MAPPING PROPERTIES OF CESÀRO SUMS OF ORDER TWO OF THE GEOMETRIC SERIES

S. E. RAUCH

1. Introduction. Previous investigations of the mappings

$$w = S_n^{(k)}(z)$$

of the unit circle $|z| \leq 1$, where

$$S_n^{(k)}(z) = \binom{n+k}{k} + \binom{n+k-1}{k} z + \cdots + \binom{k}{k} z^n$$

denotes the nth Cesàro sum of order k of the geometric series, have been made by Fejér, Schweitzer, Sidon, and Szegö. Knowledge of the properties of the sums $S_n^{(k)}(z)$ is valuable in the study of power series having coefficients monotonic of order $k+1$.

The present article provides additional asymptotic properties for

$$S_n^{(2)}(e^{i\phi}) = x_n(\phi) + iy_n(\phi).$$

The following results are established:

THEOREM 1. *For n sufficiently large, an α_n exists such that $y_n(\phi)$ is increasing for $0 < \phi < \alpha_n$ and decreasing for $\alpha_n < \phi < \pi$. Furthermore,*

$$\alpha_n = \alpha/n + O(n^{-2}), \quad \textit{where} \quad \pi < \alpha < 3\pi/2.$$

THEOREM 2. *For n sufficiently large, a β_n exists such that*

$$x_n'(\phi) \begin{cases} \leq 0, \ 0 < \phi < \beta_n, \ n \equiv 0 \pmod 3 \\ < 0, \ 0 < \phi < \beta_n, \ n \equiv 1 \pmod 3 \\ < 0, \ 0 < \phi < \beta_n, \ n \equiv 2 \pmod 3 \end{cases}$$

Received January 28, 1953.

Pacific J. Math. 4 (1954), 109-121

where

$$\beta_n = \frac{2\pi}{3} + \frac{\beta}{n} - O(n^{-3/2}),$$

and $\beta = 2\pi,\ 4\pi/3,\ 2\pi/3$ *for* $n \equiv 0,\ 1,\ 2 \pmod 3$, *respectively.*

THEOREM 3. *For n sufficiently large, the mapping of* $|z| = 1$ *by*

$$w = S_n^{(2)}(e^{i\phi})$$

is convex for $0 < \phi < \gamma_n$, *where* γ_n *is the maximum angle for which convexity holds, and* $\gamma_n = \gamma/n + O(n^{-2})$ *where* $2\pi < \gamma < 3\pi$.

2. Proof of Theorem 1.

2.1. A closed expression for $y_n'(\phi)$ has been presented by Szegö [**10**]:

(2.1) $y_n'(\phi) =$

$$\frac{1}{8\sin^2\phi/2}\left\{-(n^2+3n+3) - n\cdot\frac{\sin(n+3/2)\phi}{\sin\phi/2} + 3\,\frac{\sin^2(n+1)\phi/2}{\sin^2\phi/2}\right\}.$$

The inequality $y_n'(\phi) < 0$ is satisfied if

$$n^2+3n+3 > -n\cdot\frac{\sin(n+3/2)\phi}{\sin\phi/2} + 3\,\frac{\sin^2(n+1)\phi/2}{\sin^2\phi/2}, \tag{2.2}$$

or

$$n^2+3n+3 > n\cdot\csc\phi/2 + 3\cdot\csc^2\phi/2. \tag{2.3}$$

Let δ be fixed, $\delta > 0$, and consider the restriction $\phi > \delta/n$. For n sufficiently large, $\sin(\delta/2n) > \delta/\pi n$, and the previous inequality is maintained if δ is chosen so that

$$n^2+3n+3 > n^2\pi/\delta + 3\pi^2 n^2/\delta^2, \quad\text{or}\quad \pi/\delta + 3\pi^2/\delta^2 < 1.$$

It is sufficient for the present problem to define $\delta = 3\pi$. Hence, if $\phi \geq 3\pi/n$, then $y_n'(\phi) < 0$ and $3\pi/n > \alpha_n$. Since

$$y_n'(\phi) = \sum_{m=1}^{n} a_m \cos m\phi,$$

where

$$a_m = m\binom{n+2-m}{2},$$

it at once follows that $y_n'(\phi) > 0$ for $0 \leq \phi \leq \pi/2n$.

2.2. In the next section it is shown that in the interval $\pi/2n < \phi < 3\pi/n$ there is exactly one $\phi = \alpha_n$ such that $y_n'(\phi) = 0$ if n is sufficiently large. More precisely, for the $\phi = \alpha_n$ the second derivative does not vanish and $\alpha_n \sim \alpha/n$, $\alpha > 0$, where $\pi < \alpha < 3\pi/2$. The magnitude of α is defined as the root of a transcedental equation.

It is possible to express (2.1) in the following form:

$$(2.4) \qquad y_n'(\phi) = n^2/8 \cdot g_n(\phi) \cdot \csc^2 \phi/2,$$

where the function $g_n(\phi)$ is defined as

$$g_n(\phi) = -1 - \frac{1}{n} \cdot \frac{\sin(n+3/2)\phi}{\sin \phi/2} + \frac{3}{n^2} \cdot \frac{\sin^2(n+1)\phi/2}{\sin^2 \phi/2} - \left(\frac{3}{n} + \frac{3}{n^2}\right).$$

Let $\phi = c/n$, $\pi/2 < c < 3\pi$, and then $g_n(\phi)$ becomes a function of c, denoted by $G_n(c)$. In addition,

$$\lim_{n\to\infty} G_n(c) = -1/c^2 \cdot f(c),$$

where

$$f(c) = 2c \cdot \sin c + 6 \cdot \cos c + (c^2 - 6).$$

Furthermore, $G_n(c)$ converges uniformly to this limit for arbitrary values of c in the interval. It is sufficient to show that the function $f(c)$ has a unique simple zero in the interval $\pi/2 < c < 3\pi/2$ to assure that $g_n(c)$ has a simple zero in the same interval if n is sufficiently large.

An easy calculation yields

$$f'(c) = 8 \cdot \cos^2(c/2) \cdot (c/2 - \tan c/2).$$

Thus it is seen that $f(0)=0$, $f'(c)<0$ for $0<c<\pi$; $f(\pi)=-(12-\pi^2)<0$, $f'(c)>0$ for $\pi<c<2\pi$; $f(2\pi)>0$ and if $c>2\pi$ then $f(c)>(c-1)^2-13>0$. Since $f(3\pi/2)>0$, there is a simple positive zero, $c=\alpha$, of the function $f(c)$, $\pi<\alpha<3\pi/2$. In conclusion, $\alpha_n\sim\alpha/n$, $\pi<\alpha<3\pi/2$, for n sufficiently large.

2.3. It is not difficult to find a more precise asymptotic expression for α_n. For this purpose let $\alpha_n=c/n$, where $c=\alpha+a/n$ and a is a bounded, real constant. Let $h_n(a)$ denote $g_n(c)$ when the latter is regarded as a function of a. Let $\phi=c/n$; a simplification yields

$$-n^2h_n(a)=n^2(2c\cdot\sin c+6\cos c+c^2-6)/c^2+n(3+3\cos c-6/c\cdot\sin c)$$
$$+(5/2-13c/6\cdot\sin c-5/2\cdot\cos c)+O(1/n).$$

If

$$h(c)=3c^2[1+\cos c-2/c\cdot\sin c]$$

and

$$k(c)=c^2[5/2-13c/6\cdot\sin c-5/2\cdot\cos c],$$

then it is possible to rewrite the previous expression in the form

$$-n^2c^2h_n(a)=n^2\cdot f(c)+n\cdot h(c)+k(c)+O(1/n).$$

Let the functions $f(c)$, $h(c)$, $k(c)$ be expanded by Taylor's formula for values of c near α. Then the previous equality becomes

$$-n^2c^2h_n(a)=n[a\cdot f'(\alpha)+h(\alpha)]+a^2/2\cdot f''(\alpha)+a\cdot h'(\alpha)+k(\alpha)+O(1/n).$$

Thus one obtains

$$\lim_{n\to\infty}[-n\cdot c^2h_n(a)]=a\cdot f'(\alpha)+h(\alpha)\ \text{ and }\ f'(\alpha)\neq 0.$$

Obviously the limit has a zero for the value $a=-h(\alpha)/f'(\alpha)$, or

$$a=-3\alpha^2/8\cdot(1+\cos\alpha-2/\alpha\cdot\sin\alpha)\cdot\sec^2\alpha/2\cdot(\alpha/2-\tan\alpha/2)^{-1},$$

and α is the simple zero of the function

$$f(c)=2c\cdot\sin c+6\cdot\cos c+c^2-6$$

in the interval $\pi < \alpha < 3\pi/2$.

This shows that for n sufficiently large, $y_n'(\phi) = 0$ for

$$\phi = \alpha_n = \alpha/n + (a + \epsilon_n)/n^2,$$

where $\epsilon_n \longrightarrow 0$. Thus the assertion of Theorem 1 has been verified.

3. Proof of Theorem 2.

3.1. In the article by Szegö [10], a closed expression for $x_n'(\phi)$ is presented:

(3.1) $x_n'(\phi)$

$$= \frac{\cos \phi/2}{8 \sin^3 \phi/2}\left[-(2n+3) - (n+3/2)\frac{\cos(n+3/2)\phi}{\cos \phi/2} + \frac{3}{2}\cdot\frac{\sin(n+3/2)\phi}{\sin \phi/2}\right].$$

It immediately follows that $x_n'(\phi)$ is negative if

$$(3.2) \qquad [3/(2n+3)]^2 \cdot \csc^2 \phi/2 + \sec^2 \phi/2 < 4; \quad \cot \phi/2 > 0.$$

Let $0 < \phi \leq \pi/n$. As

$$x_n(\phi) = \sum_{m=1}^{n} b_m \cos m\phi,$$

where

$$b_m = \binom{n+2-m}{2},$$

then $x_n'(\phi) < 0$. Next consider the interval $\pi/n \leq \phi \leq 2\pi/3 - c/n$, where c is fixed, $c > 0$. Since

$$[3/(2n+3)]^2 \cdot \csc^2 \phi/2 + [1 - \sin^2 \phi/2]^{-1},$$

as a function of $\sin^2 \phi/2$, is convex from below, it obtains its maximum at one or both end-points of the interval. Thus in order to prove the inequality (3.2) it is sufficient to consider only the end-point values of $\pi/n \leq \phi \leq 2\pi/3 - c/n$. It easily follows that (3.2) is satisfied by $\phi = \pi/n$. Now study $\phi = 2\pi/3 - c/n$. Since

$$\sin^{-2}\phi/2 = O(1), \quad \cos^2\phi/2 = 1/4 \cdot (1+\sqrt{3}\cdot c/n) + O(1/n^2),$$

the left side of (3.2) then can be written as

$$[3/(2n+3)]^2 \cdot O(1) + 4[1+\sqrt{3}\cdot c/n + O(1/n^2)]^{-1}$$

$$= 4(1-\sqrt{3}\cdot c/n) + O(1/n^2),$$

which indeed is less than 4 provided n is sufficiently large. The minimum value of n is a function of c. Thus it now is established that $x_n'(\phi) < 0$ for $0 < \phi \le 2\pi/3 - c/n$, if n is sufficiently large, $n \ge n_1(c)$, where c is an arbitrary positive fixed magnitude.

3.2. Next let $\phi = 2\pi/3$. By (3.1) it follows that

$$x_n'(2\pi/3) = -(2n+3)/6\sqrt{3}\cdot(1-\cos 2\pi n/3) - 1/6\cdot\sin 2\pi n/3.$$

Three possible cases for the n arise. For $n \equiv 0 \pmod 3$, $x_n'(2\pi/3) = 0$; whereas for $n \equiv 1, 2 \pmod 3$, $x_n'(2\pi/3) < 0$. Thus the behavior of $x_n'(\phi)$ in the neighborhood of $\phi = 2\pi/3$ must be examined more fully, $n \equiv 0 \pmod 3$. Let

$$x_n'(\phi) = r(\phi)\cdot s(\phi),$$

where

$$r(\phi) = 1/8\cdot\cos\phi/2\cdot\csc^3\phi/2$$

and

$$s(\phi) = -(2n+3) - (n+3/2)\cdot\cos(n+3/2)\phi\cdot\sec\phi/2$$

$$+ 3/2\cdot\sin(n+3/2)\phi\cdot\csc\phi/2.$$

As $s = 0$ for $\phi = 2\pi/3$, then

$$x_n''(2\pi/3) = r(2\pi/3)\cdot s'(2\pi/3).$$

Upon letting $N = n + 3/2$, we see that

$$s'(2\pi/3) = 0, \quad x_n''(2\pi/3) = 0.$$

An examination of the third derivative shows that

$$x_n'''(2\pi/3) = r(2\pi/3)\cdot s''(2\pi/3).$$

As $r(2\pi/3) > 0$, sgn $x_n''(\phi) =$ sgn $s''(\phi)$. Since

$$s''(\phi) = N^3 \sec \phi/2 \cdot \cos N\phi + O(N^2),$$

then $s''(2\pi/3) = -2N^3 + O(N^2)$, and for n sufficiently large $s''(2\pi/3) < 0$. It is now known than $x_n'(\phi) < 0$ for $0 < \phi < 2\pi/3$ if n is sufficiently large.

3.3. This section extends the investigation beyond $\phi = 2\pi/3$. For this purpose let $\phi = 2\pi/3 + c/N$, where again $N = n + 3/2$. The substitution of this value of ϕ into (3.1) yields

$$(3.3)\quad \frac{8 \sin^3 \phi/2}{\cos \phi/2} \cdot x_n'(\phi) = -2N\left[1 - \frac{1}{2}\,\frac{\cos(2\pi n/3 + c)}{\cos \phi/2}\right] - \frac{3}{2} \cdot \frac{(2\pi n/3 + c)}{\sin \phi/2}.$$

Any easy calculation shows that

$$\sin \phi/2 = \sqrt{3}/2 + c/4N + c^2 \cdot O(1/N^2),$$

$$\cos \phi/2 = 1/2 - \sqrt{3} \cdot c/4N + c^2 \cdot O(1/N^2).$$

The remainder of the section will study the separate cases of $n \pmod 3$.

$n \equiv 0 \pmod 3$. Let us rewrite (3.3) as follows:

$$(3.4)\quad \frac{8 \sin^3 \phi/2 \cdot x_n'(\phi)}{\cos \phi/2 \cdot 2(1 - \cos c)}$$

$$= -N + \frac{\sqrt{3}}{2}\left[\frac{c \cdot \cos c - \sin c}{1 - \cos c}\right] + \frac{c^2}{1 - \cos c} \cdot O\left(\frac{1}{N}\right).$$

Let

$$F(c) = [c \cdot \cos c - \sin c] \cdot [1 - \cos c]^{-1}.$$

Since

$$F'(c) = \sin c \cdot [\sin c - c]\,[1 - \cos c]^{-2},$$

it is easily seen that $F(c)$ is decreasing for $0 < c < \pi$ and increasing for $\pi < c < 2\pi$. It follows that $x_n'(\phi) \leq 0$ for $0 < \phi \leq 2\pi/3 + c/N$, where $\pi < c < 2\pi - \epsilon$, ϵ a fixed positive number. Now

$$c = 2\pi - \delta/\sqrt{N},$$

δ a fixed positive number for n sufficiently large. Then

$$F(c) = 4\pi/\delta^2 \cdot N + O(1/N),$$

so that, for the above value of ϕ, (3.4) becomes

$$\frac{4 \sin^3 \phi/2 \cdot x_n'(\phi)}{\cos \phi/2 \cdot (1 - \cos c)} = -N + 2\pi\sqrt{3} \cdot N/\delta^2 + O(1).$$

In addition,

$$(1 - \cos c)^{-1} = (1 - \cos \delta/\sqrt{N})^{-1} = O(N).$$

Thus

$$x_n'(\phi) < 0 \text{ if } 2\pi\sqrt{3}/\delta^2 < 1,$$

and

$$x_n'(\phi) > 0 \text{ if } 2\pi\sqrt{3}/\delta^2 > 1.$$

Thus

$$\delta = (2\pi)^{1/2} \cdot (3)^{1/4}$$

furnishes the critical value of ϕ. It has been shown that, for $n \equiv 0 \pmod 3$,

$$x_n'(\phi) < 0 \text{ for } 0 < \phi < 2\pi/3$$

and

$$x_n'(\phi) \leq 0 \text{ for } 0 < \phi < 2\pi/3 + 2\pi/N - O(N^{-3/2}),$$

for n sufficiently large.

$n \equiv 1 \pmod 3$. It is possible to rewrite (3.3) so that the right side becomes

$$-2N[1 - \cos(c + 2\pi/3)]$$
$$+ \sqrt{3}\,[c \cdot \cos(c + 2\pi/3) - \sin(c + 2\pi/3)] + c^2 \cdot O(1/N).$$

By reasoning as in the previous case, one finds $x_n'(\phi) < 0$ for $0 \leq c \leq 4\pi/3 - \epsilon$, $\epsilon > 0$, for n sufficiently large. Let

$$c = 4\pi/3 - \delta/\sqrt{N}.$$

Then the right side of (3.3) reduces to

$$-\delta^2 + 4\pi/\sqrt{3} + O(1/\sqrt{N}).$$

Therefore

$$x_n'(\phi) < 0 \text{ if } \delta > 2 \cdot \pi^{1/2} \cdot 3^{-1/4},$$

and

$$x_n'(\phi) > 0 \text{ if } \delta < 2 \cdot \pi^{1/2} \cdot 3^{-1/4},$$

for n sufficiently large. It follows that $x_n'(\phi) < 0$ for $0 < \phi < \beta_n$, where

$$\beta_n = 2\pi/3 + 4\pi/3N - O(N^{-3/2}),$$

for n sufficiently large.

$n \equiv 2 \pmod 3$. In this case the right side of (3.3) becomes

$$-2N[1 - \cos(c + 4\pi/3)]$$
$$+ \sqrt{3}[c \cdot \cos(c + 4\pi/3) - \sin(c + 4\pi/3)] + O(1/N).$$

It follows that $x_n'(\phi) < 0$ for $0 \leq c \leq 2\pi/3 - \epsilon$, $\epsilon > 0$, for n sufficiently large. Let

$$c = 2\pi/3 - \delta/\sqrt{N}.$$

Then the right side of (3.3) is equivalent to

$$-\delta^2 + 2\pi/\sqrt{3} + O(N^{-1/2}).$$

Thus

$$x_n'(\phi) < 0 \text{ if } \delta > (2\pi)^{1/2} \cdot 3^{1/4}$$

and

$$x_n'(\phi) > 0 \text{ if } \delta < (2\pi)^{1/2} \cdot 3^{1/4},$$

for n sufficiently large. It has been shown that $x_n'(\phi) < 0$ for $0 < \phi < \beta_n$, where

$$\beta_n = 2\pi/3 + 2\pi/3N - O(N^{-3/2}),$$

for n sufficiently large.

If $n + 3/2$ is substituted for N, then the results expressed in Theorem 2 are proved.

4. Proof of Theorem 3.

4.1 The Curvature of an image is defined to be

$$1/\rho = [1 + \Re\, z \cdot f''(z)/f'(z)] \cdot [|z \cdot f'(z)|]^{-1}.$$

If the point $w = f(z)$ traverses a closed, single-valued curve in a preassigned positive direction, then the curve is called convex if

$$1 + \Re[z \cdot f''(z)/f'(z)] > 0. \tag{4.1}$$

Let us examine the inequality (4.1) for the function

$$f(e^{i\phi}) = s_n^2(e^{i\phi}) = x_n(\phi) + i \cdot y_n(\phi)$$

if $z = e^{i\phi}$. By the employment of differentiation and elementary algebraic steps after substituting the derivatives in the left side of (4.1), one obtains

$$1 + \Re[z \cdot f''(z)/f'(z)] = [x_n' \cdot y_n'' - x_n'' \cdot y_n'] \cdot [x_n'^2 + y_n'^2]^{-1}.$$

Thus the condition for the mapping to be convex is satisfied if

$$x_n' \cdot y_n'' - x_n'' \cdot y_n' > 0. \tag{4.2}$$

4.2. The next section studies the previous condition of convexity for the function $w = s_n^2(z)$, $z = e^{i\phi}$, where $\phi = \gamma/n$, $\gamma > 0$, for n sufficiently large.

In the present case the expressions for $y_n'(\gamma/n)$ and $x_n'(\gamma/n)$, for which see (2.1) and (3.1), become

$$y_n'(\gamma/n) = n^4/\gamma^4\, [-\gamma \sin\gamma - 3\cos\gamma + 3 - \gamma^2/2 + O(1/n)],$$

$$x_n'(\gamma/n) = n^4/\gamma^4\, [-2\gamma - \gamma\cos\gamma + 3\sin\gamma + O(1/n)].$$

Substitution of the latter expressions into (4.2) yields directly

$$(2\sin\gamma-\gamma\cos\gamma-\gamma)(-2\gamma-\gamma\cos\gamma+3\sin\gamma)$$
$$-(-2+\gamma\sin\gamma+2\cos\gamma)(-\gamma\sin\gamma-3\cos\gamma+3-\gamma^2/2)+O(1/n)>0.$$

Further simplification of the previous inequality, which establishes the requirement for convexity of the image of $|z|=1$, leads to the convenient form

$$\sin\gamma\,(\tan\gamma/2-\gamma/2)(6-\gamma^2/2-3\gamma\cot\gamma/2)+O(1/n)>0. \tag{4.3}$$

The remainder of the section is devoted to determining the maximum value of $\phi=\gamma/n$ which satisfies (4.3). In particular it is shown that the maximum angle $\gamma_n=\gamma/n$ for which the mapping of $|z|=1$ by $w=s_n^2(z)$ is convex, where $z=e^{i\phi}$, is determined by $2\pi<\gamma<3\pi$, for n sufficiently large.

4.3. Consider the elementary function

$$v(\gamma)=\sin\gamma\,[\tan(\gamma/2)-\gamma/2].$$

Define γ_0 by the equality $\tan(\gamma_0/2)=\gamma_0/2$. Then it is easily shown that

$$v(\gamma)\begin{cases}>0, & 0<\gamma<2\pi,\\ <0, & 2\pi<\gamma<\gamma_0,\\ >0, & \gamma_0<\gamma<3\pi,\end{cases} \tag{4.4}$$

Let us define

$$f(\gamma)=6-\gamma^2/2-3\gamma\cot(\gamma/2).$$

Then the image of $|z|=1$ is convex if

$$f(\gamma)\begin{cases}>0, & 0<\gamma<2\pi,\\ <0, & 2\pi<\gamma<\gamma_0,\\ >0, & \gamma_0<\gamma<3\pi,\end{cases} \tag{4.5}$$

for n sufficiently large.

Next it is shown that the first two inequalities for $f(\gamma)$ in (4.5) are satisfied, however, for $\gamma_0<\gamma<3\rho$, one finds that $f(\gamma)<0$. Since

$$f'(\gamma)\cdot\sin^2(\gamma/2)=-\gamma\sin^2(\gamma/2)-3/2\cdot\sin\gamma+3\gamma/2,$$

by a further differentiation with respect to γ one can obtain

$$d/d\gamma\{f'(\gamma)\cdot\sin^2(\gamma/2)\} = 1-\cos\gamma-\gamma/2\sin\gamma,$$

$$= \sin\gamma\{\tan(\gamma/2)-\gamma/2\} = v(\gamma).$$

Consider the interval $0<\gamma<2\pi$. By (4.4), $v(\gamma)>0$. Also $f'(\gamma)\cdot\sin^2(\gamma/2)=$ 0 if $\gamma=0$. Thus $f'(\gamma)\sin^2(\gamma/2)>0$ for $0<\gamma<2\pi$, and consequently $f'(\gamma)>0$ for the same interval. Finally,

$$f(0) = \lim_{\gamma\to 0^+} f(\gamma) = 0,$$

which establishes the fact that $f(\gamma)>0$ in the interval $0<\gamma<2\pi$.

In the interval $2\pi<\gamma<\gamma_0$, $v(\gamma)<0$; therefore the function $f'(\gamma)\sin^2(\gamma/2)$ is decreasing. It follows that

$$f'(\gamma_0)\cdot\sin^2(\gamma_0/2) = \gamma_0[3/2-\sin^2(\gamma_0/2)] - 3/2\cdot\sin\gamma_0,$$

and thus $f'(\gamma_0)\cdot\sin^2(\gamma_0/2) > 1/2(\gamma_0-3)>0$. Consequently $f'(\gamma)\sin^2(\gamma/2)$ and also $f'(\gamma)$ are positive in the interval $2\pi<\gamma<\gamma_0$. Hence $f(\gamma)$ is increasing. As $f(\gamma)$ has no lower bound as γ approaches 2π from above, and

$$f(\gamma_0) = 6-\gamma_0^2/2-3\gamma_0\cot(\gamma_0/2) < -\gamma_0^2/2 < 0,$$

then it can be concluded that $f(\gamma)<0$ for $2\pi<\gamma<\gamma_0$.

Finally consdier the interval $\gamma_0<\gamma<3\pi$. Since $v(\gamma)>0$, $f'(\gamma_0)\sin^2(\gamma_0/2)>$ 0, and thus $f'(\gamma_0)>0$, then $f'(\gamma)>0$ holds for $\gamma_0<\gamma\,3\pi$. Hence $f(\gamma)$ is increasing. But

$$f(3\pi) = 6-9\pi^2/2 < 0,$$

so that a γ exists such that $f(\gamma)<0$ occurs in the interval.

It was shown in (4.5) that if the image of $|z|=1$ was to be convex for $\gamma_0<\gamma<3\pi$, then $f(\gamma)>0$. Thus the image is not convex for the complete interval, which completes the proof of Theorem 3.

REFERENCES

1. E. Egerváry, *Abbildungseigenschaften der arithmetischen Mittel der geometrischen Reihe*, Math. Z. 42 (1936-1937), 221-230.

2. L. Fejér, *Potenzreihen mit mehrfach monotoner Koeffizientenfolge und ihre Legendrepolynome*, Proc. Cambridge Philos. Soc. **31** (1935), 307-316.

3. ———, *Trigonometrische Reihen und Potenzreihen mit mehrfach monotoner Koeffizientenfolge*, Trans. Amer. Math. Soc. **39** (1936), 18-59.

4. ———, *Untersuchungen über Potenzreihen mit mehrfach monotoner Koeffizientenfolge*, Acta Sci. Math. Szeged **8** (1936), 90-116.

5. L. Fejér and G. Szegö, *Über die monotone Konvergenz von Potenzreihen mit mehrfach monotoner Koeffizientenfolge*, Prace Mat. Fiz. **44** (1935), 15-25.

6. ———, *Special conformal mappings*, Duke Math. J. **18** (1951), 535-547.

7. K. Knopp, *Mehrfach monotone Zahlenfolgen*, Math. Z. **22** (1925), 75-85.

8. M. Schweitzer, *The partial sums of second order of the geometric series*, Duke Math. J. **18** (1951), 527-533.

9. S. Sidon, *Über Potenzreihen mit monotoner Koeffizientenfolge*, Acta Sci. Math. Szeged **9** (1938-1940), 244-246.

10. G. Szegö, *Power series with multiply monotonic sequences of coefficients*, Duke Math. J. **8** (1941), 559-564.

UNIVERSITY OF CALIFORNIA
SANTA BARBARA COLLEGE