ON INTEGRATION OF 1-FORMS

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1. Introduction. It has been noted by several people that in order to define the integral of some differential 1-form ω along a curve C, the latter need not be of bounded variation. For example, in the extreme (and trivial) case where ω is the differential of some function f, the integral can be defined as the difference of the values assumed by f at the end-points of C. No condition on C is necessary. H. Whithney [4], with J. H. Wolfe, by the introduction of certain norms, has found general abstract spaces of curves along which the integral of 1-forms satisfying certain conditions can be defined. In fact, H. Whitney considers integration of p-forms with $p \geq 1$. In a previous paper [2], we obtained rather awkward conditions for a decent integral to exist that depended on the number of higher derivatives of ω on C.

In this paper, we consider 1-forms ω possessing 'higher derivatives' on C in a sense somewhat different from that due to H. Whitney [3] which we used previously. A Lipschitz type condition on the remainders of the Taylor expansion is imposed (see 4.1.). We define the α-variation of a curve as the supremum of sums of αth powers of chords (see 2.7) and show that the integral of ω along C exists if the α-variation of C is bounded, where α is related to the number of 'higher derivatives' of ω on C. Under somewhat stronger hypotheses on C, we show that this integral is an anti-derivative of ω on C.

2. Notation and basic definitions. Throughout this paper, N is a positive integer and we use the following notation.

2.1. E denotes Euclidean $(N+1)$-space.

2.2. $\|x\| = \left(\sum_{i=0}^{N} x_i^2\right)^{1/2}$ for $x \in E$.

2.3. diam $U = \sup\{d : d = \|x - y\|$ for some $x \in U$ and $y \in U\}$

2.4. φ is a continuous function on the closed unit enterval to E and $C = \text{range } \varphi$.

2.5. $\mathscr{S}$ is the set of all subdivisions of the unit interval, i.e. functions T on $\{0, 1, \cdots, k\}$ for some positive integer k such that: $T(0) = 0, \quad T(k) = 1, \quad T(i-1) < T(i)$ for $i = 1, \cdots, k$

2.6. $[T/a, b] = \{i : a \leq T(i-1) < T(i) \leq b\}$

2.7. $V_\alpha(a, b) = \sup\limits_{T \in \mathscr{S}} \sum\limits_{i \in [T/a,b]} \|\varphi(T(i-1) - \varphi(T(i))\|^\alpha$

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3. Properties of V_α.

3.1. LEMMA. *If* $0 \leq a \leq b \leq c \leq 1$, *then*

$$V_\alpha(a, b) + V_\alpha(b, c) \leq \alpha(a, c) \leq V_\alpha(a, b) + V_\alpha(b, c) + (\operatorname{diam} C)^\alpha$$

3.2. LEMMA. *If* $\alpha < \beta$ and $V_\alpha(a, b) < \infty$, then $V_\beta(a, b) > \infty$.

Proof. Since $V_\alpha(a, b) < \infty$, there is an integer n such that there are at most n elements $i \in [T/a, b]$ with $\| \varphi(T(i-1)) - \varphi(T(i)) \| \geq 1$ for any $T \in \mathscr{S}$. For any other $i \in [T/a, b]$ we have

$$\| \varphi(T(i-1)) - \varphi(T(i)) \|^\beta < \| \varphi(T(i-1)) - \varphi(T(i)) \|^\alpha .$$

Hence,

$$V_\beta(a, b) < V_\alpha(a, b) + n(\operatorname{diam} C)^\beta < \infty .$$

4. Integration of 1-forms. In this section, we first define the kind of differential form we shall be dealing with. Our definition is a variant of Whitney's definition of a function m times differentiable on a closed set [3]. Next, we choose a special sequence of subdivisions and proceed to define the integral of the form over the curve C by taking sums of polynomials of degree m and then passing to the limit. Under conditions involving the generalized variation V_α, we show that the integral exists and possesses, in particular, the properties of linearity and 'anti-derivative'.

Throughout this section, m is a positive integer, $\eta \geq 0$, $K > 0$.

4.1. *The Differential Form.* Let

$$\sigma k = \sum_{i=0}^{N} k_i \text{ for any } (N+1)\text{-tuple } k .$$

A differential 1-form ω on C is a function on the set of all $(N+1)$-tuples k, for which k_i is a non-negative integer for $i = 0, \cdots, N$ and $1 \leq \sigma k \leq m$, to the set of real-valued functions on C such that

$$\omega_k(y) = \sum_{\sigma j=0}^{m-\sigma k} \omega_{k+j}(x) \frac{(y_0 - x_0)^{j_0} \cdots (y_N - x_N)^{j_N}}{j_0! \cdots j_N!} + R_k(x, y)$$

where

$$| R_k(x, y) | < K \| x - y \|^{m+\eta-\sigma k} \text{ for } x \in C \text{ and } y \in C .$$

It is important to note that, in case $m = 1$ and $\eta > 0$, ω is a differential form on C satisfying a Hölder condition. If however $m > 1$, then ω is also a closed differential form on C, that is, $d\omega = 0$ on C.

By taking $m = 1$ and $\eta = 1$, we get the sharp forms considered by Whitney. The conditions we impose on C, however, are quite different and, we feel, in practice easier to check than those obtained in [4].

4.2. *The sequence of subdivisions.* We define first, for each $(n+1)$-tuple of non-negative integers $(s_0, \cdots, s_n)$, a point $t(s_0, \cdots, s_n)$ by recursion on n and on s_n. These will be the end-points of the nth subdivision of the unit interval.

4.2.1. DEFINITION. $t(0) = 0, \qquad t(1) = 1\,,$

$$t(s_0, \cdots, s_n, 0) = t(s_0, \cdots, s_n)\,,$$
$$t(s_0, \cdots, s_n, j+1) = \sup \{u : t(s_0, \cdots, s_n, j) \leq u \leq t(s_0, \cdots, s_n + 1)$$
$$\text{and } \| \varphi(u') - \varphi(t(s_0, \cdots, s_n, j)) \| \leq \frac{1}{2^{n+1}} \text{ for } t(s_0, \cdots, s_n, j) \leq u' \leq u\}$$

for any non-negative integers n and j.

We shall denote by T the sequence of subdivisions of the unit interval such that:

$$\text{range } T_n = \{u : u = t(s_0, \cdots, s_n) \text{ for some } n\text{-tuple } (s_0, \cdots, s_n)\}\,.$$

4.2.2. LEMMA. *For any non-negative integers n and j, we have*

$$t(s_0, \cdots, s_n) \leq t(s_0, \cdots, s_n, j) \leq t(s_0, \cdots, s_n + 1)\,.$$

4.2.3. LEMMA. *For any positive integer n, $i \in [T_n/0, 1]$, $j \in [T_{n-1}/0, 1]$ we have: T_{n+1} is a refinement of T_n,* i.e. range $T_n \subset$ range T_{n+1};

if
$$T_n(i-1) \leq u \leq T_n(i)\,,$$
then
$$\| \varphi(T_n(i-1)) - \varphi(u) \| \leq \frac{1}{2^n}\,;$$

if
$$T_{n-1}(j-1) \leq T_n(i-1) < T_n(i) < T_{n-1}(j)\,,$$
then
$$\| \varphi(T_n(i-1)) - \varphi(T_n(i)) \| = \frac{1}{2^n}\,.$$

4.2.4 LEMMA. *If $F(x, y)$ is a real number whenever $0 \leq x \leq y \leq 1$, $a \in$ range T_n, $b \in$ range T_n, and $a \leq b$, then*

$$\sum_{i \in [T_{n+1}/a,b]} F(T_{n+1}(i-1), T_{n+1}(i)) = \sum_{j \in [T_n/a,b]} \sum_{i \in [T_{n+1}/T_n(j-1), T_n(j)]} F(T_{n+1}(i-1), T_{n+1}(i))\,.$$

4.3. *The integral of* ω. First, we define $\int_b^a \omega d\varphi$ as the limit of certain sums of polynomials.

4.3.1. *Definitions.*

$$\begin{aligned}
P'(x, y) &= \sum_{\sigma k=1}^{m} \omega_k(x) \frac{(y_0 - x_0)^{k_0} \cdots (y_N - x_N)^{k_N}}{k_0\,! \cdots k_N\,!} ,\\
P(a, b) &= P'(\varphi(a), \varphi(b)),\\
S_n(a, b) &= \sum_{i \in [T_n/a, b]} P(T_n(i-1), T_n(i)) ,\\
\int_a^b \omega d\varphi &= \lim_{n\to\infty} S_n(a, b) .
\end{aligned}$$

Next, in order to prove the existence of $\int_a^b \omega d\varphi$ and some of its properties under conditions involving $V_\alpha(a, b)$ for some $\alpha < m + \eta$, we introduce the following.

4.3.2. *Definitions.*

$$R(x, y, z) = P'(x, y) + P'(y, z) - P'(x, z) .$$

$$M = K \sum_{\sigma k=1}^{m} \frac{1}{k_0\,! \cdots k_N\,!} .$$

$$\beta = m + \eta .$$

4.3.3. LEMMA. *If* $x, y, z \in C$, $\|x - y\| \leq \delta$ *and* $\|y - z\| \leq \delta$, *then*

$$|R(x, y, z)| < M\delta^\beta .$$

Proof. Let $h(v) = P'(x, v)$ for $v \in E$. Then, h is a polynomial of degree m. Let $\boldsymbol{O}_r = \{k : k$ is an $(N+1)$-tuple of non-negative integers and $1 \leq \sigma k \leq r\}$.
For $k \in \boldsymbol{O}_r$ and $p \in \boldsymbol{O}_r$, let $p \geq k$ iff $p_i \geq k_i$ for $i = 0, \cdots, N$, and let

$$D_k h(v) = \frac{\partial^{\sigma k} h(v)}{\partial^{k_0} v_0 \cdots \partial^{k_N} v_N} ,$$

then

$$D_k h(v) = \sum_{\substack{p \in \boldsymbol{O}_m \\ p \geq k}} \omega_p(x) \frac{(v_0 - x_0)^{p_0 - k_0} \cdots (v_N - x_N)^{p_N - k_N}}{(p_0 - k_0)\,! \cdots (p_N - k_N)\,!} .$$

Hence, by Taylor's formula

$$h(z) = h(y) + \sum_{k \in \boldsymbol{O}_m} D_k h(y) \frac{(z_0 - y_0)^{k_0} \cdots (z_N - y_N)^{k_N}}{k_0\,! \cdots k_N\,!} = h(y) +$$

$$+\sum_{k\in O_m}\left\{\left[\sum_{\substack{p\in O_m\\ p\geq k}}\omega_p(x)\frac{(y_0-x_0)^{p_0-k_0}\cdots(y_N-x_N)^{p_N-k_N}}{(p_0-k_0)!\cdots(p_N-k_N)!}\right]\right.$$
$$\left.\cdot\frac{(z_0-y_0)^{k_0}\cdots(z_N-y_N)^{k_N}}{k_0!\cdots k_N!}\right\}.$$

On the other hand from 4.3.1 and 4.1 we have

$$P'(y,z)=\sum_{k\in O_m}\left\{\left[\omega_k(x)+\sum_{j\in O_{m-\sigma k}}\omega_{k+j}(x)\frac{(y_0-x_0)^{j_0}\cdots(y_N-x_N)^{j_N}}{j_0!\cdots j_N!}+R_k(x,y)\right]\right.$$
$$\left.\cdot\frac{(z_0-y_0)^{k_0}\cdots(z_N-y_N)^{k_N}}{k_0!\cdots k_N!}\right\}$$
$$=\sum_{k\in O_m}\left\{\left[\sum_{\substack{k\in O_m\\ p\geq k}}\omega_p(x)\frac{(y_0-x_0)^{p_0-k_0}\cdots(y_N-x_N)^{p_N-k_N}}{(p_0-k_0)!\cdots(p_N-k_N)!}+R_k(x,y)\right]\right.$$
$$\left.\cdot\frac{(z_0-x_0)^{k_0}\cdots(z_N-y_N)^{k_N}}{k_0!\cdots k_N!}\right]\Bigg\}$$
$$=h(z)-h(y)+\sum_{k\in O_m}R_k(x,y)\frac{(z_0-y_0)^{k_0}\cdots(z_N-y_N)^{k_N}}{k_0!\cdots k_N!}.$$

Making use of the condition on $R_k(x,y)$ stated in 4.1, we get

$$|P'(x,y)+P'(y,z)-P'(x,z)|<\sum_{k\in O_m}\frac{K\|y-x\|^{\beta-\sigma k}\|z-y\|^{\sigma k}}{k_0!\cdots k_N!}\leq M\delta^\beta.$$

4.3.4 LEMMA. *Suppose* $\|x(0)-x(i)\|\leq A$ and $\|x(i-1)-x(i)\|\leq A$ *for* $i=1,\cdots,p$, *whereas* $\|x(i-1)-x(i)\|=A/r$ *for* $i=1,\cdots,p-1$, *where all* $x(i)\in C$. *Then*

$$\left|\sum_{i=1}^p P'(x(i-1),x(i))-P'(x(0),x(p))\right|<Mr^\alpha A^{\beta-\alpha}\sum_{i=1}^p\|x(i-1)-x(i)\|^\alpha.$$

Proof. $\left|\sum_{i=1}^p P'(x(i-1),x(i))-P'(x(0),x(p))\right|$

$$\leq\sum_{i=2}^p|P'(x(0),x(i-1))+P'(x(i-1),x(i))-P'(x(0),x(i))|$$
$$=\sum_{i=2}^{p-1}|R(x(0),x(i-1),x(i))|<(p-1)MA^\beta=(p-1)Mr^\alpha A^{\beta-\alpha}\left(\frac{A}{r}\right)^\alpha$$
$$=Mr^\alpha A^{\beta-\alpha}\sum_{i=1}^{p-1}\|x(i-1)-x(i)\|^\alpha\leq Mr^\alpha A^{\beta-\alpha}\sum_{i=1}^p\|x(i-1)-x(i)\|^\alpha.$$

4.3.5 LEMMA. *Let* $n>1$, $a\in$ range T_n, $b\in$ range T_n, $a\leq b$, $[T_{n-1}/a,b]=0$. *Then* $|S_n(a,b)-P(a,b)|<M5^\beta V_\beta(a,b)$.

Proof. Let

$$a'=\sup\{u:u\in\text{range }T_{n-1}\text{ and }u\leq a\}$$
$$b'=\sup\{u:u\in\text{range }T_{n-1}\text{ and }u\leq b\}.$$

First, suppose $a \leq b' \leq b$. Then $a' < a$ and, by 4.2.3

$$\|\varphi(u) - \varphi(a')\| \leq \frac{1}{2^{n-1}} \qquad \text{for } a' \leq u \leq b'$$

$$\|\varphi(u) - \varphi(b')\| \leq \frac{1}{2^{n-1}} \qquad \text{for } b' \leq u \leq b\,.$$

Hence

$$\|\varphi(T_n(i)) - \varphi(a)\| \leq \frac{2}{2^{n-1}} \qquad \text{for } i \in [T_n/a, b]\,,$$

$$\|\varphi(T_n(i)) - \varphi(b')\| \leq \frac{1}{2^{n-1}} \qquad \text{for } i \in [T_n/b', b]\,,$$

$$\|\varphi(T_n(i-1)) - \varphi(T_n(i))\| = \frac{1}{2^n} \qquad \text{for } i \in [T_n/a, b],\ T_n(i) \neq b',\ T_n(i) \neq b\,.$$

Replacing α by β in 4.3.4 and using 4.3.3 and 3.1, we see that

$$\begin{aligned}
&|S_n(a, b) - P(a, b)| = |S_n(a, b') + S_n(b', b) - P(a, b)| \\
&\leq |S_n(a, b') - P(a, b')| + |S_n(b', b) - P(b', b)| + |P(a, b') + P(b', b) - P(a, b)| \\
&< M4^\beta V_\beta(a, b') + M2^\beta V_\beta(b', b) + MV_\beta(a, b) \leq M5^\beta V_\beta(a, b)\,.
\end{aligned}$$

Next suppose $b' < a$. Then, for $i \in [T_n/a, b]$,

$$\|\varphi(T_n(i)) - \varphi(a)\| \leq \frac{2}{2^{n-1}}\,,$$

$$\|\varphi(T_n(i-1)) - \varphi(T_n(i))\| = \frac{1}{2^n}\,.$$

Hence, by 4.3.4,

$$|S_n(a, b) - P(a, b)| < M4^\beta V_\beta(a, b)\,.$$

4.3.6 Lemma. Let $a \in \text{range}\, T_n$, $b \in \text{range}\, T_n$, $a < b$. *Then,*

$$|S_{n+1}(a, b) - S_n(a, b)| < M2^\alpha V_\alpha(a, b)\left(\frac{1}{2^{\beta-\alpha}}\right)^n.$$

Proof. Using 4.2.4, 4.2.3 and 4.3.4, we see that

$$\begin{aligned}
&|S_{n+1}(a, b) - S_n(a, b)| \\
&= \left|\sum_{j \in [T_n/a, b]} \left[\sum_{i \in [T_{n+1}/T_n(j-1), T_n(j)]} P(T_{n+1}(i-1), T_{n+1}(i)) - P(T_n(j-1), T_n(j))\right]\right| \\
&< \sum_{j \in [T_n/a, b]} \left[M2^\alpha\left(\frac{1}{2^n}\right)^{\beta-\alpha} \sum_{i \in [T_{n+1}/T_n(j-1), T_n(j)]} \|\varphi(T_{n+1}(i -)) - \varphi(T_{n+1}(i))\|^\alpha\right] \\
&= M2^\alpha\left(\frac{1}{2^n}\right)^{\beta-\alpha} \sum_{i \in [T_{n+1}/a, b]} \|\varphi(T_{n+1}(i-1)) - \varphi(T_{n+1}(i))\|^\alpha \leq M2^\alpha V_\alpha(a, b)\left(\frac{1}{2^{\beta-\alpha}}\right)^n.
\end{aligned}$$

4.3.7. Theorem. *If* $0 \leq a \leq b \leq 1$, $\alpha < \beta$, $V_\alpha(a, b) < \infty$, *then*

$$\left|\int_n^b \omega d\varphi\right| < \infty \, .$$

Proof. Let

$$a'_n = \inf\{u : u \in \text{range}\, T_n \text{ and } a \leq u\} \, ,$$
$$b'_n = \sup\{u : u \in \text{range}\, T_n \text{ and } u \leq b\} \, .$$

If $a = b$, the theorem is trivial. If $a < b$, for n sufficiently large, we have

$$a \leq a'_{n+1} \leq a'_n \leq b'_n \leq b'_{n+1} \leq b \, ,$$
$$[T_n/a, a'_n] = 0 \qquad \text{and } [T_n/b'_n, b] = 0 \, ,$$
$$\|\varphi(a'_{n+1})\varphi - (a'_n)\| \leq \frac{2}{2^n} \qquad \text{and } \|\varphi(b'_n) - \varphi(b'_{n+1})\| \leq \frac{1}{2^n} \, .$$

Hence

$$|S_{n+1}(a, b) - S_n(a, b)| = |S_{n+1}(a'_{n+1}, b'_{n+1}) - S_n(a'_n, b'_n)|$$
$$= |S_{n+1}(a'_{n+1}, a'_n) + S_{n+1}(a'_n, b'_n) + S_{n+1}(b'_n, b'_{n+1}) - S_n(a'_n, b'_n)|$$
$$\leq |S_{n+1}(a'_{n+1}, a'_n) - P(a'_{n+1}, a'_n)| + |S_{n+1}(a'_n, b'_n) - S_n(a'_n, b'_n)|$$
$$+ |S_{n+1}(b'_n, b'_{n+1})| - P(b'_n, b'_{n+1})| + |P(a'_{n+1}, a'_n)| + 1P(b'_n, b'_{n+1}| < (\text{by } 4.3.5,\ 4.3.6)$$
$$< M5^\beta V_\beta(a'_{n+1}, a'_n) + M2^\alpha V_\alpha(a'_n, b'_n)\left(\frac{1}{2^{\beta-\alpha}}\right)^n + M5^\beta V_\beta(b'_n, b'_{n+1}) + M'\frac{2}{2^n} + M'\frac{1}{2^n} \, ,$$

where

$$M' = \sup_{\substack{x \in C \\ 1 \leq \sigma k < m}} |\omega_k(x)| \sum_{\sigma k=1}^{m} \frac{1}{k_0 ! \cdots k_N !} \, .$$

Therefore, for any positive integer p we have

$$|S_{n+p}(a, b) - S_n(a, b)| \leq \sum_{q=0}^{p-1} |S_{n+q+1}(a, b) - S_{n+q}(a, b)|$$
$$< M5^\beta \sum_{q=0}^{\infty} [V_\beta(a'_{n+q+1}, a'_{n+q}) + V_\beta(b'_{n+q}, b'_{n+n+q+1})] + M2^\alpha V_\alpha(a, b) \sum_{q=0}^{\infty} \left(\frac{1}{2^{\beta-\alpha}}\right)^{n+q}$$
$$+ 3M' \sum_{q=0}^{\infty} \frac{1}{2^{n+q}} < M5^\beta (V_\beta(a, a'_n) + V_\beta(b'_n, b)) + M\frac{2^3}{2^{3-\alpha}-1} V_\alpha(a, b)\left(\frac{1}{2^{3-\alpha}}\right)^n + \frac{6M'}{2^n} \, .$$

Since, by 3.2, $V_\beta(a, b) < \infty$, with the help of 3.1 we see that $V_\beta(a, a'_n) \to 0$ and $V_\beta(b'_n, b) \to 0$ as $n \to \infty$. Thus, the $S_n(a, b)$ form a Cauchy sequence and $\left|\int_a^b \omega d\varphi\right| < \infty$.

4.3.8. THEOREM. *Suppose* $\delta > 0$, $\alpha < \beta$, $L < \infty$, $\|\varphi(a) - \varphi(b)\| < 1$, and

$$V_\alpha(a, b) < L \|\varphi(a) - \varphi(b)\|^\alpha$$

whenever $0 \leq a \leq b \leq 1$ *and* $b - a < \delta$. *Then, for some* $M' < \infty$,

$$\left|\int_a^b \omega d\varphi - P(a, b)\right| < M' \,||\, \varphi(a) - \varphi(b) \,||^\alpha$$

whenever $0 \leq a \leq b \leq 1$ and $b - a < \delta$.

Proof. Given $0 \leq a \leq b \leq 1$ and $b - a < \delta$, let

$$a'_q = \inf\{u : u \in \text{range}\, T_q \text{ and } a \leq u\}\ ,$$
$$b'_q = \sup\{u : u \in \text{range}\, T_q \text{ and } u \leq b\}\ ;$$

and let n be the integer such that $[T_{n-1}/a, b] = 0$, $[T_n/a, b] \neq 0$.

Given $\varepsilon > 0$, we can choose p so that

$$\left|\int_a^b \omega d\varphi - S_{n+p}(a'_{n+p}, b'_{n+p})\right| < \varepsilon$$

and

$$|\, P(a, b) - P('_{n+p}, b'_{n+p}) \,| < \varepsilon$$

and

$$|\; ||\, \varphi(a) - \varphi(b) \,|| - ||\, \varphi(a'_{n+p}) - \varphi(b'_{n+p}) \,|| \;| < \varepsilon\ .$$

Hence we need only to show that

$$|\, S_{n+p}(a'_{n+p}, b'_{n+p}) - P(a'_{n+p}, b'_{n+p}) \,| < M' \,||\, \varphi(a'_{n+p}) - \varphi(b'_{n+p}) \,||^\alpha$$

for some $M' < \infty$ and all positive integers p.

We can check that

$$\begin{aligned}
&|\, S_{n+p}(a'_{n+p}, b'_{n+p}) - P(a'_{n+p}, b'_{n+p}) \,| \\
&\leq |\, S_n(a'_n, b'_n) - P(a'_n, b'_n) \,| + |P(a'_{n+p}, a'_n) + P(a'_n, b'_n) - P(a'_{n+p}, b'_n)| \\
&+ |\, P(a'_{n+p}, b'_n) + P(b'_n, b'_{n+p}) - P(a'_{n+p}, b'_{n+p}) \,| \\
&+ \sum_{k=0}^{p-1} \{|P(a'_{n+p}, a'_{n+k+1}) + P((a'_{n+k+1}, a'_{n+k}) - P(a'_{n+p}, a'_{n+k}) \,| \\
&+ |\, P(b'_{n+k}, b'_{n+k+1}) + P(b'_{n+k+1}, b'_{n+p}) - P(b'_{n+k}, b'_{n+p}) \,| \\
&+ |\, S_{n+k+1}(a'_{n+k+1}, a'_{n+k}) - P(a'_{n+k+1}, a'_{n+k}) \,| \\
&+ |\, S_{n+k+1}(b'_{n+k}, b'_{n+k+1}) - P(b'_{n+k+1}, b'_{n+k}) \,| \\
&+ |\, S'_{n+k+1}(a'_{n+k}, b'_{n+k}) - S_{n+k}(a'_{n+k}, b'_{n+k}) \,|\}\ .
\end{aligned}$$

Now, we observe that

$$||\, \varphi(u) - \varphi(v) \,|| \leq \frac{2}{2^{n+k}} \qquad \text{for } a'_{n+p} \leq u \leq v \leq a'_{n+k}\ ,$$

$$||\, \varphi(u) - \varphi(v) \,|| \leq \frac{1}{2^{n+k}} \qquad \text{for } b'_{n+k} \leq u \leq v \leq b'_{n+p}\ ,$$

$$[T_{n+k}/a'_{n+k+1}, a'_{n+k}] = 0\ ,$$
$$[T_{n+k}/b'_{n+k}, b'_{n+k+1}] = 0\ .$$

Hence by 4.3.5, 4.3.3, 4.3.6 we have

$$
\begin{aligned}
&| S'_{n+p}(a'_{n+p}, b'_{n+p}) - P(a'_{n+p}, b'_{n+p}) | \\
&< M5^{\beta} V_{\beta}(a'_n, b'_n) + MV_{\beta}(a'_{n+p}, b'_n) + MV_{\beta}(a'_{n+p}, b'_{n+p}) \\
&+ M\sum_{k=0}^{p-1} \Big\{ V_{\alpha}(a'_{n+p}, a'_{n+k})\Big(\frac{2}{2^{n+k}}\Big)^{\beta-\alpha} + V_{\alpha}(b'_{n+k}, b'_{n+p})\Big(\frac{1}{2^{n+k}}\Big)^{\beta-\alpha} \\
&+ 5^{\beta} V_{\beta}(a'_{n+k+1}, a'_{n+k}) + 5^{\beta} V_{\beta}(b'_{n+k}, b'_{n+k+1}) + 2^{\alpha} V_{\alpha}(a'_{n+k}, b'_{n+k})\Big(\frac{1}{2^{\beta-\alpha}}\Big)^{n+k} \Big\} \\
&< M5^{\beta} V_{\beta}(a'_{n+p}, b'_{n+p}) + 2MV_{\beta}(a'_{n+p}, b'_{n+p}) \\
&+ MV_{\alpha}(a'_{n+p}, b'_{n+p})(2^{\beta-\alpha} + 1 + 2^{\alpha}) \sum_{k=0}^{\infty} \Big(\frac{1}{2^{\beta-\alpha}}\Big)^{n+k} \\
&< MV_{\alpha}(a'_{n+p}, b'_{n+p})\Big[5^{\beta} + 2 + (2^{\beta-\alpha} + 1 + 2^{\alpha}) \sum_{k=0}^{\infty} \Big(\frac{1}{2^{\beta-\alpha}}\Big)^{n+k} \Big] \\
&< M' \,\| \varphi(a'_{n+p}) - \varphi(b'_{n+p}) \|^{\alpha}
\end{aligned}
$$

where

$$M' = ML\Big[5^{\beta} + 2 + (2^{\beta-\alpha} + 1 + 2^{\alpha}) \sum_{k=0}^{\infty} \Big(\frac{1}{2^{\beta-\alpha}}\Big)^{n+k} \Big] < \infty .$$

4.3.9. THEOREM. *If* $0 \leq a \leq b \leq c \leq 1, \left| \int_a^b \omega d\varphi + \int_b^c \omega d\varphi \right| < \infty$, then

$$\int_a^c \omega d\varphi = \int_a^b \omega d\varphi + \int_b^c \omega d\varphi .$$

Proof. Let

$$
\begin{aligned}
a'_n &= \sup\{u : u \in \text{range } T_n \text{ and } u \leq b\} \\
b'_n &= \inf\{u : u \in \text{range } T_n \text{ and } b \leq u\} .
\end{aligned}
$$

We have $\lim_{n\to\infty} P(a'_n, b'_n) = 0$ and for sufficiently large n

$$S_n(a, c) = S_n(a, b) + P(a'_n, b'_n) + S_n(b, c) .$$

Taking the limit on both sides we get the desired result.

4.3.10. REMARK. If ω and ω' are both 1-forms in the sense of 4.1, then so is $(\omega + \omega')$ and

$$\int_a^b (\omega + \omega')d\varphi = \int_a^b \omega d\varphi + \int_a^b \omega' d\varphi$$

provided the right hand side is bounded. This is an immediate consequence of the definitions.

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