A NUMERICAL CONDITION FOR MODULARITY
OF A LATTICE

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1. Introduction. In this note a simple numerical condition (θ) is presented which is necessary for modularity of a finite lattice L. Though not sufficient (θ) appears to be a condition imposing a strong tendency toward modularity.

NOTATION. Covering, proper inclusion, and inclusion will be denoted by $>$, $\supset$, $\supseteq$ respectively. $N[S]$ will denote the order of the set S. The unit and zero elements will be denoted by u and z respectively.

DEFINITION 1. A finite lattice L is upper semi-modular [1: p. 100] if and only if

(ξ') $\qquad\qquad a$ and $b > a \cap b$ imply $a \cup b > a$ and b.

L is lower semi-modular if and only if

(ξ'') $\qquad\qquad a \cup b > a$ and b imply a and $b > a \cap b$.

DEFINITION 2. In a finite lattice let $C(a) = \{x \in L \,|\, x < x \cup a > a\}$ and $D(a) = \{x \in L \,|\, x > x \cap a < a\}$.

2. Tests for modularity An immediate consequence of Definitions 1 and 2 is the following theorem.

THEOREM 1. *In a finite lattice L condition (ξ') is equivalent to $D(a) \subseteq C(a)$ for all $a \in L$ and both imply $N[D(a)] \leq N[C(a)]$. Dually, (ξ'') is equivalent to $D(a) \supseteq C(a)$ for all $a \in L$ and both imply $N[D(a)] \geq N[C(a)]$. Moreover, modularity, (ξ') and (ξ''), is equivalent to $D(a) = C(a)$ for all $a \in L$ and both imply the condition (θ):*

(θ) $\qquad\qquad N[D(a)] = N[C(a)]$ for all $a \in L$.

The contrapositive of the last statement of Theorem 1 serves as a useful test for non-modularity :

THEOREM 2. *If there exists $a \in L$ for which $N[D(a)] \neq N[U(a)]$, then L is non-modular.*

When either (ξ') or (ξ'') is known to hold in L, the verification of the condition (θ) is a test often easiest to apply. It merely requires counting coverings.

THEOREM 3. *In a finite lattice L (ξ') and (θ) together imply modularity and dually (ξ'') and (θ) together likewise imply modularity.*

Proof. From Theorem 1 condition (ξ') implies $D(a) \subseteq C(a)$, and along with (θ) we obtain $D(a) = C(a)$ for all $a \in L$. Hence L is modular.

Condition (θ) appears to be a very strong condition toward modularity. It would be useful to know a much weaker but easily applicable condition than (ξ') or (ξ'') to serve along with (θ) as a set of necessary and sufficient conditions for modularity.

3. Near-modular lattices.

DEFINITION 3. A finite lattice L is *near-modular*, henceforth abbreviated NM, if and only if (θ) is valid and the Jordan-Dedekind chain condition is satisfied.

REMARK. It is conceivable that the JD chain condition is implied by (θ), though no proof was readily found. The imposition of the JD condition seems desirable, since it is satisfied in all finite semi-modular and modular lattices. Hence each element of a NM lattice L will possess a uniquely determined rank.

THEOREM 4. *In a NM lattice L we have $D(x) = C(x)$ whenever x is a point (atom) or dual point. Condition (ξ') is satisfied by all pairs of points and (ξ'') by all pairs of dual points.*

Proof. Let p be an arbitrary point of L and q any element in $C(p)$. By consideration of rank, q is also a point which is distinct from p. Hence $q > q \cap p = z$, the zero element of L, and $q \in D(p)$. Thus $C(p) \subseteq D(p)$. Equality of orders yields $C(p) = D(p)$. Any pair of points p and q cover their meet z so that $q \in D(p)$. Hence $q \in C(p)$ so that (ξ') is valid for p and q. The remainder of Theorem 4 follows by duality.

COROLLARY. *All NM lattices of rank less than 4 are modular.*

THEOREM 5. *There exist NM lattices of rank 4 that are nonmodular.*

The smaller example L_4 of the two examples found was constructed from the finite projective geometry PG(2, 2) as follows. If the points of PG(2, 2) are designated by 1, 2, 3, 4, 5, 6, 7, the lines, considered as sets of points, can be taken as 356, 467, 571, 612, 723, 134, 245, and $u = 1234567$. For L_4 take $u = 1234567$, and the 7 dual points as 1247, 2351, 3462, 4573, 5614, 6725, 7136, namely the complementary sets to the dual points of PG(2, 2). The remaining elements of L_4 are generated by

taking all point set meets of its dual points. The lines of L_4 are the $21 = \binom{7}{2}$ pairs of points 12, 13, $\cdots$, 67; the points are 1, 2, 3, 4, 5, 6, 7; and z is the null set.

The automorphism group of L_4 is easily seen from the manner of its construction to be the same as that for PG(2, 2), of order 168. L_4 possesses dual automorphisms, one of which carries the planes in the order indicated above into the points 7, 6, 5, 4, 3, 2, 1 respectively. Moreover, L_4 is a complemented point (atomic) lattice. It possesses no non-trivial homomorphic images, since all prime quotients are projective.

When the above procedure of construction of L_4 from PG(2, 2) was applied to PG(2, 3), PG(2, 2^2), PG(3, 2) and other PG(k, p^n), the lattices obtained were all found to violate (θ). Some of them violated also the JD chain condition.

The structure of L_4 suggested a method of obtaining additional examples of non-modular NM lattices as follows. Let L_t consist of z; n points $p_1, p_2, \cdots, p_n$ where $n = 1 + \binom{t}{2}; \binom{n}{2}$ lines consisting of all pairs of points: $p_1 p_2, \cdots, p_{n-1} p_n$; n planes of which the first is the set of t points $p_{i_1} p_{i_2} \cdots p_{i_r} \cdots p_{i_t}$ where $i_r = 1 + \binom{r}{2}$, ($r = 1, 2, \cdots, t$) and the remaining planes are obtained from the first by repeated applications of the cyclic permutation (123 $\cdots$ n) to the subscripts; and $u = p_1 p_2 \cdots p_n$. This procedure yields for $t = 1, 2, 3$ the Boolean algebras B^1, B^2, B^4 respectively. For $t = 4$, the lattice L_4 described above is obtained. For $t = 5$ a second example, L_5, of a non-modular NM lattice of length 4 is obtained. For $t \geq 6$ one fails to obtain a lattice. It can readily be shown by consideration of certain congruences that for $t \geq 6$ there always exist at least two pairs of planes, as described in the construction, which intersect in three or more points and other pairs of planes that intersect in less than two points. When two planes have three points p, q, r in common, the lines pq, pr, and qr have each of the planes as upper bound, but fail to have a least upper bound.

4. **Extensions,** In this section, methods of construction of other NM lattices from given ones are presented.

THEOREM 6. *The direct product of* NM *lattices is also an* NM *lattice.*

Proof. Let $L = L_1 \times L_2 \times \cdots \times L_n$ where the components are NM lattices. Represent each $a \in L$ in the usual way as the n-tuple ($a_1, \cdots, a_n$) with $a \in L_i$ ($i = 1, \cdots, n$), so that $a \cup b$ and $a \cap b$ are obtained by taking joins and meets respectively component-wise. Let $C(a)$ and $D(a)$ be the functions of Definition 2. Define $H(a)$ as the set of elements covering $a \in L$ and $K(a)$ as the set of elements covered by a. Let

$C(a_i)$, $D(a_i)$, $H(a_i)$, $K(a_i)$ be the corresponding sets with respect to $a_i \in L_i$. Now $a>b$ in L if and only if $a_j>b_j$ for some j and $a_i=b_i$ for $i\neq j$. It follows readily that

$$(1) \qquad N[C(a)]=\sum_{i=1}^{n} N[C(a_i)]+\sum_{i\neq j} N[H(a_i)]\cdot N[K(a_j)]$$

$$(2) \qquad N[D(a)]=\sum_{i=1}^{n} N[D(a_i)]+\sum_{i\neq j} N[K(a_j)]\cdot N[H(a_i)]$$

The last summations of the two equations are equal. By hypothesis (θ_i): $N[C(a_i)]=N[D(a_i)]$ for $i=1, \cdots, n$. Hence (θ) is valid in L.

NOTATION. $L=a/b$ indicates a lattice with unit element a and zero element b. The set sum and product of lattices L_1 and L_2, considered only as sets of elements, will be denoted by L_1+L_2 and $L_1 \cdot L_2$ respectively.

LEMMA 1. *If $L_1=a_1/z$ possesses a dual ideal a_1/b_1 isomorphic to an ideal a_2/b_2 of a second lattice $L_2=u/b_2$, then by identifying as x each pair of elements $x_1 \in a_1/b_1$ and $x_2 \in a_2/b_2$ that correspond under the isomorphism, a lattice $L=u/z$ can be constructed having L_1 as an ideal and L_2 as a dual ideal such that $L=L_1+L_2$ and $a/b\equiv L_0=L_1 \cdot L_2$.*

The elements of L are taken as the identified elements $x \in a/b$ and the remaining elements of L_1 and L_2. Join $\cup$ and meet $\cap$ in L are defined in terms of $\cup_1$, $\cap_1$ in L_1 and $\cup_2$, $\cap_2$ in L_2 according to the cases:

$$\left.\begin{array}{l} r \cup s=r \cup_i s \\ r \cap s=r \cap_i s \end{array}\right\} \quad r, s \in L_i, \ (i=1, 2)$$

$$\left.\begin{array}{l} r \cup s=s \cup r=(r \cup_1 b) \cup_2 s \\ r \cap s=s \cap r=r \cap_1(a \cap_2 s) \end{array}\right\} \quad r \in L_1, s \in L_2 \ .$$

The verification of the lattice postulate is routine and is omitted. This method of extension was first employed systematically by M. Hall and R. P. Dilworth [2; Lemma 4.1].

In Lemmas 2, 3, 4 and Theorem 7 following, let $L=u/z$, ideal $L_1=a/z$, dual ideal $L_2=u/b$, and quotient sublattice $L_0=a/b$ be related as in Lemma 1: $L=L_1+L_2$ and $L_0=L_1 \cdot L_2$. We note that $L-L_2, L_0, L-L_1$ is a partitioning of L into disjoint subsets.

LEMMA 2. *If $s>r$ in L, then s and r are both in L_1 or both in L_2.*

Proof. Obviously impossible is the case $s \notin L_2$, $r \notin L_1$. Assume that $s \notin L_1$, $r \notin L_2$; that is, $b \subseteq s \nsubseteq a, a \supseteq r \nsupseteq b$. Then $s=s \cup r=s \cup(b \cup r) \supset b \cup r$, otherwise $s \subseteq b \cup r \subseteq a$, a contradiction. Furthermore $b \cup r \supset r$, otherwise $b \subseteq r$,

a contradiction. Thus the covering $s>r$ is violated and the only possible cases are as stated.

LEMMA 3.   *If* $D_i(x)$, $C_i(x)$ *are the functions of Definition* 2 *relative to* L_i $(i=0,1,2)$, *then*

$$(3)\quad D_0(x)=D_1(x)\cdot D_2(x)$$
$$(3')\quad C_0(x)=C_1(x)\cdot C_2(x)$$
$$(4)\quad D(x)=D_1(x)+D_2(x)$$
$$(4')\quad C(x)=C_1(x)+C_2(x)$$

Proof. (3) holds since $r\in D_0(x)$: $r>r\cap x<x$ with r, $r\cap x$, x all in $L_0=L_1\cdot L_2$ if and only if $r>r\cap x<x$ with $r,r\cap x,x$ all in both L_1 and L_2; that is, $r\in D_1(x)$, $D_2(x)$, $D_1(x)\cdot D_2(x)$. Next, $r\in D_1(x)$: $r>r\cap x<x$ with $r, r\cap x, x$ all in L_1 implies $r\in D(x)$: $r>r\cap x<x$ with $r, r\cap x, x$ all in L. Thus $D_1(x)\subseteqq D(x)$. Similarly $D_2(x)\subseteqq D(x)$ so that $D_1(x)+D_2(x)\subseteqq D(x)$. For demonstration of the less trivial reverse inclusion let $r\in D(x)$: $r>r\cap x<x$ in L. If $r\cap x\notin L_2$, both r and x are in L_1, along with $r\cap x_1$, by Lemma 2. For this case $r\in D_1(x)$. If $r\cap x\in L_2$, then certainly also are both r and x. Whence $r\in D_2(x)$. We thus obtain $D(x)\subseteqq D_1(x)+D_2(x)$ and therefore (4). Dually (3′) and (4′) are valid.

LEMMA 4.   *The following statements of orders are valid:*

$$(5)\qquad N[D(x)]+N[D_0(x)]=N[D_1(x)]+N[D_2(x)]$$
$$(5')\qquad N[C(x)]+N[C_0(x)]=N[C_1(x)]+N[C_2(x)]$$

This follows immediately from Lemma 3.

THEOREM 7.   *If any three of* L, L_1, L_2, L_0, *related as in* Lemmas 1–4, *are near-modular, then all are near-modular.*

Proof. Equality of three pairs of corresponding members of (5) and (5′) implies equality of the remaining pair.

REMARKS. It is no doubt possible to construct non-modular NM lattices in other ways; for example, by piecing together several NM lattices to become the ideals of L and several others to become the dual ideals of L. Such a construction would require perhaps a more precise knowledge of the basic structure of a NM lattice.

A sublattice, and even a quotient sublattice, of a NM lattice is not necessarily near-modular. It is an open question whether or not the homomorphic image of a NM lattice is near-modular.

REFERENCES

1. G. Birkhoff, *Lattice theory*, rev. ed., Amer. Math. Soc. Colloquium Publications, New York, 1948.
2. M. Hall and R. P. Dilworth, *The imbedding theorem for modular lattices*, Ann. of Math., **45** (1944), 450–456.

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