THE RELATIONS BETWEEN A SPECTRAL OPERATOR AND ITS SCALAR PART

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1. Introduction. It is shown in Dunford's theory of spectral operators, that every spectral operator T can be decomposed into the sum of a scalar operator S, and a generalized nilpotent N [1]. We study here properties which are inherited by S from T. The main results are:

1. If the spectral operator T is compact, weakly compact, or has a closed range, then respectively S is compact, weakly compact, or has a closed range.

2. The relations between the point spectra, continuous spectra, and residual spectra of S and T are investigated.

3. If the sum of two commuting spectral operators is spectral, then the sum of their scalar parts is scalar.

2. Notation. Most of the notation is taken from [1]. Let X be a complex Banach space. A spectral measure is a set function $E(\cdot)$, defined on Borel sets in the complex plane, whose values are projections on X, which satisfy:

(α) For any two Borel sets σ and δ $E(\sigma)E(\delta)=E(\sigma \cap \delta)$.

(β) Let Φ be the void set and p the complex plane.

Then

$$E(\Phi)=0 \text{ and } E(p)=I .$$

(γ) There exists a constant M such that $|E(\sigma)|\leq M$, for every Borel set σ.

(δ) The vector valued set function $E(\cdot)x$ is countable additive for each $x \in X$.

The operator T is a spectral operator, whose resolution of the identity is the spectral measure $E(\cdot)$ if

(a) for every Borel set σ $E(\sigma)T=TE(\sigma)$.

(b) Let T_α denote the restriction of T to the subspace $E(\alpha)X$, ($T_\alpha = T|E(\alpha)X$) then

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$$\sigma(T_\alpha)\subset\bar{\alpha}$$

where $\sigma(A)$ is the spectrum of A.

Throughout the paper T denotes a spectral operator, $E(\cdot)$ its resolution of the identity, S its scalar part given by $S=\int_p \lambda E(d\lambda)$, N its radical given by $N=T-S$. The operator N is a generalized nilpotent, and the operators N, S, T, $E(\alpha)$ commute [1]. A spectral operator is of finite type, if for some integer n, $N^{n+1}=0$. We shall denote $N\cdot E(\langle 0\rangle)$ by N_0, hence $N_0=TE(\langle 0\rangle)=E(\langle 0\rangle)T$.

3. **Topological properties.** In this section, several topological properties will be shown to be valid for S whenever they are valid for T. The following lemma will be used.

LEMMA 1. *S is in the uniformly closed operator algebra generated by the projections $E(\alpha)$ with $0\notin\bar{\alpha}$.*

Proof. $S=\int_{\sigma(T)}\lambda E(d\lambda)$ and $\sigma(T)$ is bounded, see [1] Theorem 1. Given $\varepsilon>0$ let $\sigma(T)$ be divided into the disjoint sets $\alpha_0, \alpha_1, \cdots, \alpha_n$ with

$$\begin{aligned} &0\in\alpha_0\,, \qquad 0\notin\bar{\alpha}_i\,, \qquad i=1,2,\cdots,n \text{ and}\\ &\operatorname{diam}(\alpha_i)<\varepsilon \qquad\qquad i=0,1,2,\cdots,n\,.\end{aligned}$$

Let $\lambda_0=0$ and $\lambda_i\in\alpha_i$. Then

$$\left|S-\sum_{i=1}^n\lambda_iE(\alpha_i)\right|=\left|\int_{\sigma(T)}\left(\lambda-\sum_{i=0}^n\lambda_i\chi_{\alpha_i}(\lambda)\right)E(d\lambda)\right|\,.$$

If $\lambda\in\sigma(T)$ then

$$\left|\lambda-\sum_{i=0}^n\lambda_i\chi_{\alpha_i}(\lambda)\right|\leqq\varepsilon\,.$$

Now by [1], p. 330, for every bounded measurable function defined on $\sigma(T)$

$$\left|\int_{\sigma(T)}f(\lambda)E(d\lambda)\right|\leqq\sup\{|f(\lambda)|\,, \qquad \lambda\in\sigma(T)\}\cdot 4M\,.$$

Hence

$$\left|S-\sum_{i=1}^n\lambda_iE(\alpha_i)\right|\leqq 4M\varepsilon\,.$$

THEOREM 1. *Let $\mathfrak{A}$ be a uniformly closed right (left) ideal in the algebra of operators on X. If T belongs to $\mathfrak{A}$ so do S, N, and $E(\alpha)$ with $0\notin\bar{\alpha}$.*

Proof. By condition b of §2 T_α with $0\notin\bar{\alpha}$ possesses a bounded everywhere defined inverse T_α^{-1}. Let us define P_α by $P_\alpha x = T_\alpha^{-1}E(\alpha)x$, $x \in X$, $0 \notin \overline{\alpha}$. P_α is a bounded everywhere defined operator. Now

$$TP_\alpha x = T(T_\alpha^{-1}E(\alpha)x) = (TT_\alpha^{-1})(E(\alpha)x) = E(\alpha)x \ .$$

Also

$$P_\alpha Tx = T_\alpha^{-1}E(\alpha)Tx = T_\alpha^{-1}TE(\alpha)x = (T_\alpha^{-1}T)E(\alpha)x = E(\alpha)x \ .$$

Hence if $0 \notin \overline{\alpha}$ then $E(\alpha) \in \mathfrak{A}$. Note that this fact remains true even if $\mathfrak{A}$ is not uniformly closed. Now by Lemma 1 $S \in \mathfrak{A}$ and therefore $N \in \mathfrak{A}$ too.

COROLLARY 1. *If T is compact then so are S, N and $E(\alpha)$ $(0 \notin \overline{\alpha})$.*

COROLLARY 2. *If T is weakly compact then so are S, N and $E(\alpha)$ with $0 \notin \overline{\alpha}$.*

COROLLARY 3. *If $TX \subset Y$ where Y is a closed subspace of X, then $SX \subset Y$ and $NX \subset Y$ and $E(\alpha)X \subset Y$, $0 \notin \overline{\alpha}$. Hence*

$$SX \cup NX \cup \cup (E(\alpha)X \mid 0 \notin \overline{\alpha}) \subset \overline{TX}$$

and if the range of T is separable so are the ranges of S, N and $E(\alpha)$, $0 \notin \overline{\alpha}$.

COROLLARY 4. *If $A_0T = 0$ $(TA_0 = 0)$ then $A_0S = A_0N = 0$ and $A_0E(\alpha) = 0$, $0 \notin \overline{\alpha}$ $(SA_0 = NA_0 = E(\alpha)A_0 = 0$ if $0 \notin \overline{\alpha})$. In particular T is a spectral operator of finite type if and only if some power of N annihilates T.*

COROLLARY 5. *If $Tx = 0$ then $Nx = Sx = E(\alpha)x = 0$ where $\overline{\alpha}$ does not contain 0.*

COROLLARY 6. *If (x_n) is a bounded sequence of vectors, and the sequence (Tx_n) has a limit then the sequences (Sx_n), (Nx_n) and $(E(\alpha)x_n)$ with $0 \notin \overline{\alpha}$ have limits.*

To prove these corollaries one has to note that:
(a) The classes of compact and weakly compact operators are uniformly closed two-sided ideals. (See [3] Chapter 6).
(b) The classes of operators A satisfying $AX \subset Y$ or $A_0A = 0$ are uniformly closed right ideals.
(c) The classes of operators A satisfying $Ax = 0$ or $AA_0 = 0$ or the limit of Ax_n exists are uniformly closed left ideals.

REMARK TO COROLLARY 6. By the proof of Theorem 1 the sequence $(E(\alpha)x_n)$, $0 \notin \overline{\alpha}$, has a limit whenever the sequence (Tx_n) has, even if the sequence (x_n) is not bounded.

THEOREM 2. *$AT=0$ if and only if $AE(p-\langle 0\rangle)=0$ $(A=AE(\langle 0\rangle))$ and $AN_0=0$. Similarly $TA=0$ if and only if $E(p-\langle 0\rangle)A=N_0A=0$.*

Proof. If $AN_0=AE(p-\langle 0\rangle)=0$ then $AE(\alpha)=AE(p-\langle 0\rangle)E(\alpha)=0$ if $0\notin\bar{\alpha}$, thus by Lemma 1 $AS=0$. Now

$$AN=ANE(\langle 0\rangle)+ANE(p-\langle 0\rangle)=AN_0+(AE(p-\langle 0\rangle))N=0 .$$

Thus $AT=AS+AN=0$. Conversely if $AT=0$ then $AN_0=ATE(\langle 0\rangle)=0$, and $AE(\alpha)=0$ if $0\notin\bar{\alpha}$. Now for each $x\in X$

$$AE(p-\langle 0\rangle)x=\lim AE\left\{z\,\middle|\,\frac{1}{n}\leqq|z|\right\}x=0$$

by countable additivity.

The second half of the theorem is proved in the same way.

Using Corollary 5 one can prove in the same way that $Tx=0$ if and only if $N_0x=E(p-\langle 0\rangle)x=0$.

COROLLARY 1. *If $E(\langle 0\rangle)=0$, then $AT=0$ or $TA=0$ if and only if $A=0$.*

Proof. By Theorem 2 if $AT=0$ or $TA=0$ then $A=AE(\langle 0\rangle)$ or $A=E(\langle 0\rangle)A$.

COROLLARY 2. *If $E(\langle 0\rangle)=0$ then $\overline{TX}=X$.*

Proof. If $\overline{TX}\neq X$ then there exists a bounded functional $x^*\neq 0$ such that $x^*(TX)=0$. Let $Ax=x^*(x)x_1$ where x_1 is any vector different from 0. $AT=0$ and $A\neq 0$ which contradicts Corollary 1.

THEOREM 3. *If T has a closed range so does S.*

1. *Proof.* Let $E(\langle 0\rangle)=0$ then Corollary 2 of Theorem 2 shows that $\overline{TX}=X$. But by assumption $\overline{TX}=TX$, thus $TX=X$. Also, the operator T is one-to-one by [1] p. 327 and thus T possesses a bounded everywhere defined inverse. Thus $0\notin\sigma(S)=\sigma(T)$ and $SX=X$.

2. Let $E(\langle 0\rangle)\neq 0$. The operator $T_{p-\langle 0\rangle}$ is a spectral operator whose resolution of the identity $F(\cdot)$ is given by $F(\alpha)=E(\alpha)E(p-\langle 0\rangle)=E(\alpha-\langle 0\rangle)$, hence $F(\langle 0\rangle)=0$. Now if $T_{p-\langle 0\rangle}x_n\to y(y\in E(p-\langle 0\rangle)X)$, then, there exists a vector x in X such that $Tx=y$, because T has a closed range. Therefore

$$T_{p-\langle 0\rangle}(E(p-\langle 0\rangle)x)=TE(p-\langle 0\rangle)x=E(p-\langle 0\rangle)Tx=E(p-\langle 0\rangle)y=y .$$

Hence $T_{p-\langle 0\rangle}$ satisfies the same conditions assumed for T in the first part and therefore $0 \notin \sigma(T_{p-\langle 0\rangle})$ and

$$S_{p-\langle 0\rangle}X = E(p-\langle 0\rangle)X\ , \quad \text{but} \quad S_{p-\langle 0\rangle}X = SX\ ,$$

so S has a closed range.

By the proof of the last theorem it follows that if T has a closed range then $0 \notin \sigma(T_{p-\langle 0\rangle})$, hence 0 is an isolated point of the spectrum of T.

THEOREM 4. *The operator T has a closed range if and only if*

1. *0 is an isolated point of $\sigma(T)$.*
2. *The operator N_0 has a closed range.*

Proof. We proved that Condition 1 is necessary. Now if $N_0 x_n \to y$ then $E(\langle 0\rangle)N_0 x_n \to E(\langle 0\rangle)y$ but $E(\langle 0\rangle)N_0 = N_0$ thus $E(\langle 0\rangle)y = y$. Also $N_0 = TE(\langle 0\rangle)$ and T has a closed range, thus if $T(E(\langle 0\rangle)x_n) \to y$ then for some x, $Tx = y$. Hence $TE(\langle 0\rangle)x = N_0 x = E(\langle 0\rangle)y = y$. Conversely if 1. and 2. are satisfied let $Tx_n \to y$. Then

$$\begin{aligned} TE(p-\langle 0\rangle)x_n + TE(\langle 0\rangle)x_n &= TE(p-\langle 0\rangle)x_n + N_0 x_n \\ &\to y = E(p-\langle 0\rangle)y + E(\langle 0\rangle)y\ . \end{aligned}$$

Multiplying this equation by $E(p-\langle 0\rangle)$ and $E(\langle 0\rangle)$ one gets the following two equations

$$\begin{aligned} TE(p-\langle 0\rangle)x_n &\to E(p-\langle 0\rangle)y \\ N_0 x_n &\to E(\langle 0\rangle)y \end{aligned}$$

By 1. $T_{p-\langle 0\rangle}$ possesses a bounded everywhere defined inverse. Hence, for some x_1 in $E(p-\langle 0\rangle)X$, $Tx_1 = E(p-\langle 0\rangle)y$.
By 2. for some vector x_2, $N_0 x_2 = E(\langle 0\rangle)y$. Thus

$$T(x_1 + E(\langle 0\rangle)x_2) = Tx_1 + N_0 x_2 = y.$$

4. Properties of spectral points. Let A be a bounded linear operator on X, define

$\sigma_p(A) = \{\lambda \mid \lambda I - A$ is not one-to-one$\}$

$\sigma_c(A) = \{\lambda \mid \lambda I - A$ is one-to-one and $(\lambda I - A)X$ is dense in X, but not equal to $X\}$.

$\sigma_r(A) = \{\lambda \mid \lambda I - A$ is one-to-one and $\overline{(\lambda I - A)X} \neq X\}$.

(See [6] p. 292.)

The sets $\sigma_p(A)$, $\sigma_c(A)$ and $\sigma_r(A)$ are disjoint and

$$\sigma(A) = \sigma_p(A) \cup \sigma_c(A) \cup \sigma_r(A)\ .$$

THEOREM 1. *If T is a spectral operator of finite type, then $\lambda \in \sigma_p(T)$ if and only if $E(\langle\lambda\rangle) \neq 0$, and $\lambda \in \sigma_c(T)$ if and only if $E(\langle\lambda\rangle)=0$, and $\lambda \in \sigma(T)$. Thus $\sigma(T)=\sigma_p(T) \cup \sigma_c(T)$.*

Proof. If $E(\langle\lambda\rangle)\neq 0$ let $x \in E(\langle\lambda\rangle)X$, $x \notin 0$, then

$$Sx=\int_{\sigma(T)} \mu E(d\mu)x=\int_{\sigma(T)} \mu E(d\mu)E(\langle\lambda\rangle)x=\lambda x .$$

Let ν be the first integer such that $N^\nu x=0$, then

$$TN^{\nu-1}x=SN^{\nu-1}x+N^\nu x=N^{\nu-1}Sx=\lambda N^{\nu-1}x ,$$

therefore $\lambda \in \sigma_p(T)$. If $E(\langle\lambda\rangle)=0$ then Corollary 2 of Theorem 2, §3, applied to $\lambda I-T$, shows that $\overline{(\lambda I-T)X}=X$. Also, by [1] Lemma 1, $\lambda I-T$ is one-to-one and thus $\lambda \in \sigma_c(T)$.

THEOREM 2. *$\sigma_c(S)\subset\sigma_c(T)$ and $\sigma_p(T) \cup \sigma_r(T)\subset\sigma_p(S)$.*

Proof. If $\lambda \in \sigma_c(S)$ then $E(\langle\lambda\rangle)=0$, and by the last part of the proof of Theorem 1, $\lambda \in \sigma_c(T)$. Thus $\sigma_c(S)\subset\sigma_c(T)$ and

$$\sigma_p(T) \cup \sigma_r(T)=\sigma(T)-\sigma_c(T)\subset\sigma(T)-\sigma_c(S)=\sigma(S)-\sigma_c(S)=\sigma_p(S) .$$

If $E(\langle\lambda\rangle)=0$ then $\lambda \in \sigma_c(T)$. Let us examine therefore the case where $E(\langle\lambda\rangle)\neq 0$. To simplify notation assume that $\lambda=0$.

THEOREM 3. *Let $E(\langle 0\rangle)\neq 0$ then*
1. *$0 \in \sigma_p(T)$ if N_0 is not one-to-one on $E(\langle 0\rangle)X$.*
2. *$0 \in \sigma_c(T)$ if N_0 is one-to-one on $E(\langle 0\rangle)X$ and $\overline{N_0(E(\langle 0\rangle)X)}=E(\langle 0\rangle)X$.*
3. *$0 \in \sigma_r(T)$ if N_0 is one-to-one on $E(\langle 0\rangle)X$ and $\overline{N_0(E(\langle 0\rangle)X}\neq E(\langle 0\rangle)X$.*

Proof.

1. If there exists a vector x such that $x\neq 0$, $x=E(\langle 0\rangle)x$ and $N_0x=0$ then

$$Tx=TE(\langle 0\rangle)x=N_0x=0$$

2. The operator $T_{p-\langle 0\rangle}$ is one-to-one on $E(p-\langle 0\rangle)X$ by [1] Lemma 1. Now if N_0 is one-to-one on $E(\langle 0\rangle)X$ then T is one-to-one on X: If $Tx=0$ then $E(\langle 0\rangle)Tx=N_0x=N_0E(\langle 0\rangle)x=0$ and $TE(p-\langle 0\rangle)x=T_{p-\langle 0\rangle}E(p-\langle 0\rangle)x=0$. Thus $E(\langle 0\rangle)x=0$ and $E(p-\langle 0\rangle)x=0$, but then $x=E(\langle 0\rangle)x+E(p-\langle 0\rangle)x=0$. Now by Corollary 2 of Theorem 2, §3

$$\overline{T_{p-\langle 0\rangle}E(p-\langle 0\rangle)X}=E(p-\langle 0\rangle)X$$

and by assumption

$$\overline{N_0X} = E(\langle 0\rangle)X$$

but

$$\overline{TX} \supset \overline{T_{p-\langle 0\rangle}E(p-\langle 0\rangle)X}$$

and

$$\overline{TX} \supset \overline{N_0X}$$

therefore

$$\overline{TX} \supset X\,.$$

3. By Part 2, T is one-to-one. Let x be a vector in $E(\langle 0\rangle)X$ whose distance from N_0X is greater than some positive number r. Let y be any vector in X. Then

$$|x - Ty| = |x - N_0y - TE(p-\langle 0\rangle)y|\,.$$

Hence

$$\begin{aligned}|x-Ty| &\geqq \frac{1}{M}\,|E(\langle 0\rangle)[x - N_0y - TE(p-\langle 0\rangle)y]| \\ &= \frac{1}{M}\,|x - N_0E(\langle 0\rangle)y| \geqq \frac{r}{M}\,.\end{aligned}$$

Hence

$$x \notin \overline{TX}\,.$$

The next theorem is valid for separable spaces only.

THEOREM 4. *If X is separable, then $\sigma_p(T) \cup \sigma_r(T)$ is countable.*

Proof. Theorems 1 and 2 show that $\sigma_p(T) \cup \sigma_r(T) \subset \sigma_p(S) = \{\lambda \mid E(\langle\lambda\rangle) \neq 0\}$. For any λ in $\sigma_p(S)$ let x_λ be a vector satisfying $|x_\lambda| = 1$ and $E(\langle\lambda\rangle)x_\lambda = x_\lambda$. Now if $\lambda_1 \neq \lambda_2$ then

$$|x_{\lambda_1} - x_{\lambda_2}| \geqq \frac{1}{M}\,|E(\langle\lambda_1\rangle)(x_{\lambda_1} - x_{\lambda_2})| = \frac{|x_{\lambda_1}|}{M} = \frac{1}{M}\,.$$

The set $\{x_\lambda \mid \lambda \in \sigma_p(S)\}$ is separable because X is, hence the set is countable.

We conclude this discussion by studying another subset of the spectrum.

DEFINITION. Let A be a bounded linear operator on X, then $\sigma_0(A)$

$=\{\lambda \mid \text{there exists a sequence } (x_n) \text{ such that } |x_n|=1 \text{ and } (\lambda I-A)x_n \to 0\}$. See [5] p. 51.

LEMMA 1. $\sigma_p(S) \subset \sigma_0(T)$.

Proof. Let $x \neq 0$ satisfy $Sx=\lambda x$. If for some n, $N^n x=0$, let us take the first such integer. Then

$$TN^{n-1}x=(S+N)N^{n-1}x=N^{n-1}Sx=\lambda N^{n-1}x ,$$

and thus $\lambda \in \sigma_p(T) \subset \sigma_0(T)$. If for every n, $N^n x \neq 0$ then

$$T\frac{(N^n x)}{|N^n x|}=(S+N)\frac{N^n x}{|N^n x|}=\lambda\frac{N^n x}{|N^n x|}+\frac{N^{n+1}x}{|N^n x|} .$$

It is enough to show that for some subsequence n_i

$$\frac{|N^{n_i+1}x|}{|N^{n_i}x|} \to 0 .$$

Let us assume, to the contrary, that for some $\varepsilon>0$ $|N^{n+1}x| \geqq \varepsilon |N^n x|$ for all n, then

$$|x| \leqq \frac{|Nx|}{\varepsilon} \leqq \frac{|N^2 x|}{\varepsilon^2} \leqq \cdots \leqq \frac{|N^n x|}{\varepsilon^n} ,$$

but this would imply that

$$\lim_{n\to\infty} \sqrt[n]{|N^n|}=\lim_{n\to\infty} \sqrt[n]{|N^n|}\sqrt[n]{|x|} \geqq \limsup_{n\to\infty} \sqrt[n]{|N^n x|}$$

$$\geqq \limsup_{n\to\infty} \varepsilon\sqrt[n]{|x|}=\varepsilon .$$

But N is a generalized nilpotent and thus $\lim_{n\to\infty}\sqrt[n]{|N^n|}=0$.

THEOREM 5. $\sigma(T)=\sigma_0(T)$.

Proof. By Theorem 2 and Lemma 1 $\sigma_p(T) \cup \sigma_r(T) \subset \sigma_p(S) \subset \sigma_0(T)$. Thus it is enough to show that $\sigma_c(T) \subset \sigma_0(T)$. Let $\lambda \in \sigma_c(T)$ we may assume that $\lambda=0$. If $0 \notin \sigma_0(T)$ then $|Tx| \geqq \varepsilon |x|$, $x \in X$, for some positive ε. This implies that TX has a closed range, but $\overline{TX}=X$ hence $TX=X$, which contradicts the assumption that $0 \in \sigma_c(T)$.

Let us conclude this section with a few examples.

1. Define in l_1 the generalized nilpotent operator N by

$$N(x_1, x_2, x_3, \cdots)=(x_2, 0, x_4, 0, \cdots)$$

and let $S=0$. S is compact while T is not weakly compact.

2. Let X be the space of continuous functions on $[0, 1]$ vanishing

at the point 0. Define N by $Nf=g$, $g(x)=\int_0^x f(s)\,ds$, and let $S=0$. S has a closed range while T does not. $0 \in \sigma_p(S)$ but $0 \in \sigma_c(T)$.

3. Let N be defined as in 2, and $S=I$. T and S have closed ranges but the range of N is not closed.

5. Decompositions of spectral operators. Let $T_1, \cdots, T_n$ be n commuting operators. There exists a minimal algebra of operators $\mathfrak{A}$, with the properties:

1. $T_i \in \mathfrak{A}$, $i=1, 2, \cdots, n$.
2. If $U \in \mathfrak{A}$ and U^{-1} is a bounded everywhere defined operator then $U^{-1} \in \mathfrak{A}$.
3. The algebra $\mathfrak{A}$ is uniformly closed.

This algebra will be called the full algebra generated by $T_1, \cdots, T_n$, and it is a commutativealgebra. Let $\Delta_{\mathfrak{A}}$ denote the space of homomorphisms from $\mathfrak{A}$ to the algebra of complex numbers. By Condition 2, and the Gelfand theory [4], if $U \in \mathfrak{A}$ then $\sigma(U)=\{\mu(U) \mid \mu \in \Delta_{\mathfrak{A}}\}$; thus if $\mu(U)=0$ for each $\mu \in \Delta_{\mathfrak{A}}$ then U is a generalized nilpotent.

LEMMA 1. *Every scalar operator S is the sum S_1+iS_2 where S_1 and S_2 are scalar operators and*

1. $S_1S_2=S_2S_1$.
2. *$\sigma(S_1)$ and $\sigma(S_2)$ are sets of real numbers.*
3. *The Boolean algebra of projections generated by the resolutions of the identity of S_1 and S_2 is bounded.*

Proof. Let $E(\cdot)$ be the resolution of the identity of S; then

$$S=\int zE(dz)=\int(x+iy)E(dz)=\int xE(dz)+i\int yE(dz)$$
$$=\int \lambda E_1(d\lambda)+i\int \lambda E_2(d\lambda)$$

where

$$E_1(\alpha)=E\{z \mid z=x+iy \text{ and } x \in \alpha\}$$

$$E_2(\alpha)=E\{z \mid z=x+iy \text{ and } y \in \alpha\}$$

Conditions 1, 2, and 3 are readily verified.

THEOREM 1. *Let T be a spectral operator. Then there exist two operators R and J such that*

1. *$T=R+iJ$ and $RJ=JR$*
2. *The sets $\sigma(R)$ and $\sigma(J)$ are real sets.*
3. *R is a scalar operator and J is a spectral operator.*

4. *The Boolean algebra of projections generated by the resolutions of the identity of R and J is bounded.*

If R_1 and J_1 satisfy Conditions 1 *and* 2, *then they are spectral operators and there exists a generalized nilpotent M such that*

$$R_1=R+M\ , \qquad J_1=J+iM\ .$$

REMARK. By the last assertion and Theorem 8 of [1] Conditions 1, 2, and 3 insure uniqueness. We shall call R the real part of T and J the imaginary part of T.

Proof. Let $T=S+N$. Using the notation of Lemma 1, put $R=S_1$, $J=S_2-iN$, and Conditions 1., 2., 3., and 4. follow by Lemma 1. Now, if R_1 and J_1 satisfy 1., and 2., then by Theorem 5 of [1], the operators R, J, R_1, J_1 commute. Let $\mathfrak{A}$ be the full algebra generated by these operators, if $\mu\in\Delta_{\mathfrak{A}}$ then

$$0=\mu(T-T)=\mu(R-R_1)+i\mu(J-J_1)$$

but $\mu(R-R_1)$ and $\mu(J-J_1)$ are real numbers by Condition 2. Hence

$$\mu(R-R_1)=\mu(J-J_1)=0\ .$$

Thus if $M=R-R_1$ then M is a generalized nilpotent and $J-J_1=iM$.

LEMMA 2. *Every scalar operator S can be written as the product of two scalar operators T_1 and T_2 which satisfy*

1. $T_1T_2=T_2T_1=S$.

2. *$\sigma(T_1)$ is a set of non-negative numbers and $\sigma(T_2)$ is a subset of the unit circle.*

3. *The Boolean algebra of projections generated by the resolutions of the identity of T_1 and T_2 is bounded.*

Proof. It follows from the multiplicative property of the spectral measure $E(\cdot)$ of S that

$$S=\int\lambda E(d\lambda)=\int|\lambda|E(d\lambda)\int\operatorname{sgn}\lambda E(d\lambda)\ .$$

Thus $S=T_1T_2$ where

$$T_1=\int|\lambda|E(d\lambda)=\int\mu E_1(d\mu)\ \text{if}\ E_1(\cdot)$$

is defined by

$$E_1(\alpha)=E\{\lambda\mid|\lambda|\in\alpha\}$$

and

$$T_2=\int \operatorname{sgn} \lambda E(d\lambda)=\int \mu E_2(d\mu)$$

where

$$E_2(\alpha)=E\{\lambda \mid \operatorname{sgn} \lambda \in \alpha\} .$$

It is easy to verify Conditions 1, 2, and 3.

THEOREM 2. *Let T be a spectral operator. Then there exist two operators P and U such that*

1. $T=PU=UP$.
2. *$\sigma(P)$ is a set of non-negative numbers and $\sigma(U)$ is a subset of the unit circle.*
3. *U is a scalar operator and P is spectral.*
4. *The Boolean algebra of projections generated by the resolutions of the identity of P and U is bounded.*

If P_1 and U_1 satisfy 1. and 2., then they are spectral operators and $U_1=U+N_1$ $P_1=P+N_2$ where N_1 ond N_2 are generalized nilpotents and

$$N_2=\sum_{n=0}^{\infty}(-N_1U^{-1})^{n+1}P .$$

REMARK. By the last assertion Conditions 1, 2, and 3 insure uniqueness. The operator P will be called the absolute value of T and U the argument of T.

Proof. Let $T=S+N$. Using the notation of Lemma 2 put $P=(T_1+T_2^{-1}N)$ and $U=T_2$, then $PU=T$ because $T_2N=NT_2$ (Theorem 8 of [**1**]). Now, Conditions 1, 2, 3, and 4 follow by Lemma 2. Let P_1 and U_1 satisfy 1 and 2; then by Theorem 8 of [1], P_1, U_1, P, U commute. Let $\mathfrak{A}$ be the full algebra generated by these operators. If $\mu \in \Delta_{\mathfrak{A}}$ then $\mu(T)=\mu(P)\mu(U)=\mu(P_1)\mu(U_1)$ and by Condition 2 $\mu(P)=\mu(P_1)$ and $\mu(U)=\mu(U_1)$. Thus $N_1=U_1-U$ and $N_2=P_1-P$ are generalized nilpotents. Now

$$T=UP=(U+N_1)(P+N_2)=UP+N_1P+UN_2+N_1N_2$$

or

$$-PN_1=(U+N_1)N_2$$

hence

$$\begin{aligned} N_2&=-(U+N_1)^{-1}N_1P \\ &=-\left(\sum_{n=0}^{\infty}(-1)^n(U^{-1})^{n+1}N_1^n\right)N_1P=\sum_{n=0}^{\infty}(-U^{-1}N_1)^{n+1}P . \end{aligned}$$

In order to apply these theorems we need the following result.

THEOREM 3. *A spectral operator T is a scalar operator whose spectrum lies on the unit circle if and only if: T^{-1} is a bounded everywhere defined operator, and there exists a constant M such that*

$$|T^n| \leqq M \qquad n = \pm 1, \pm 2, \cdots .$$

Proof. If $T = \int_{|\lambda|=1} \lambda E(d\lambda)$ then

$$|T^n| = \left| \int_{|\lambda|=1} \lambda^n E(d\lambda) \right| \leqq 4 \sup \{|E(\alpha)| \, | \alpha \text{ a Borel set}\} ,$$

by [1], p. 341. Conversely assume that $|T^n| \leqq M$ $n = \pm 1, \pm 2, \cdots$ then

$$R(\lambda\,;T) = \begin{cases} \sum_{n=0}^{\infty} \dfrac{T^n}{\lambda^{n+1}} , & |\lambda| > 1 \\ -\sum_{n=0}^{\infty} \lambda^n (T^{-1})^{n+1} , & |\lambda| < 1 \end{cases}$$

because that two series converge. Thus $\sigma(T) \subset \{\lambda \mid |\lambda| = 1\}$ and $|R(\lambda\,;T)| \leqq M/|1 - |\lambda||$ if $|\lambda| \neq 1$. By Lemma 3.16 of [2] if $T = S + N$, where S is scalar and N is a generalized nilpotent, then $N^2 = 0$. Hence

$$T^n = S^n + nNS^{n-1} .$$

Therefore $nN = (T^n - S^n)S^{-(n-1)}$.

Thus nN is a bounded sequence of operators and therefore $N = 0$.

LEMMA 3. *Let S_1 and S_2 be two commuting scalar operators with real spectra, if $S_1 + S_2$ is spectral then it is scalar.*

Proof. Let $S_1 + S_2 = S + N$ where S is scalar and N is a generalized nilpotent. By Theorem 3 the operator $e^{i(S+N)} = e^{iS_1} \cdot e^{iS_2}$ is a scalar operator, but

$$e^{i(S+N)} = e^{iS} e^{iN} = e^{iS} + iNe^{iS} \sum_{n=1}^{\infty} \frac{(iN)^{n-1}}{n!} ,$$

hence

$$iNe^{iS} \sum_{n=1}^{\infty} \frac{(iN)^{n-1}}{n!} = 0$$

but the operator $ie^{iS} \sum_{n=1}^{\infty} \dfrac{(iN)^{n-1}}{n!}$ possesses an inverse and thus $N = 0$.

THEOREM 4. *Let S_1 and S_2 be two commuting scalar operators, if $S_1 + S_2$ is spectral then*

1. *$S_1 + S_2$ is a scalar operator.*

2. *The real (imaginary) part of* S_1+S_2 *is the sum of the real (imaginary) parts of* S_1 *and* S_2.

Proof. Let S_1, S_2 and S_1+S_2 be decomposed into real and imaginary parts as in Theorem 1. Then

$$S_1=R_1+iJ_1\,, \qquad S_2=R_2+iJ_2\,, \qquad S_1+S_2=R+iJ$$

where R_1, J_1, R_2, J_2 and R are scalar operators, while J is spectral, and would be scalar if and only if S_1+S_2 is a scalar operator. The operators R_1, J_1, R_2, J_2 commute and thus by the Gelfand theory [4] R_1+R_2 and J_1+J_2 have real spectra. By Theorem 1 $R_1+R_2=R+M$ and $J_1+J_1=J+iM$, where M is a generalized nilpotent. By Lemma 3 the operator R_1+R_2 is a scalar operator, but R is scalar too, thus by Theorem 8 of [1] $M=0$. Now $J_1+J_2=J$ which is a spectral operator and, again, by Lemma 3, J is scalar. Thus S_1+S_2 is scalar and $R_1+R_2=R$, $J_1+J_2=J$.

THEOREM 5. *Let* S_1 *and* S_2 *be two commuting scalar operators. If* S_1S_2 *is spectral then*

1. S_1S_2 *is a scalar operator.*
2. *The absolute value (argument) of* S_1S_2 *is the product of the absolute values (arguments) of* S_1 *and* S_2.

Proof. Let S_1, S_2 and S_1S_2 be decomposed as in Theorem 2.

$$S_1=P_1U_1\,, \qquad S_2=P_2U_2 \qquad S_1S_2=PU\,.$$

The operators U_1, U_2, U, P_1 and P_2 are scalar, and P is a spectral operator, which is scalar if and only if S_1S_2 is scalar. Using commutativity of the operators in question and Theorem 2 we derive that

$$P_1P_2=P+N_2\,, \qquad U_1U_2=U+N_1\,,$$

where N_1 and N_2 are generalized nilpotents and $N_2=\sum_{n=0}^{\infty}(-N_1U^{-1})^{n+1}P$. By Theorem 3, $N_1=0$ and hence $N_2=0$ too, which proves the second assertion. In order to complete the proof it remains to show that P_1P_2 is scalar. Now P is spectral, let $P=P_1P_2=S+M$ where S is scalar and M a generalized nilpotent. Let $E(\cdot)$ and $F(\cdot)$ be the resolutions of the identity of P_1 and P_2 respectively. Denote $E\{\lambda\,|\,\lambda>\varepsilon_1\}=E_{\varepsilon_1}$ and $F\{\lambda\,|\,\lambda>\varepsilon_2\}=F_{\varepsilon_2}$, then the spectrum of $E_{\varepsilon_1}P_1F_{\varepsilon_2}P_2=SE_{\varepsilon_1}F_{\varepsilon_2}+ME_{\varepsilon_1}F_{\varepsilon_2}$ on $E_{\varepsilon_1}F_{\varepsilon_2}X$ is contained in the set $\{\lambda\,|\,\lambda\geqq\varepsilon_1\varepsilon_2\}$ by the Gelfand theory. The operator $\log(E_{\varepsilon_1}P_1E_{\varepsilon_2}P_2)$ is thus well defined and it is not difficult to show that it is equal to $\log(E_{\varepsilon_1}P_1)+\log(E_{\varepsilon_2}P_2)$. This sum is spectral by [1], p. 340, and by Theorem 4 it is scalar. Thus $E_{\varepsilon_1}P_1F_{\varepsilon_2}P_2$ is scalar and therefore $ME_{\varepsilon_1}F_{\varepsilon_2}=0$. By countable additivity $ME_0F_0=0$ but $P_1E_0=P_1$ and $P_2F_0=P_2$. Thus

$$P_1P_2=P_1E_0P_2F_0=SE_0F_0+ME_0F_0=SE_0F_0\ ,$$

but $P_1P_2=S+M$, hence $S+M=SE_0F_0$, therefore $S=SE_0F_0$ and $M=0$ by Theorem 8 of [1]. Hence $P_1P_2=S$ is a scalar operator.

REMARK. From Theorems 4 and 5 it follows that the sum or product of two commuting spectral operators is spectral, if and only if, the sum or product of their scalar parts is scalar.

A decomposition of a non-spectral operator A into real and imaginary parts is possible in some cases.

THEOREM 6. *Let A be an operator and $\sigma(A)\subset K$ where K satisfies*

1. *There exists a function f which is analytic and one-to-one in a neighborhood of K.*
2. *The image of K is a subset of the unit circle.*
3. *The inverse function of f exists and is analytic in a neighborhood of the unit circle, let us denote this function by g.*
4. $g(\bar{z})=\overline{g(z)}$ *if* $|z|=1$.

Then $A=A_1+iA_2$ where $\sigma(A_1)$ and $\sigma(A_2)$ are sets of real numbers and $A_1A_2=A_2A_1$. If $A=B_1+iB_2$ where B_1 and B_2 satisfy the same conditions then $B_1=A_1+N$ and $B_2=A_2+iN$ and N is a generalized nilpotent.

Proof. Let $\varphi(z)=g(1/f(z))$ then φ is analytic in a neighborhood of K and for $z\in K$, $\varphi(z)=\bar{z}$. Define

$$A_1=\frac{A+\varphi(A)}{2} \quad \text{and} \quad A_2=\frac{A-\varphi(A)}{2i}\ .$$

If $\mathfrak{A}$ is the full algebra generated by A and $\mu\in\Delta_{\mathfrak{A}}$,

$$\mu(A_1)=\frac{\mu(A)+\varphi(\mu(A))}{2}$$

is the real part of $\mu(A)$, and $\mu(A_2)$ is the imaginary part of $\mu(A)$. Thus the first part of the theorem is proved. The second part is proved as in Theorem 1.

We conclude this section by a study of roots of operators. The operator B is said to be an nth root of A if $B^n=A$. The operators A and B commute $AB=BA=B^{n+1}$. Let $\mathfrak{A}$ be the full algebra generated by B. If $\mu\in\Delta_{\mathfrak{A}}$ then $\mu(B)^n=\mu(A)$ thus

$$\sigma(B)\subset(\sigma(A))^{1/n}$$

Thus if $B^n=I$ then $\sigma(B)\subset\{\lambda\,|\,\lambda^n=1\}$ and hence is a finite set. By Theorem VII. 3.20 of [3], B is spectral and by Theorem 3, B is a scalar operator. Thus

$$B=\sum_{k=0}^{n-1} e^{2k\pi i/n}E_k \qquad \text{where } E_k^2=E_k,\ E_kE_j=0$$

if $k\neq j$, and $\sum_{k=0}^{n-1} E_k=I$.

THEOREM 7. *Let S be a scalar operator with real spectrum whose resolution of the identity is $E(\cdot)$. Let $S_1=\int\lambda^{1/n}E(d\lambda)$ where* $\arg\lambda^{1/n}=(\arg\lambda)/n$. *If S_2 satisfies $S_2^n=S$, then $\sigma(S_2)\subset(\sigma(S))^{1/n}$, and if $\sigma(S_2)\subset\{\lambda^{1/n}\mid\lambda\in\sigma(S)$ and $\arg\lambda^{1/n}=(\arg\lambda)/n\}$ then*

$$S_2=S_1+N \quad \textit{and} \quad N=NE(\langle 0\rangle) \quad \textit{and} \quad N^n=0\ .$$

Proof. The operators S_1 and S_2 commute by [1] p. 329. Let $\mathfrak{A}$ be the full algebra generated by them. If $\mu\in\Delta_{\mathfrak{A}}$ then $\mu(S_1)=\mu(S_2)$ and thus $S_2-S_1=N$ is a generalized nilpotent. Now

$$(1)\qquad S=S_2^n=S_1^n+nNS_1^{n-1}+\frac{n(n-1)}{2}N^2S_1^{n-2}+\cdots+N^n=S_1^n$$

therefore

$$N\Big(nS_1^{n-1}+\frac{n(n-1)}{2}NS_1^{n-2}+\cdots+N^{n-1}\Big)=0$$

but by Corollary 4 of Theorem 1, Section 3, $NS_1^{n-1}=0$. Thus by Theorem 2 of §3, $N=NE(\langle 0\rangle)$, but then $NS_1^q=0$ for every integer q. Instead of (1) we have, therefore,

$$S=S_1^n+N^n \quad \text{or} \quad N^n=0$$

which completes the proof.

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