a single point, denoted by $x'$. Choose $p \in C - x_1$. Since

$$d(x', p) < d(x', x_1) = d(x', y(x')),$$

it is clear that the perpendicular bisector $B$ of the segment $x_1 p$ intersects the segment $x_1 x'$. Hence

$$B \cdot x_1 x' = z(p) \in S.$$

THEOREM 4. *Suppose the inner evolute of the boundary $C$ of the compact convex set $S$ is contained in $S - C$. Then $y(x)$ is single-valued on $C$.*

*Proof.* Suppose there exists an $x \in C$ such that $y(x)$ is not single-valued. Choose $u$, $v \in Y(x)$. The circle with center $x$ and radius $d(x, u)$ contains $S$ and is tangent to $C$ at $u$ and $v$. Hence the arc $uv$ of $C - x$ contains a point $w$ of minimal distance from $x$. The circle with center $x$ and radius $d(x, w)$ is tangent to $C$ at $w$, while a neighboring arc of $w$ on $C$ lies outside or on that circle. Hence $C$ has a unique normal at $w$ and $E_i(w) \geq x$, so that $E_i(w)$ is on or outside $C$.

Theorems 3 and 4 do not determine the single-valuedness of $y(x)$ on $S$ if $E_i$, $E_o$, and $E_s$ lie in $S$ and contain points of $C$. This situation can be described as follows:

THEOREM 5. *Let $S$ be a compact convex set with boundary $C$ such that $E_s$ (and hence each of the evolutes) of $C$ lies in $S$. Then $y(x)$ fails to be single-valued on $C$ if and only if there exists a point $x \in C$ which lies on $E_i$, $E_o$, and $E_s$, and which is the center of a circular arc contained in $C$.*[1]

*Proof.* To prove sufficiency, suppose there exists a point $x \in C$ which is the center of a circular arc $C_1 \subset C$, and suppose $y(x)$ is single-valued on $C$. Then according to Theorem 2 the single-valuedness of $y(x)$ implies $x \in X(y)$ for each $y \in C_1$. Hence $C_1 \subseteq Y(x)$, a contradiction.

To prove necessity, assume $y(x)$ is not single-valued on $C$. Choose $u$, $v \in Y(x)$, and let $w$ be a nearest point to $x$ of the arc $C_1$ of $C - x$ joining $u$ and $v$. In the proof of Theorem 4 we saw that $E_i(w) \geq x$; but since the evolutes are

[1] By "center of a circular arc" we mean the center of the circle to which the arc belongs.

in $S$, we have

$$E_i(w) = E_o(w) = E_s(w) = x.$$

(Since $E_s$ is bounded, $C$ can contain no straight line segments.) Hence the circle with center $x$ and radius $d(x, w)$ contains $S$. Thus $d(x, w) \geq d(x, u)$. From the definition of $w$ it now follows that $d(x, z) = d(x, u)$ for each $z \in C_1$. Hence $C_1$ Hence $C_1$ is circular arc in $C$ with center at $x$.

As seen earlier, if $S$ is a simply connected set containing at least two points, then $y(x)$ is not single-valued on $S$. The situation is described more fully in the following theorem.

THEOREM 6. *Let $S$ be a compact convex set in $E_2$ with boundary $C$. Then $y(z)$ is single-valued if $z \neq E_s(x)$ for all $x \in C$; and $y(z)$ is not single-valued if $z = E_s(x)$, $z \neq E_o(x)$ for some $x \in C$.*

*Proof.* Assume $y(z)$ is not single-valued; then there exist distinct points $u \in Y(z)$, $v \in Y(z)$, and the circle with center $z$ and radius $d(z, u)$ contains $S$ and is tangent to $C$ at $u$ and $v$. Hence $E_s(u) = E_s(v) = z$.

Now suppose there exists an $x \in C$ such that $z = E_s(x)$, $z \neq E_o(x)$. Then, since $C$ is compact, there exists a point $u \neq x$, $u \in C$, such that

$$d(z, u) = d(z, x) = d(z, y(z)).$$

Hence $u \in Y(z)$, $x \in Y(z)$. Thus Theorem 6 is proved.

A few remarks about the four evolutes may be desirable at this point. The inferior and superior centers of curvature, $E_l(x)$ and $E_s(x)$, are determined by properties in the large. In fact, $E_l$ contains the set of centers of those circles which are in $S$ and which are tangent to $C$ at not less than two points. Similarly $E_s$ contains the sets of centers of those circles which contain $C$ and which are tangent to $C$ at not less than two points.

Since a convex curve $C$ has curvature almost everywhere, we have $E_i(x) = E_o(x)$ for almost all $x \in C$. Let us define

$$E \equiv \sum E_i(x)\, E_o(x),$$

($x$ ranges over $C$), where, as usual, $E_i(x)\, E_o(x)$ denotes a closed segment. The number of normals to $C$ through a point $x \in E_2$, as a function of $x$, is the same in each component of the complement of $E$. In the case where $S$ is a compact convex set for which $E$ is bounded, there are exactly two normals to $C$ through each point $x$ in the unbounded component of the complement of $E$ (the lines joining $y$ to the nearest and farthest points on $C$). However, from each point $y \notin E$ on $E_l(E_s)$ there are at least four normals to $C$. [According to Theorem 6, there are at least two normals to the two or more points of tangency $u$, $v$ of the inscribed (circumscribed) circle with center at $y$. In addition, there are lines joining $y$ to nearest (farthest) points on each of the two arcs of $C$ joining $u$ and $v$.] Thus $E_l$ and $E_s$ do not intersect the unbounded component of $\overline{E}$. These statements imply the following:

THEOREM 7. *Let $C$ be the boundary of a compact convex set $S \subset E_2$. Then $E_s \subset S$ if and only if $E_o \subset S$. Also $E_s \subset S - C$ if and only if $E_o \subset S - C$.*

AN EXAMPLE. Consider the family of ellipses $C(e)$,

$$b^2 x_1^{\,2} + a^2 x_2^{\,2} = a^2 b^2, \ a \geqq b.$$

If the eccentricity $e$ satisfies the condition $e \leq \sqrt{2}/2$, then $y(x)$ is single-valued on $C(e)$. If $e > \sqrt{2}/2$, then $y(x)$ is not single-valued at $x = (0, \pm b)$. In each case the inner and outer evolutes coincide; they form the familiar astroid with cusps at

$$\xi = (a_1, 0), \ \eta = (-a_1, 0), \ \tau = (0, b_1) \text{ and } \rho = (0, -b_1),$$

where $a_1 < a$, and $b_1 < b$ for $e < \sqrt{2}/2$ while $b_1 > b$ for $e > \sqrt{2}/2$. The superior evolute $E_s$ is the closed line-segment $\rho\tau$, and $E_l$ is the closed line-segment $\xi\eta$. If $e \neq 0$, then $y(x)$ is single-valued on the complement of the open segment $\rho\tau - \rho - \tau$.

### 3. Sets on which $Y(x)$ contains at least two points.

THEOREM 8. *Let $S \subset E_2$ be a compact set of diameter $d$, and let $D$ denote the set of end-points of diameters of $S$. If $Y(x)$ has at least two elements for each $x \in D$, then $Y(x)$ consists of exactly two points for $x \in D$, and $D$ contains a finite number of points. The $d$-convex hull of $S$ coincides with the $d$-convex hull of $D$.* [Since the latter is a Reuleaux polygon (see below), $D$ must contain an odd number of points.]

*Proof.* Let $\Sigma \equiv \{C(x)\}$ be the family of circular boundaries $C(x)$ with centers $x \in D$ and with radii $d$. Let $x \in D$; then

$$Y(x) = C(x) \cdot D.$$

Since

$$\text{diam } Y(x) \leq \text{diam } S = d,$$

there exists a *smallest* arc $A$ of $C(x)$ which contains $Y(x)$, and which has a length not exceeding $\pi d/6$. Let $x_1$ and $x_2$ be the end-points of $A$. If a circle $C(x') \in \Sigma$ were to intersect $A - x_1 - x_2$, then $C(x')$ would separate $x_1$ and $x_2$ since

$$\text{length } A \leqq \pi d/6.$$

But this contradicts the fact that $S \subset C(x')$. For any $x \in D$, we have $z = y(x)$ if and only if $x = y(z)$. Hence every $x \in D$ is a point of intersection of at least two circles of $\Sigma$. These facts imply that $Y(x) \equiv \{x_1, x_2\}$.

Define

$$H \equiv \prod\nolimits_{x \in D} K(x),$$

where $K(x)$ is the closed circular disk with center $x$ and with radius $d$. Then each $x \in D$ lies in the interior of all $K(x) \subset H$ except $K(x_1)$ and $K(x_2)$, where $Y(x) = \{x_1, x_2\}$. Hence $x$ is a corner-point of the boundary of $H$. As above, let $A_1$ and $A_2$ be the smallest arcs of $C(x_1)$ and $C(x_2)$ containing $Y(x_1)$ and $Y(x_2)$, respectively. We have shown that $A_1 \cdot A_2 = \{x\}$; and $A_1$ and $A_2$ are in the boundary of $H$. Thus $x$ is an isolated corner of the boundary of $H$. Hence $D$ contains a finite number of points, and by definition the boundary of $H$ is the boundary of the $d$-convex hull of $D$. It is clearly a Reuleaux polygon, that is, a convex circular polygon whose arcs have radii $d$, and whose vertices are the centers of these arcs [2, pp. 130-131].

Finally, each of the circles in $\Sigma$ contains $S$, and hence $S \subset H$.

COROLLARY 3. *Let $S$ be a set satisfying the conditions of Theorem 8. Then $Y(x) \subseteq D$ for each $x \in S$.*

This is an immediate consequence of the fact that $D$ consists of the vertices of $H$.

THEOREM 9. *Let $S \subset E_2$ be a compact set such that $Y(x)$ has at least two elements for each $x \in S$. Then $S$ lies in the union of a finite number of line-segments. Moreover, if $Y(x)$ has exactly two elements for each $x \in S$, then $S$ cannot be connected.*

*Proof.* Since $Y(x) \subseteq D$ for each $x \in S$, the fact that $Y(x)$ has a least two elements implies that $x$ lies on the perpendicular bisector of the line joining two elements of $D$. Thus $S$ is a subset of the set obtained by taking the union of the intersections of these perpendicular bisectors with $H$.

Since the set $H$ has at least three corners $x_1$, $x_2$ and $x_3$, let $S_i$ $(i = 1, 3)$ consist of those $x \in S$ such that $\{x_i, x_2\} \subseteq Y(x)$. Each set $S_i$ is nonempty since $S_i$ contains the center of the smaller arc of $H$ joining $x_i$ and $x_2$. From the continuity of $d(x, y(x))$, it follows that $S_i$ is closed. Hence if $S$ is connected, then $S_1 \cdot S_2 \neq 0$ (since $S$ is compact), and thus there exists an $x' \in S$ such that $Y(x') \supseteq \{x_1, x_2, x_3\}$. This establishes the theorem.

We also obtain the following result due to Bing [1].

COROLLARY 4. *Let $S$ be a bounded set in $E_2$ containing at least two points, and having the property that with every two points $x \in S$, $y \in S$ there exists a $z \in S$ such that the triangle $x\ y\ z$ is equilateral. Then $S$ is the set of vertices of an equilateral triangle.*

*Proof.* The closure $\bar{S}$ of $S$ must also satisfy the hypothesis stated. Consider the set $D$ of Theorem 8 relative to $\bar{S}$. If $x \in D$, and $\{y, z\} \subseteq Y(x)$, then $d(y, z) = d$, so that $x$, $y$, $z$ form the vertices of a Reuleaux polygon, and therefore by Theorem 8 we have $D = \{x, y, z\}$. Now let $u$ be the centroid of the triangle $x$, $y$, $z$. By Theorem 9, $S$ is contained in the segments $xu$, $yu$, and $zu$. Suppose $v \in (S \cdot xu - x)$; then $Y(v) = \{y, z\}$. But $v$, $y$, $z$ is not equilateral; hence $S \cdot xu = x$. Similarly, $S \cdot yu = y$, $S \cdot zu = z$. Consequently, $S = \{x, y, z\}$.

**4. Remarks and problems.** Several questions are raised by our theorems.

(1) If we try to characterize disconnected sets in $E_2$ for which $y(x)$ is single-valued, we see that this condition is not very restrictive. In fact, given any set $S$ which contains at least one point of the boundary of its $r$-convex hull $H$ for some radius $r$, we can adjoin a single point $z$ to $S$, such that $z$ lies on an interior normal to $H$ at a point of $H \cdot S$, and such that $y(x)$ relative to $S + \{z\}$ is single-valued on $S + \{z\}$.

(2) The characterization of connected sets $S$ in $E_n$ $(n > 2)$ for which $y(x)$ is single-valued on $S$ offers considerable difficulties. The mapping $y(x)$ still yields a continuous map of $S$ into the boundary of its convex hull, but it need no longer be an onto mapping. For example, the torus, both the solid and its surface, will have single-valued $y(x)$ for suitable ratios of the two radii. The argument that a nontrivial compact $S$ which contains no indecomposable continua cannot be simply-connected holds, however, regardless of dimension, since every continuous mapping of such a simply-connected set $S$ into itself has fixed points [4].

(3) The generalization of the discussion of multivalued $y(x)$ suggests the

following problem: Let $S$ be a compact set in $E_n$ such that $Y(x)$ has at least $k$ elements for $x \in S$. Does it follow that $S$ lies in the union of a finite number of $(n-k+1)$-dimensional planes? (Note that in the case $k = 1$ this is no restriction, while for $k > n+1$ there would be no sets $S$.) Are there any sets for which $k = n+1$?

It seems likely that this generalization is false, since the argument which proved the finiteness of the set $D$ in Theorem 8 fails for $n > 2$.

In the case $k \geq n$, all points of $D$ are vertices of their $d$-conves hull. Thus in this case $D$ must surely be denumerable.

(4) Is it possible to generalize Corollary 4, as follows:

If the bounded set $S$ in $E_n$ contains at least two points; and if, for some $k \geq 2$, with every two points $x, y \in S$ there are $k-1$ points in $S$ which together with $x, y$ form the vertices of a regular $k$-simplex, does it follow that $S$ is the set of vertices of a regular $l$-simplex, where $k \leq l \leq n$?

(5) Another question raised by Corollary 4 is the following:

What are the sets (bounded sets, compact sets) $S$ in $E_2$ which have the property that with every $x, y \in S$ there is a $z \in S$ such that $x\ y\ z$ is an isosceles triangle with vertex $z$ and prescribed verticle angle $\alpha$?

For $\alpha < \pi/3$, a nontrivial set with the stated property obviously cannot be bounded. For $\alpha = \pi/3$, the question for the bounded case is answered by Corollary 4. For $\alpha > \pi/3$, there is a considerable variety of bounded sets, although none of them can be finite. In fact, for $\alpha > \pi/3$ every $S$ must be dense in itself; and thus, if closed, it must be perfect. The case $\alpha = \pi$ has been discussed by J. W. Green and W. Gustin [3]; for closed sets $S$, this case characterizes convexity.

An easy argument shows that for compact $S$, and $\pi/3 < \alpha \leq \pi/2$, the entire line-segment joining two farthest points of $S$ must be contained in $S$.

It may also be worth remarking that if $S$ has the foregoing property for an angle $\alpha$, then its complement has the same property for the angle $\pi - \alpha$. Thus the case $\alpha = \pi/2$ is especially noteworthy, since in this case the class of all $S$ with the stated property is closed under the operation of taking complements.

(6) Finally, one should compare the theorems about $Y(x)$ with those for $M(x)$, where $M(x)$ denotes the subset of $S$ whose points have minimum distance from $x$. In particular the theorem of Motzkin [6, 7] (see also Jessen [5]) states that a closed set $S$ is convex if and only if $M(x)$ is a single point for all $x$. This theorem does not correspond to any of the results on $Y(x)$ in § 1. In fact, the

analogous assumption, concerning a (not necessarily closed) set $S$ in $E_n$, that $y(x)$ be single-valued *for all* $x$, is satisfied if and only if $S$ consists of a single point.

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