THE ASYMPTOTIC DISTRIBUTION OF THE EIGENVALUES FOR A CLASS OF MARKOV OPERATORS

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1. Introduction. This paper is an extension of the preceding paper "Markov Operators and their Associated Semi-groups" (hereafter referred to as MO) by R. K. Getoor. Throughout this paper we will retain the terminology, notations, and all the assumptions of §2 of MO. Let G be an open subset of X with $m(\partial G) = 0$ and suppose that for each $t > 0$, $f(t, x, y)$ is in $L_2(G \times G, m \times m)$. This is condition (K) in §6 of MO. Assume further that $f(t, x, y) = f(t, y, x)$ for all t, x, y and, for simplicity, that $f(t, x, y) > 0$ for all t, x, y. These assumptions will be retained throughout this paper. It is proved in §6 of MO that under these conditions there is a non-decreasing sequence $\{\lambda_j\}$ of non-negative numbers tending to infinity and a complete orthonormal set $\{\varphi_j\}$ in $L_2(G, m)$ such that the series

$$\sum_{j=1}^{\infty} e^{-\lambda_j t}\varphi_j(x)\varphi_j(y)$$

converges absolutely. It is further proved that if $k(t, x, y)$ denotes this sum (with $k(t, x, y) = 0$ if x or y is not in G) then $K(V, G; t, x, A) = \int_A k(t, x, y)dm(y)$ for all $t > 0$, x in G, A in $\mathscr{B}(X)$.

Intuitively one can think of k as the transition density of a Markov process that is obtained from $x(t)$ by "killing" $x(t)$ at the boundary of G and upon which a "local death rate" $V(x)$ is imposed. From this interpretation one would expect $k(t, x, y)$ to behave, in some sense, like $f(t, x, y)$ at least for small t and y close enough to x, provided x is in G and V is bounded. In the terminology of Kac [4] "the boundary and death rate aren't felt for small t". In §2 we make this statement precise by proving that if V is bounded and a certain regularity condition is imposed on f, then for all x in G, $k(t, x, y)f(t, x, y)^{-1} \to 1$ as $t \to 0$ for almost all y in a suitable neighborhood of x (Theorem 2.1). From this we are then able to show the somewhat surprising fact that $k(t, x, x)f(t, x, x)^{-1} \to 1$ as $t \to 0$ for *all* x in G (Theorem 2.2). Using these facts we derive the asymptotic distribution of the eigenvalues $\{\lambda_j\}$ for a wide class of processes (Theorem 2.3). In §3 we apply this theory to the symmetric stable processes on the real line and to the Ornstein-Uhlenbeck processes.

Received July 8, 1958. The first author was supported, in part, by the Office of Naval Research.

2. The main theorems. Let $\{(\mathscr{X}, \mathscr{B}(\mathscr{X}), P_x)\}_{x \in X}$ be the probability spaces constructed in §2 of MO. Let $G(t) = \{x(\cdot): x(\tau) \in \bar{G}; 0 \leqq \tau \leqq t\}$ and let $H(t)$ be the complement in $\mathscr{X}$ of $G(t)$, that is

$$H(t) = \{x(\cdot): x(\tau) \notin \bar{G} \text{ for some } \tau \leqq t\} .$$

It was shown in MO that $G(t)$, and hence $H(t)$, are in $\mathscr{B}(\mathscr{X})$.

From the definition of k above and the orthonormality of $\{\varphi_j\}$ it follows that

$$k(t+s, x, y) = \int k(t, x, z)k(s, z, y)dm(z) \tag{2.1}$$

for all t, s, x, y, and that $k(t, x, y) = k(t, y, x)$ for all t, x, y. Since $K(V, G; t, x, A) \leqq p(t, x, A)$ it follows that for each t and x, $0 \leqq k(t, x, y) \leqq f(t, x, y)$ a.e. (m), and from (2.1) and the symmetry of k and f it follows that these inequalities hold for all y. From now on we will assume that V is bounded on $\bar{G}$. In this case we have, for x in G, $e^{-Mt}K(0, G; t, x, A) \leqq K(V, G; t, x, A) \leqq K(0, G; t, x, A)$ where M is any upper bound of V on $\bar{G}$. If, for the moment, we let k and k' denote the densities of $K(V, G; t, x, A)$ and $K(0, G; t, x, A)$ respectively, defined by the corresponding series above, then for each t and x

$$e^{-Mt}k'(t, x, y) \leqq k(t, x, y) \leqq k'(t, x, y) \qquad \text{a.e. } (m) \tag{2.2}$$

and since k and k' each satisfy (2.1) and are symmetric these inequalities hold for all y.

In the remainder of this section we will assume that the density f satisfies the following condition:

(D) for every compact set A and every $\eta > 0$ there are numbers $t_0 > 0$ and $M > 0$ such that $f(\sigma, x, y)f(t, x, z)^{-1} \leqq M$ for all $\sigma \leqq t \leqq t_0$, x in A, y and z in X with $\rho(x, y) \geqq \eta$, $\rho(x, z) < \eta$. (ρ is the metric on X.)

In the applications, where X is the real line and ρ is the usual metric we will verify this condition for certain familiar process densities.

THEOREM 2.1. *For each x in G there is an open neighborhood $U \subset G$ of x such that $k(t, x, y)f(t, x, y)^{-1} \to 1$ as $t \to 0$ for almost all y in U.* (Note that an assumption of MO is that the support of m is X and hence $m(U) > 0$ whenever U is open and non-empty.)

Proof. In view of (2.2) and the remark following it we may assume $V \equiv 0$. Let $q(t, x, y) = f(t, x, y) - k(t, x, y)$. Then

$$\begin{aligned} \int_A q(t, x, y)dm(y) &= P_x[H(t) \cap \{x(\cdot): x(t) \in A\}] \\ &= Q(G; t, x, A) . \end{aligned}$$

Fix x in G and let $S_\varepsilon(x)$ be an open ε-neighborhood of x which is wholly contained in G. Let $\delta > 0$, be such that $4\delta < \varepsilon$ and $S_{2\delta}(x)$ has compact closure. Now if $\{x_k\}$ is a countable dense subset of X then for every $r_0 \geqq 1$ $\{S_{1/r}(x_k);\, r \geqq r_0, k \geqq 1\}$ is a countable family of sets which generates $\mathscr{B}(X)$. Thus we can construct a sequence $\{\mathscr{M}_n\}$ of finite partitions of X into $\mathscr{B}(X)$ sets such that for every n, $\mathscr{M}_{n+1}$ is a refinement of $\mathscr{M}_n$, $\mathscr{B}(X)$ is generated by the sets in these partitions, and any set in any of these partitions which intersects $S_\delta(x)$ is contained in $S_2(x)$. Since $Q(G; t, x, \cdot)$ is absolutely continuous with respect to $p(t, x, \cdot)$ and since $q(t, x, y)f(t, x, y)^{-1}$ is the derivative of Q with respect to p, it follows from known theorems on derivatives (see [2], pp. 343-344) that for almost all y in the sense of $p(t, x, \cdot)$ and hence for almost all y in the sense of m

$$\lim_{n\to\infty} \frac{Q(G; t, x, B_n)}{p(t, x, B_n)} = \frac{q(t, x, y)}{f(t, x, y)} \tag{2.3}$$

where B_n denotes that element of $\mathscr{M}_n$ which contains y. (The quotients on the left are taken to be 0 whenever the denominator vanishes.)

Given any $t > 0$ let $\{T_k\}$ $(T_k = \{t_{k1} < \cdots < t_{kk}\})$ with $t_{k1} = 0$ and $t_{kk} = t$ be an increasing sequence of subsets of $[0, t]$ becoming dense in $[0, t]$ as $k \to \infty$. Let

$$\Lambda_{kj} = \{x(\cdot): x(t_{kj}) \notin \bar{G}, x(t_{kl}) \in \bar{G}; l = 1, \cdots, j-1\}$$

and let $\Lambda_k = \bigcup_{j=1}^{k} \Lambda_{kj}$. For each k the Λ_{kj}'s are disjoint and $\Lambda_k \subset \Lambda_{k+1}$. Moreover $\bigcup_{k=1}^{\infty} \Lambda_k = H(t)$ so that for any $B \in \mathscr{B}(X)$ we have

$$Q(G, t, x, B) = \lim_{k\to\infty} \sum_{j=1}^{k} P_x[\Lambda_{kj} \cap \{x(\cdot): x(t) \in B\}] .$$

For each x in G and A in $\mathscr{B}(X)$ define $(p(0, x, A) = I_A(x))$

$$\mu_{kj}(x, A) = \int_{\bar{G}} \cdots \int_{\bar{G}} p(t_{k1}, x, dx_1) \cdots p(t_{kj} - t_{k(j-1)}, x_{j-1}, A) .$$

Then $\mu_{kj}(x, X - \bar{G}) = P_x[\Lambda_{kj}]$ and

$$\begin{aligned} &P_x[\Lambda_{kj} \cap \{x(\cdot): x(t) \in B\}] \\ &\qquad = \int_{X-\bar{G}} \mu_{kj}(x, dx_j) \int_B f(t - t_{kj}, x_j, y) dm(y) \end{aligned}$$

provided $t_{kj} < t$. On the other hand if $t_{kj} = t$ and $B \subset G$ the left side of this last equation is 0, so for convenience we define the right side to be 0 in this case. If B_n is in $\mathscr{M}_n$ and $B_n \subset G$

$$\text{(2.4)} \qquad \frac{Q(G; t, x, B_n)}{p(t, x, B_n)} = \lim_{k\to\infty} \frac{\sum_{j=1}^{k} \int_{X-\bar{G}} \mu_{kj}(x, dx_j) \int_{B_n} f(t - t_{kj}, x_j, z) dm(z)}{\int_{B_n} f(t, x, z) dm(z)} .$$

We wish to apply condition (D) with $A = \overline{S_{2\delta}(x)}$ and $\eta = 2\delta$. Let y be in $S_\delta(x)$ and let B_n be that element of $\mathscr{M}_n$ which contains y. By construction $B_n \subset S_{2\delta}(x)$ so if z is in B_n and x_j is in $X - \bar{G}$ then $\rho(x, z) < 2\delta$ and $\rho(x_j, z) > 2\delta$. Thus for sufficiently small t the right side of (2.4) does not exceed $M \cdot P_x[H(t)]$. This estimate depends on B_n only through the fact that $B_n \subset S_{2\delta}(x)$ so combining this with (2.3) we see that

$$q(t, x, y) f(t, x, y)^{-1} \leqq M P_x[H(t)]$$

for almost all y in $S_\delta(x)$ provided t is small enough (how small not depending on y). Then for almost all y in $S_\delta(x)$ we have

$$\text{(2.5)} \qquad 1 \geqq \frac{k(t, x, y)}{f(t, x, y)} \geqq 1 - M P_x[H(t)] .$$

By the right continuity of the paths $P_x[H(t)] \to 0$ as $t \to 0$ and so if we take $U = S_\delta(x)$ the proof of Theorem 2.1 is complete.

THEOREM 2.2. *For all x in G, $k(t, x, x) f(t, x, x)^{-1} \to 1$ as $t \to 0$.*

Proof. If x and δ are as in the preceding proof then

$$1 \geqq \frac{k(2t, x, x)}{f(2t, x, x)} = \frac{\int k(t, x, y) k(t, y, x) dm(y)}{\int f(t, x, y) f(t, y, x) dm(y)}$$

$$\geqq \frac{\left(\int_{S_\delta(x)} k^2(t, x, y) dm(y)\right)\left(\int_{S_\delta(x)} f^2(t, x, y) dm(y)\right)^{-1}}{1 + \left(\int_{X - S_\delta(x)} f(t, x, y) p(t, x, dy)\right)\left(\int_{S_\delta(x)} f(t, x, y) p(t, x, dy)\right)^{-1}} .$$

By (2.5) the expression in the numerator is not less than $(1 - M P_x[H(t)])^2$. Applying condition (D), with $A = \{x\}$ and $\eta = \delta$ to the second term in the denominator we find that for sufficiently small t it does not exceed $N \cdot p(t, x, X - S_\delta(x)) p(t, x, S_\delta(x))^{-1}$ where N is a fixed positive number. The right continuity of the paths implies that this last expression approaches 0 as $t \to 0$, and since $P_x[H(t)] \to 0$ as $t \to 0$ Theorem 2.2 is established.

Let $N(\lambda)$ be the number of the eigenvalues $\{\lambda_j\}$ which do not exceed λ, that is $N(\lambda) = \sum_{\lambda_j \leq \lambda} 1$. We next prove the following theorem concerning the asymptotic behavior of $N(\lambda)$.

THEOREM 2.3. *Suppose*

$$m(G) < \infty, \int_G f(t, x, x)dm(x) < \infty$$

for all sufficiently small t, *and*

$$\left[\int_G f^2(t, x, x)dm(x)\right]^{1/2}\left[\int_G f(t, x, x)dm(x)\right]^{-1}$$

remains bounded as $t \to 0$. *Then*

$$\int_G k(t, x, x)dm(x)\left(\int_G f(t, x, x)dm(x)\right)^{-1} \to 1 \qquad \text{as } t \to 0.$$

If in addition $\int_G f(t, x, x)dm(x) \sim At^{-\gamma}$ as $t \to 0$ *for some* A *and* $\gamma > 0$ *then* $N(\lambda) \sim A\lambda^{\gamma}(\Gamma(1+\gamma))^{-1}$ *as* $\lambda \to \infty$.

Proof. We have

$$\frac{\int_G q(t, x, x)dm(x)}{\int_G f(t, x, x)dm(x)} = \frac{\int_G \frac{q(t, x, x)}{f(t, x, x)} f(t, x, x)dm(x)}{\int_G f(t, x, x)dm(x)} \tag{2.6}$$

$$\leqq \left(\int_G \frac{q^2(t, x, x)}{f^2(t, x, x)}dm(x)\right)^{1/2} \frac{\left(\int_G f^2(t, x, x)dm(x)\right)^{1/2}}{\int_G f(t, x, x)dm(x)}$$

$m(G)$ is finite, $q(t, x, x)f(t, x, x)^{-1}$ is bounded by 1 and by Theorem 2.2 approaches 0 as $t \to 0$ for all x in G. The second factor in the last expression in (2.6) remains bounded as $t \to 0$, so

$$\left(\int_G q(t, x, x)dm(x)\right)\left(\int_G f(t, x, x)dm(x)\right)^{-1} \to 0 \qquad \text{as } t \to 0 .$$

This yields the first assertion of Theorem 2.3. From the definition of k it follows that

$$\int_G k(t, x, x)dm(x) = \sum_{j=1}^{\infty} e^{-\lambda_j t} = \int_0^{\infty} e^{-\lambda t}dN(\lambda) .$$

Thus by the first part of the theorem and the additional hypothesis of the second part we have

$$\int_0^{\infty} e^{-\lambda t}dN(\lambda) \sim \int_G f(t, x, x)dm(x) \sim At^{-\gamma} \qquad \text{as } t \to 0 .$$

The conclusion of the theorem then follows by applying the Karamata tauberian theorem [6. p. 192].

3. Applications. In this section we apply the results of §2 to the symmetric stable processes and the Ornstein-Uhlenbeck processes on the real line. First consider the symmetric stable process of index α $(0 < \alpha \leqq 2)$. Here $X = R^1$, m is Lebesgue measure, and $f(t, x, y) = g(t, x - y)$ where

$$g(t, x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ixu}e^{-t|u|^{\alpha}}du \; . \tag{3.1}$$

It is well known that the symmetric stable processes satisfy the conditions of §2 of MO and clearly f is symmetric in x and y. For each t, $f(t, x, y)$ is uniformly bounded so condition (K) in §6 of MO is satisfied if $m(G)$ is finite (in particular if $\bar{G}$ is compact). We wish to verify condition (D) for the density f. To this end we state three lemmas.

LEMMA 3.1. *For each $t > 0$, $g(t, x)$ decreases as $|x|$ increases.*

LEMMA 3.2. *Suppose φ, a real valued function defined on $[0, \infty)$, has N continuous derivatives and that $\varphi, \varphi^{(1)}, \cdots, \varphi^{(N)}$ are all absolutely integrable on $[0, \infty)$. Suppose that for each $n \leqq N - 1$, $\varphi^{(n)}(u) \to 0$ as $u \to \infty$. Then if $0 < \lambda < 1$ we have*

$$\begin{aligned} &\int_0^{\infty} u^{\lambda-1}\varphi(u)\cos bu\,du \\ &= -\sum_{n=0}^{N-1}\frac{\Gamma(n+\lambda)}{n!}\cos\left(\frac{\pi}{2}(n+\lambda-2)\right)\varphi^{(n)}(0)b^{-n-\lambda} + O(b^{-N}) \text{ as } b \to \infty . \end{aligned} \tag{3.2}$$

LEMMA 3.3. *For each $x \neq 0$, $g(t, x)$ is an increasing function of t in the domain $0 < t < B_{\alpha}|x|^{\alpha}$ where B_{α} is a positive constant independent of x.*

Lemma 3.1 is reasonably well known and a proof may be found in [7, Th. 11.8, p. 32]. Lemma 3.2 is a trivial modification of a theorem of Erdélyi [3, p. 48], to which we refer the reader. Lemma 3.3 is doubtless well known, but we are unable to find an explicit reference to it in the literature and so we give a proof.

Proof of Lemma 3.3. We fix $x \neq 0$ and look at the derivative $dg/dt = -(\pi)^{-1}\int_0^{\infty}(\cos xu)ue^{\alpha-tu^{\alpha}}du$. Making the change of variable $tu^{\alpha} = y^{\alpha}$ we obtain

$$\frac{dg}{dt} = -\frac{1}{\pi}t^{-1-1/\alpha}h_{\alpha}(b) \tag{3.3}$$

where $b = |x|t^{-1/\alpha}$ and

$$h_{\alpha}(b) = \int_0^{\infty} y^{\alpha}e^{-y^{\alpha}}\cos by\,dy \; .$$

If $0 < \alpha < 1$ we apply Lemma 3.2 with $N = 2$, $\lambda = \alpha$, and $\varphi(y) = ye^{-y^\alpha}$. φ clearly satisfies the assumptions of Lemma 3.2 and $\varphi(0) = 0$, $\varphi'(0) = 1$ so we obtain

$$\begin{aligned}(3.4) \qquad h_\alpha(b) &= -\Gamma(1+\alpha)\cos\left[\frac{\pi}{2}(\alpha-1)\right]b^{-1-\alpha} + O(b^{-2}) \\ &= -A(\alpha)b^{-1-\alpha} + O(b^{-2}) \qquad \text{as } b \to \infty\end{aligned}$$

where $A(\alpha) = \Gamma(1+\alpha)\cos\left[\frac{\pi}{2}(\alpha-1)\right] > 0$. If $1 < \alpha < 2$ we take $N=3$, $\lambda = \alpha - 1$, and $\varphi(y) = y^2e^{-y^\alpha}$ and obtain

$$(3.5) \qquad h_\alpha(b) = -A(\alpha)b^{-1-\alpha} + O(b^{-3}) \qquad \text{as } b \to \infty\ .$$

If $0 < \alpha < 1$ then (3.4) implies that there are constants M_α and b_α such that $|h_\alpha(b) + A(\alpha)b^{-1-\alpha}| \leqq M_\alpha b^{-2}$ if $b > b_\alpha$. Thus

$$\left|\frac{dg}{dt} - \frac{A(\alpha)}{\pi}|x|^{-1-\alpha}\right| \leqq M'_\alpha|x|^{-2}t^{-1+1/\alpha}$$

provided $|x|t^{-1/\alpha} > b_\alpha$ or equivalently $0 < t < b'_\alpha|x|^\alpha$. Then dg/dt will be positive if $M'_\alpha|x|^{-2}t^{-1+1/\alpha} < A(\alpha)\pi^{-1}|x|^{-1-\alpha}$ or equivalently if $0 < t < M''_\alpha|x|^\alpha$. Thus if we take $B_\alpha = \min(b'_\alpha, M''_\alpha)$ Lemma 3.3 is established for $0 < \alpha < 1$. If $1 < \alpha < 2$ a similar analysis beginning with (3.5) yields the desired result. Finally $g(t, x) = \pi^{-1}t(t^2 + x^2)^{-1}$ if $\alpha = 1$ and

$$g(t, x) = (2\sqrt{\pi t})^{-1}\exp(-x^2/4t)$$

if $\alpha = 2$ and the conclusion of the lemma is easily verified in these cases.

Now to verify condition (D) let A, a compact subset of R^1, and $\eta > 0$ be given. If $t_0 < B_\alpha\eta^\alpha$ where B_α is the constant of Lemma 3.3, if $|x-y| > \eta$ and if $0 < \sigma < t < t_0$ then $f(\sigma, x, y) = g(\sigma, x-y) \leqq g(t, x-y)$, and if $|x - z| \leqq \eta$ then

$$\frac{f(\sigma, x, y)}{f(t, x, z)} = \frac{g(\sigma, x-y)}{g(t, x-z)} \leqq \frac{g(t, x-y)}{g(t, x-z)} \leqq 1$$

the last inequality being a consequence of Lemma 3.1. In this case these estimates do not depend on x being in A.

Since

$$f(t, x, x) = g(t, 0) = (\pi)^{-1}\int_0^\infty e^{-tu^\alpha}du = (\alpha\pi)^{-1}t^{-1/\alpha}\Gamma(1/\alpha)\ ,$$

if $m(G) < \infty$ then the conditions of Theorem (2.3) are satisfied and we have

(3.6) $$N(\lambda) \sim \frac{\lambda^{1/\alpha}}{\pi} m(G) .$$

This is the asymptotic distribution of the eigenvalues for the symmetric stable process of index α on an open set G of finite Lebesgue measure with V bounded. This should be compared with the results of Kac [5]. (Kac's V is different from ours. His $V \equiv 1$ yields our results with our $V \equiv 0$.)

Next we turn to the Ornstein-Uhlenbeck processes. It is well known [1] that these processes satisfy the conditions of §2 of MO (in fact the paths can be taken to be continuous.) The transition density relative to Lebesgue measure of the $0 - U$ process with parameter $\beta > 0$ is given by

(3.7) $$\frac{1}{\sqrt{2\pi(1-\rho^2)}} \exp\left[-\frac{1}{2}\frac{(y-\rho x)^2}{1-\rho^2}\right]$$

where $\rho = \rho(t) = e^{-\beta t}$, $\beta > 0$, $t > 0$. This density is not symmetric, but if we introduce the measure m defined by $dm(y) = e^{-y^2/2}dy$ then the transition density relative to m is

(3.8) $$f(t, x, y) = \frac{1}{\sqrt{2\pi(1-\rho^2)}} \exp\left[-\frac{1}{2}\frac{\rho^2x^2 - 2\rho xy + \rho^2y^2}{1-\rho^2}\right]$$

which is symmetric. We now verify condition (D) for this density. Let the compact set A and number η be given. Then

(3.9) $$\frac{f(\sigma, x, y)}{f(t, x, z)} = \frac{(1-\rho^2)^{-1/2}}{(1-\theta^2)^{-1/2}} \frac{\exp\left[-\frac{1}{2}\frac{(y-x)^2}{1-\rho^2}\right]\exp\left[-\frac{xy}{1+\rho}\right]\exp\left[\frac{x^2}{2}\right]\exp\left[\frac{y^2}{2}\right]}{\exp\left[-\frac{1}{2}\frac{(z-x)^2}{1-\theta^2}\right]\exp\left[-\frac{xz}{1+\theta}\right]\exp\left[\frac{x^2}{2}\right]\exp\left[\frac{z^2}{2}\right]}$$

where $\rho = e^{-\beta\sigma}$ and $\theta = e^{-\beta t}$. The fourth factors in the numerator and denominator cancel. If we consider only x, y, and z such that x is in A, $|x - z| < \eta$ and $|y - x| \geqq \eta$ then the third and fifth factors in the denominator are bounded away from 0 and the second factor is no smaller than $\exp\left[-\frac{1}{2}\frac{\eta^2}{1-\theta^2}\right]$. Thus there exists a positive constant N_1 such that

(3.10) $$\frac{f(\sigma, x, y)}{f(t, x, z)} \leqq N_1 \frac{(1-\rho^2)^{-1/2}}{(1-\theta^2)^{-1/2}} \frac{\exp\left[-\frac{1}{2}\frac{(y-x)^2}{1-\rho^2}\right]\exp\left[-\frac{xy}{1+\rho}\right]\exp\left[\frac{y^2}{2}\right]}{\exp\left[-\frac{1}{2}\frac{\eta^2}{1-\theta^2}\right]}$$

The product of the exponentials in the numerator is precisely

$$\exp\left[-\frac{1}{2}\frac{(\rho y - x)^2}{1-\rho^2}\right].$$

If $|y| > 2\max_{x \in A}|x| + 2\eta$ *and* $\rho > 1/2$ *then* $(\rho y - x)^2 > \eta^2$. *But for any* other y such that $|x - y| \geqq \eta$, the second and third exponentials in the numerator of (3.10) are uniformly bounded while the first exponential does not exceed $\exp\left[-\frac{1}{2}\frac{\eta^2}{1-\rho^2}\right]$. Thus if t_0 is such that $e^{-\beta t_0} > 1/2$, then for $\sigma < t \leqq t_0$, x in A, $|x - y| \geqq \eta$, and $|x - z| < \eta$ we have

$$\frac{f(\sigma, x, y)}{f(t, x, y)} < N_2 \frac{(1-\rho^2)^{-1/2}\exp\left[-\frac{1}{2}\frac{\eta^2}{1-\rho^2}\right]}{(1-\theta^2)^{-1/2}\exp\left[-\frac{1}{2}\frac{\eta^2}{1-\theta^2}\right]} \tag{3.11}$$

where N_2 is a positive constant. The right side of (3.11) is easily seen to be uniformly bounded for $0 < \sigma < t \leqq t_0$ and thus condition (D) is verified.

For this density $f(t, x, x) = b(t)\exp(\rho x^2/1+\rho)$ where $b(t) = [2\pi(1-\rho^2)]^{-1/2}$ and $\rho = \rho(t) = e^{-\beta t}$. One verifies easily that if $\mu(G) < \infty$, where μ denotes Lebesgue measure, then condition (K) as well as all the hypotheses of Theorem 2.3 are satisfied. In particular since ρ increases to 1 as, $t \to 0$ we have

$$\int_G f(t, x, x)dm(x) = b(t)\int_G e^{(\rho x^2/1+\rho)}e^{-(x^2/2)}dx$$

$$\sim b(t)\mu(G) \sim \frac{\mu(G)}{2\sqrt{\beta\pi}}t^{-1/2} \qquad \text{as } t \to 0 .$$

So applying Theorem 2.3 we obtain for the $0-U$ process with parameter β

$$N(\lambda) \sim \frac{\mu(G)\lambda^{1/2}}{\pi\sqrt{\beta}} . \tag{3.12}$$

If G is the open interval (a, b) then the infinitesimal generator Ω'_G is given by the differential operator $\Omega'_G\varphi = \beta[\varphi'' + (x\varphi)'] - V\varphi$ on an appropriate domain in $L_2[G, m]$ subject to the boundary conditions $\varphi(a) = \varphi(b) = 0$. If $\beta = 1$ notice that (3.12) reduces to (3.6) with $\alpha = 2$. If $\alpha = 2$ in (3.6) the corresponding infinitesimal generator is given by $\varphi'' - V\varphi$ on an appropriate domain with the same boundary conditions. Thus the term $(x\varphi)'$ does not affect the asymptotic distribution of the eigenvalues, which is certainly what one would expect. The λ_j are the eigenvalues of $-\Omega'_G$ in each case. See Theorem 6.3 of MO.

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