AN INVERSION OF THE STIELTJES TRANSFORM

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A generalized Lambert transform, or L-transform, is an integral of the form

$$H(x)=\int_0^\infty \sum_{k=1}^\infty a_k e^{-kxt}\psi(t)dt \ .$$

In this paper we shall invert the integral transform

$$(1) \qquad G(x)=\int_0^\infty \frac{\psi(t)t\,dt}{x^2+t^2} \qquad 0<x<\infty$$

by reducing it by means of a certain summation to an L-transform and then applying an inversion theorem for L-transforms.

From this we deduce an inversion formula for the Stieltjes transform. This is given in Theorem 3.

1. The inversion of the transform (1). We shall need the following theorem on L-transforms which is the case $r=1$ of Theorem 7.7 in [1].

THEOREM 1. *Let $\{a_k\}_{k=1}^\infty$ be a bounded sequence of non-negative numbers with $a_1>0$. Let $\{b_n\}_{n=1}^\infty$ be the (unique) sequence such that*

$$\sum_{d|m} a_d b_{m/d}=\begin{cases}1, & m=1\\ 0, & m=2,\ 3,\ \cdots\ ,\end{cases}$$

the summation running over all divisors d of m. If the b_n are also bounded and if

1. $K(t)=\sum_{k=1}^\infty a_k e^{-kt}$ $(0<t<\infty)$
2. $H(x)=\int_0^\infty K(xt)\psi(t)dt$ *converges for some* $x>0$
3. $\int_0^1 \frac{|\psi(t)\log t|}{t}dt<\infty$

then

$$\lim_{p\to\infty}\frac{(-1)^p}{p\,!}\left(\frac{p}{t}\right)^{p+1}\sum_{n=1}^\infty b_n n^p H^{(p)}\left(\frac{np}{t}\right)=\psi(t) \ \textit{almost everywhere} \ (0<t<\infty)\ .$$

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Now let

$$G(x)=\int_0^\infty \frac{\psi(t)t\,dt}{x^2+t^2}$$

where we assume

$$\int_0^\infty \frac{|\psi(t)|}{t}dt<\infty \quad \text{and} \quad \int_0^1 \frac{|\psi(t)\log t|}{t}dt<\infty \ .$$

To reduce $G(x)$ to an L-transform we define

$$H_N(x)=\frac{1}{x}\left[\frac{G(0)}{2}+\sum_{k=1}^{N}(-1)^k G\left(\frac{k\pi}{x}\right)\right] \qquad N=1,\ 2,\ \cdots$$

Then

$$(2) \qquad H_N(x)=\int_0^\infty \psi(t)\left[\frac{1}{2xt}+\sum_{k=1}^{N}\frac{(-1)^k xt}{(xt)^2+(k\pi)^2}\right]dt \ .$$

For $N=1,\ 2,\ \cdots$ we have

$$\left|\sum_{k=1}^{N}\frac{(-1)^k xt}{(xt)^2+(k\pi)^2}\right|<\frac{xt}{(xt)^2+\pi^2}<\frac{1}{xt} \qquad (0<xt<\infty).$$

(This is because the terms of the sum alternate in sign and decrease in absolute value so that the modulus of the sum is less than that of its first term.) Hence for any $x>0$

$$\int_0^\infty \left|\psi(t)\left[\frac{1}{2xt}+\sum_{k=1}^{N}\frac{(-1)^k xt}{(xt)+(k\pi)^2}\right]\right|dt\leqslant\frac{3}{2x}\int_0^\infty \frac{|\psi(t)|}{t}dt<\infty \ .$$

This, by dominated convergence, allows us to let N become infinite under the integral sign in (2) and we obtain

$$H(x)\equiv\lim_{N\to\infty}H_N(x)=\int_0^\infty \psi(t)\left[\frac{1}{2xt}+\sum_{k=1}^{\infty}\frac{(-1)^k xt}{(xt)^2+(k\pi)^2}\right]dt \qquad (x>0).$$

But for $z>0$

$$\frac{1}{2z}+\sum_{k=1}^{\infty}\frac{(-1)^k z}{z^2+(k\pi)^2}=\frac{\text{cosech } z}{2}=\frac{1}{e^z-e^{-z}}=\sum_{k=1}^{\infty}e^{-(2k-1)z}\ ,$$

(see [3 ; 113]). Thus

$$(3) \qquad H(x)=\int_0^\infty \sum_{k=1}^{\infty}e^{-(2k-1)xt}\psi(t)dt=\int_0^\infty K(xt)\psi(t)dt$$

where $K(t)=\sum_{k=1}^{\infty}a_k e^{-kt}$ and

(4) $$a_{2k-1}=1\ ,\ a_{2k}=0\ , \qquad k=1,\ 2,\ \cdots$$

It was shown in [2; 556] that the sequence $\{b_n\}_{n=1}^{\infty}$ defined in Theorem 1 corresponding to the a_k in (4) is

(5) $$b_{2n-1}=\mu_{2n-1}\ ,\ b_{2n}=0\ , \qquad n=1,\ 2,\ \cdots$$

Here the μ_n are the Moebius numbers defined as $\mu_1=1$, $\mu_n=(-1)^s$ if n is the product of s distinct primes and $\mu_n=0$ if n is divisible by a square factor. The b_n are bounded, so that we may apply Theorem 1 (with the a_k and b_n as in (4) and (5)) to invert the L-transform (3) and obtain $\psi(t)$ for almost all $t>0$. These results are summarized in Theorem 2.

THEOREM 2. *Let*

$$G(x)=\int_0^\infty \frac{\psi(t)t\,dt}{x^2+t^2}$$

where

$$\int_0^\infty \frac{|\psi(t)|}{t}dt<\infty$$

and

$$\int_0^1 \frac{|\psi(t)\log t|}{t}dt<\infty$$

Then

$$H(x)=\lim_{N\to\infty}\frac{1}{x}\left[\frac{G(0)}{2}+\sum_{k=1}^{\infty}(-1)^k G\left(\frac{k\pi}{x}\right)\right]$$

exists for all positive x and

$$H(x)=\int_0^\infty \sum_{k=1}^{\infty} e^{-(2k-1)xt}\psi(t)dt\ .$$

Moreover

$$\lim_{p\to\infty}\frac{(-1)^p}{p\,!}\left(\frac{p}{t}\right)^{p+1}\sum_{n=1}^{\infty}\mu_{2n-1}(2n-1)^p H^{(p)}\left[\frac{(2n-1)p}{t}\right]=\psi(t)$$

almost everywhere $(0<t<\infty)$.

2. The inversion of the Stieltjes transform. Let

(6) $$F(x)=\int_0^\infty \frac{\varphi(t)}{x+t}dt$$

where

$$\int_0^\infty \frac{|\varphi(t)|}{t}\,dt<\infty \quad \text{and} \quad \int_0^1 \frac{|\varphi(t)\log t|}{t}\,dt<\infty \ .$$

Let $G(x)=\frac{1}{2}F(x^2)$, $\psi(t)=\varphi(t^2)$. Then

$$G(x)=\frac{1}{2}\,F(x^2)=\frac{1}{2}\int_0^\infty \frac{\varphi(t)}{x^2+t}dt=\int_0^\infty \frac{\varphi(t^2)t\,dt}{x^2+t^2}=\int_0^\infty \frac{\psi(t)t\,dt}{x^2+t^2}\ ;$$

also

$$\int_0^\infty \frac{|\psi(t)|}{t}dt=\int_0^\infty \frac{\varphi(t^2)}{t}dt=\frac{1}{2}\int_0^\infty \frac{\varphi(t)}{t}dt<\infty\ ;$$

similarly

$$\int_0^1 \frac{|\psi(t)\log t|}{t}dt<\infty\ .$$

The assumptions of Theorem 2 thus hold. We can therefore use Theorem 2 to obtain $\psi(t)=\varphi(t^2)$ for almost all $t>0$. This gives us $\varphi(t)$ for almost all $t>0$ and thus effects an inversion of the Stieltjes transform (6).

THEOREM 3. *Let*

$$F(x)=\int_0^\infty \frac{\varphi(t)}{x+t}\,dt$$

where

$$\int_0^\infty \frac{|\varphi(t)|}{t}\,dt<\infty$$

and

$$\int_0^1 \frac{|\varphi(t)\log t|}{t}dt<\infty\ .$$

Let

$$G(x)=\frac{1}{2}F(x^2)$$

and

$$H(x)=\frac{1}{x}\left[\frac{G(0)}{2}+\sum_{k=1}^\infty(-1)^kG\left(\frac{k\pi}{x}\right)\right]$$

(the sum converging by Theorem 2). *Then*

$$\lim_{p\to\infty}\frac{(-1)^p}{p\,!}\left(\frac{p}{\sqrt{t}}\right)^{p+1}\sum_{n=1}^{\infty}\mu_{2n-1}(2n-1)^p H^{(p)}\left[\frac{(2n-1)p}{\sqrt{t}}\right]=\varphi(t)$$

almost everywhere $(0<t<\infty)$.

Of course, the Stieltjes transform has been inverted under less restrictive conditions on $\varphi(t)$. We believe the interest of this note lies in the use of the μ_n as an inverting device.

References

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